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Abrupt Bifurcation, Chaotic Scattering, and Anti-integrable Limit

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1 Background

This note is based on a talk which the author gave at the RIMS conference.

The concept of anti-integrable limit was introduced to prove the existence of chaotic orbits for the standard map by Aubry and Abramovici in 1990 [1], and has been developed (e.g. [9]) and applied to many areas of dynamical systems. For instance, it was employed to prove the existence of invariant cantori for symplectic maps [16], breathers for coupled oscillators [15], multibump and chaotic trajectories for time-dependent Lagrangian systems [5, 7], horseshoes for the Hénon map and high dimensional Hénon-like maps [10, 18, 20], periodic and chaotic trajectories of the second species for the 3-body problem [6], periodic and chaotic trajectories for two charges in a uniform magnetic field [17], and to estimate lower bounds for the topological entropy of convex billiards with small inner scatterers [11].

In [8], using the concept of anti-integrable limit, the author proved the following.

Theorem 1. *For any given homotopy class with respect to the origin, there corresponds a unique periodic orbit for the Sinai billiard provided the circular scatterer is small enough.*

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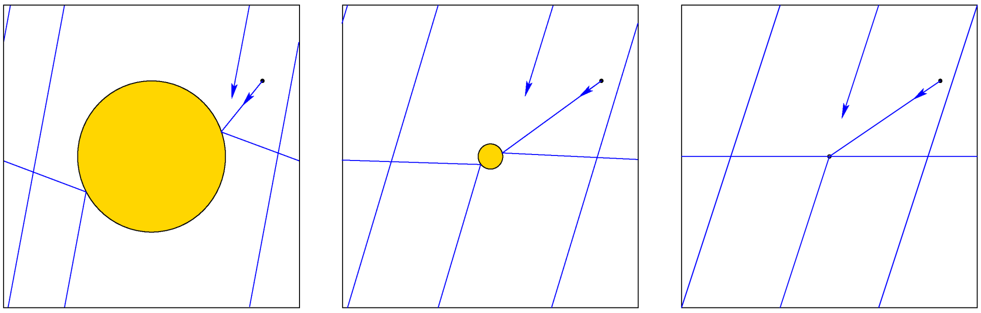

Figure 1: Continuation of a billiard orbit as the radius of the inner circular scatterer varies.

Figure 1 illustrates the underlying ideas of Theorem 1. Think about the following question: Given a periodic orbit of a Sinai billiard, does the orbit change continuously within a topology when the radius of the inner disc changes? and what is the limiting behaviour when the radius becomes zero?

It is well known that the topological entropy of the Sinai billiard is infinite and the Sinai billiard is equivalent to the billiard system of a square table with a disc scatterer placed in the centre of the square. This naturally motives us to consider another question: With a fixed disc in the centre, what will happen to the topological entropy if we smooth out the corners of a square billiard table? In other words, we ask that what is the topological entropy of a convex billiard table having one inner circular scatterer? In [11], the author obtained the following result.

Theorem 2. *For any positive real number χ, for a generic convex billiard table with a convex inner scatterer, the return map to the scatterer induced from the billiard flow has topological entropy at least χ provided that the inner scatterer is sufficiently small.*

Similar to the Sinai billiard case, we can also ask whether a periodic orbit of a convex billiard table with an inner disc scatterer changes continuously within a topology when the radius of the inner disc changes? and what is the limiting behaviour when the disc becomes a point? Figure 2 illustrates one possibility. In the figure, the left or the middle billiard system has an inner circular scatterer of non-zero radius, whereas the system on the right has a point scatterer. Suppose there are three periodic orbits (two period-2 orbits and one period-6 orbit) bouncing off the boundaries of the scatterer and the billiard table as indicated in the system on the left. Because all these orbits must collide perpendicularly with the boundary curves, it is easy to see that all these three periodic orbits persist when the inner disc scatterer becomes smaller and smaller (with the centre of the disc fixed). In the limiting situation when the scatterer becomes a point, these periodic orbits become piecewise straight lines, emanating from and return back to that point, subject to the law of reflection on the boundary of the table.

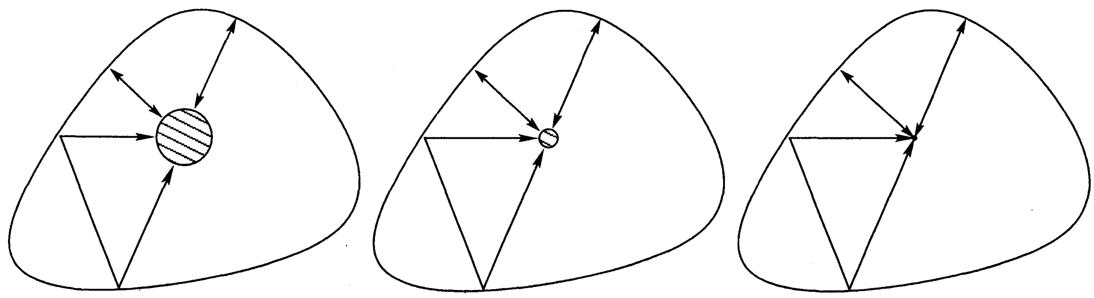

Figure 2: Three continuous families of periodic orbits parameterised by radius of the inner scatterer.

In Figure 3, the billiard table on the left has a point scatterer O_1. In the table, denote the oriented piecewise straight line segments emanating from point O_1 to point Ω_3 and return back to O_1 by Γ_1. (Note that, by complying with the law of reflection, Γ_1 intersects perpendicularly with the boundary of billiard table at Ω_3.) In the same table, denote the oriented piecewise straight line segments emanating from O_1 to Ω_1 then to Ω_2 then back to O_1 by $\Gamma_2^{\circlearrowleft}$, while the oriented piecewise straight line segments from O_1 to Ω_2 then to Ω_1 then return to O_1 by $\Gamma_2^{\circlearrowright}$. An infinite product of Γ_1, $\Gamma_2^{\circlearrowleft}$ and $\Gamma_2^{\circlearrowright}$ is called an *anti-integrable orbit.* For example, $\Upsilon_1 = \cdots \cdot \Gamma_1 \cdot \Gamma_1 \cdot \Gamma_1 \cdot \Gamma_2^{\circlearrowleft} \cdot \Gamma_2^{\circlearrowleft} \cdot \Gamma_2^{\circlearrowleft} \cdot \cdots$ is a one, and $\Upsilon_2 = \cdots \cdot \Gamma_1 \cdot \Gamma_2^{\circlearrowleft} \cdot \Gamma_2^{\circlearrowright} \cdot \Gamma_1 \cdot \Gamma_2^{\circlearrowleft} \cdot \Gamma_2^{\circlearrowright} \cdot \cdots$ is another one. Given an anti-integrable orbit, does there exist a true orbit close to the anti-integrable one in some topology when the inner scatterer is a small disc? More specifically, given Υ_1 for the zero-radius disc scatterer case (the anti-integrable limit), when the radius is not zero but small will there be a heteroclinic orbits backwards asymptotic to a period-2 orbit and forwards asymptotic to a period-3 orbit? (See the middle sub-figure of Figure 3.) Given the anti-integrable orbit Υ_2, will there be a period-8 orbit close to Υ_2 when the radius of inner scatterer is small? (See the right sub-figure of Figure 3.) In [11], the author gave a positive answer to the aforementioned questions. The answer implies that the complexity of a convex billiard table with small inner convex scatterers is at least as that of the anti-integrable orbits at the anti-integrable limit.

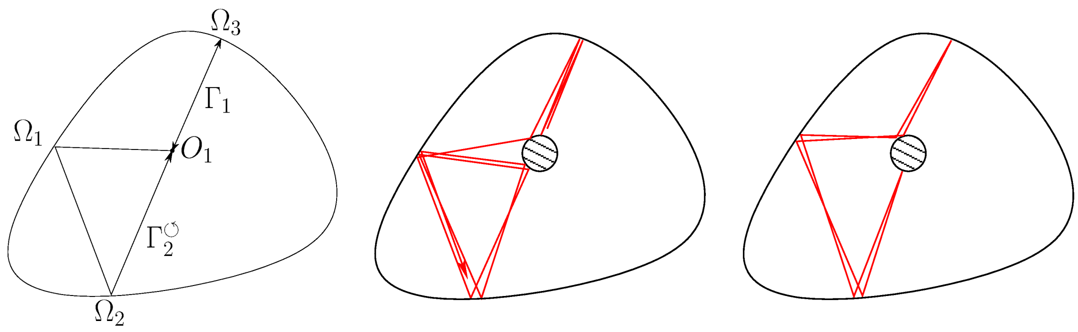

Figure 3: Anti-integrable orbits and their continuations.

Based on the foregoing results on chaotic scattering on billiards, one principal aim of this note is to provide an explanation, in terms of the anti-integrable limit, how the chaotic orbits occur in chaotic scattering after the abrupt bifurcation found numerically by Bleher, Ott, and Grebogi in 1989.

2 Abrupt bifurcation

In 1989, Bleher, Ott and Grebogi [4] investigated the motion of a particle scattered by a potential $V : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$V(x, y) = x^2y^2 \exp(-(x^2 + y^2)). \tag{1}$$

When the total energy E of the particle is slightly less than $E_c := \max_{(x,y)\in\mathbb{R}^2} V(x, y)$, their study suggested that there exists a bounded hyperbolic invariant set of the form of a suspension of a Cantor set in the energy level, on which chaotic scattering occurs, whereas there are no bounded orbits when $E > E_c$. They called the bifurcation at $E = E_c$ an *abrupt bifurcation.* Note that there was a numerical computation which suggests that the topological entropy jumps from $\ln 3$ down to 0 when E increases from below to above E_c [14].

In oder to explain the phenomenon, they invoked results from the literatures [12, 13, 19] about heteroclinic intersection of stable and unstable manifolds of periodic orbits, and numerically verified a sufficient condition that guarantees the intersection [3].

In [2], we developed a general rigorous approach to proving the abrupt bifurcation in chaotic scatterings, in particular constructing hyperbolic suspensions of topological Markov chains. The point is to show that the limit $E \to E_c$ from below can be interpreted as a non-degenerate anti-integrable limit and that a rigorous analysis is possible in general. Our main result is described in the next section.

3 Setting and main results

The system considered is governed by a Lagrangian of the form

$$L : T\mathbb{R}^2 \to \mathbb{R}, \quad (x, v) \mapsto L(x, v) = \frac{1}{2}|v|^2 - V(x),$$

where the potential V is C^4 in x. We assume the system has properties as follows:

(V1) There is a finite set O_{max} of non-degenerate local maximum points for V with equal height E_c, without loss of generality $E_c = 0$.

(V2) There is a finite set $\{\Gamma_k\}_{k\in K}$ of non-degenerate hetero/homoclinic trajectories to O_{max} for some finite index set K.

Define $M_E := \{x \in \mathbb{R}^2 | \ V(x) < E\}$ and let

$$J(\gamma) := \int_0^1 \sqrt{2(E - V(\gamma(s))} \ |\dot\gamma(s)| \ ds$$

be the Maupertuis functional on the set of absolutely continuous curves $\gamma : [0, 1] \to M_E$ with fixed ends $\gamma(0)$ and $\gamma(1)$.

Let

$$\Omega = \Omega(M_0, O_{e_0}, O_{e_1}) := \{\gamma : [0, 1] \to M_0 | \ \gamma(0) = O_{e_0}, \ \gamma(1) = O_{e_1}\}$$

be the space of absolutely continuous curves with fixed endpoints O_{e_0}, $O_{e_1} \in O_{max}$. The condition that γ is a non-degenerate heteroclinic orbit connecting O_{e_0} to O_{e_1} (γ is a homoclinic orbit if $O_{e_1} = O_{e_0}$) is satisfied provided that γ is a non-degenerate critical point of the Maupertuis functional J on the subset of elements belonging to Ω with the parametrization $|\dot\gamma|^2/2 + V(\gamma) = 0$.

An *anti-integrable trajectory* is defined to be a bi-infinite product of curves $\cdots \cdot \Gamma_{k_{i-1}} \cdot \Gamma_{k_i} \cdot \Gamma_{k_{i+1}} \cdot \cdots$ with $k_i \in K$ and subject to $\lim_{t\to\infty} \Gamma_{k_i}(t) = \lim_{t\to-\infty} \Gamma_{k_{i+1}}(t)$ for every $i \in \mathbb{Z}$. Define $O_{e_i} = \lim_{t\to\infty} \Gamma_{k_i}(t) = \lim_{t\to-\infty} \Gamma_{k_{i+1}}(t) \in O_{max}$, and

$$\xi^+_{k_i} := \lim_{t\to\infty} \frac{\dot\Gamma_{k_i}(t)}{|\dot\Gamma_{k_i}(t)|}, \qquad \xi^-_{k_i} := \lim_{t\to-\infty} \frac{\dot\Gamma_{k_i}(t)}{|\dot\Gamma_{k_i}(t)|}.$$

Let O be a non-degenerate maximum point of $V : \mathbb{R}^2 \to \mathbb{R}$. The potential is said to be *elliptic* around O if $D^2V(O)$ has distinct eigenvalues, otherwise it is said to be *circular* around O.

Definition 3. *An anti-integrable trajectory* $\cdots \cdot \Gamma_{k_{-1}} \cdot \Gamma_{k_0} \cdot \Gamma_{k_1} \cdots$, $k_i \in K$, *is called* **admissible** *for* $E < 0$ *(resp.* $E > 0$*) if, for every* $i \in \mathbb{Z}$, *we have*

$$\langle \xi^+_{k_i}, \xi^-_{k_{i+1}} \rangle < 0 \qquad (\textit{resp.} > 0) \tag{2}$$

when O_{e_i} *is circular. When* O_{e_i} *is elliptic, besides (2) we require that both* $\xi^+_{k_i}$ *and* $\xi^-_{k_{i+1}}$ *are tangent to the "long axis" of the elliptic maximum point* O_{e_i}.

By the admissibility condition, the collection of all anti-integrable trajectories over $\{\Gamma_k\}_{k\in K}$ admissible for $E < 0$ (resp. $E > 0$) determines a topological Markov graph G^-_K (resp. G^+_K) with vertices belonging to K.

In [2], we proved

Theorem 4. *There exist* $\epsilon^- < 0 < \epsilon^+$ *such that for any* $\epsilon^- < E < 0$ *(resp.* $0 < E < \epsilon^+$*), on a cross section on the energy level of* E, *there is a uniformly hyperbolic invariant set* $\mathcal{A}_E$ *for a Poincaré map* $\mathcal{P}_E$. *The restriction of* $\mathcal{P}_E$ *to* $\mathcal{A}_E$ *is topologically conjugate to the topological Markov chain* G^-_K *(resp.* G^+_K*).*

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