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On Logistic Equations with Diffusion and Nonlocal Terms

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1 Introduction

This article is concerned with the following logistic equation with diffusion and nonlocal terms:

$$\text{(P)}\qquad \begin{cases} u_t = d\Delta u + u\left(a - f(u) - \displaystyle\int_\Omega k(x,y)g(u(y,t))dy\right) & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, a, d are positive constants, $k \in C(\overline{\Omega}\times\overline{\Omega})$ is a nonnegative function and u_0 ($\not\equiv 0$) is a nonnegative function. Here u denotes the population density of a certain biological species, d is a diffusion coefficient and a is an intrinsic growth rate of the species. We assume that

(A.1) f and g are strictly increasing functions of class C^1 for $u \geq 0$ such that

$$f(0) = g(0) = 0 \quad \text{and} \quad \lim_{u\to\infty} f(u) = \infty.$$

The standard logistic diffusion equation is given in the following form without nonlocal term

$$u_t = d\Delta u + u(a - bu), \tag{1.1}$$

where $-bu$ represents the self-inhibitory effect due to the competition. If the first equation of (P) is replaced by (1.1), then it is well known that there exists a unique global solution u satisfying

$$\lim_{t\to\infty} u(\cdot,t) = \begin{cases} 0 & \text{uniformly in } \Omega \text{ if } 0 < a \leq d\lambda_1, \\ \theta & \text{uniformly in } \Omega \text{ if } a > d\lambda_1, \end{cases}$$

where λ_1 is the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition and θ is a unique positive stationary solution of (1.1)(which exists if and only if $a > d\lambda_1$). When we discuss the movement of an individual species, it is sometimes determined

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by surrounding conditions around the point where the species stays. For example, if we consider movements of animals, each individual species mutually interacts by seeing, hearing and smelling around themselves. Under certain circumstances, interaction by chemical means may take place. So it will be reasonable to take account of nonlocal effects in the study of population dynamics. Roughly speaking, there are two ways to add nonlocal terms to (1.1). The first one is to consider a nonlocal effect in a reaction term like

$$u_t = d\Delta u + u\,(a - bu - \ell_0(u))\,, \tag{1.2}$$

where

$$\ell_0(u) = \int_\Omega k(x,y)u(y)dy$$

with a nonnegative continuous function $k(x,y)$. The second way is to consider the case where a diffusion coefficient depends on a nonlocal term. One of such examples is given by

$$u_t = d(\ell_0(u))\Delta u + u\,(a - bu)\,, \tag{1.3}$$

where $d(\ell)$ is a positive continuous function and $\ell_0(u)$ is given by the above relation.

In the present article, we will discuss logistic diffusion equations with nonlocal terms in the form of (P), which is generalization of (1.2). Our main purpose is to study the similarity and difference between local problems ($k \equiv 0$) and nonlocal problems ($k \not\equiv 0$) in the following issues :

(a) Existence and uniqueness of bounded global solutions for (P),
(b) Asymptotic behavior of global solutions as $t \to \infty$,
(c) Structure of solutions for the corresponding stationary problem:

$$\text{(SP)} \qquad \begin{cases} d\Delta u + u\left(a - f(u) - \displaystyle\int_\Omega k(x,y)g(u(y))dy\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

For semilinear elliptic equations with nonlocal terms, there are a lot of works (see, e.g, [1], [2], [3], [6], [9], [11], [18]). In most papers, existence of positive solutions has been established with use of bifurcation theory or the Leray-Schauder degree theory. Here we will give two constructive methods to look for a positive stationary solution to (SP). Furthermore, we intend to investigate the stability of positive stationary solutions of (SP).

The contents of the present paper are as follows. In Section 2, we will show that (P) admits a unique global solution for any nonnegative initial data in a suitable class. Section 3 is devoted to the analysis of (SP) in case $k(x,y) = p(x)q(y)$ and the existence of stationary solutins is shown by an elementary approach. Moreover, we will study the stability of such a statioanry solution by putting some additional conditions on p, q and g. In Section 4, we will study (SP) for general kernel k and derive a necessary and sufficient condition for the existence of positive solutions of (SP). Our approach is based on the bifurcation theory in this section. We will show the non-degeneracy of any positive stationary solution under a certain special condition, which implies the uniqueness of a positive solution as well as its linearized stability.

Notation. We denote by $L^p(\Omega)$ the space of measurable functions $u: \Omega \to R$ such that $|u(x)|^p$ is integrable over Ω with norm

$$\|u\|_p := \left\{ \int_\Omega |u(x)|^p \, dx \right\}^{1/p}.$$

For $p = 2$, we simply write $\|\cdot\|$ in place of $\|\cdot\|_2$. By $W^{k,p}(\Omega)$, we denote the Sobolev space of functions $u \to R$ such that u and its distributional derivatives up to order k belong to $L^p(\Omega)$. Its norm is defined by

$$\|u\|_{W^{k,p}}^p = \sum_{|\rho| \le k} \|D^\rho u\|_p^p,$$

where ρ denotes a multi-index for derivatives.

2 Existence of global solutions

We will discuss (P) in the framework of $L^p(\Omega)$ with $p > 1$. Define a closed linear operator A in $L^p(\Omega)$ by

$$Au = -d\Delta u \quad \text{with domain} \quad D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Then it is well known that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t \ge 0}$ in $L^p(\Omega)$ (see, e.g., [12, 14]). Our problem (P) can be written as

$$\begin{cases} u_t + Au = F(u, \ell(u)), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where

$$F(u, v) = u(a - f(u) - v) \qquad \text{with} \quad \ell(u) = \int_\Omega k(x, y) g(u(y)) dy.$$

For (2.1) we can prove the following local existence theorem:

Theorem 2.1. *Let $p > \max\{1, N/2\}$. For any $u_0 \in L^p(\Omega)$, there exists a positive number T such that* (2.1) *has a unique solution u in the class*

$$u \in C([0, T]; L^p(\Omega)) \cap C((0, T]; W^{2,p}(\Omega)) \cap C^1((0, T]; L^p(\Omega)).$$

Proof. The proof is standard. The first procedure is to rewrite (2.1) in the form of integral equation

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(u(s), \ell(u(s))) ds. \tag{2.2}$$

The second procedure is to apply Banach's fixed point theorem to (2.2) in order to show the existence and uniqueness of a local solution. For details, see [12] or [14]. $\square$

In what follows we assume

$$u_0 \in L^\infty(\Omega) \quad \text{and} \quad u_0 \ge 0 \ (\not\equiv 0) \tag{2.3}$$

and establish the global existence theorem.

Theorem 2.2. *Let $p > \max\{1, N/2\}$ and assume* (2.3). *Then* (P) *has a unique solution u satisfying $u(x,t) > 0$ for $(x,t) \in \Omega \times (0,\infty)$ and*

$$u \in C([0,\infty); L^p(\Omega)) \cap C((0,\infty); W^{2,p}(\Omega)) \cap C^1((0,\infty); L^p(\Omega)).$$

Moreover, u satisfies

$$\sup_{(x,t)\in\Omega\times(0,\infty)} u(x,t) \le \max\{\|u_0\|_\infty, m\},$$

where m is a unique positive number satisfying $a = f(m)$.

Proof. Since $u_0 \ge 0$ and $u_0 \not\equiv 0$, it is easy to show by the strong maximum principle for parabolic equations (see [15]) that $u(\cdot,t) > 0$ as long as it exists. Therefore, u satisfies

$$u_t \le d\Delta u + u(a - f(u)) \qquad \text{in } \Omega \times [0,T),$$

where T is a maximal existence time. The comparison theorem for parabolic equations implies that

$$u(x,t) \le \max\{\|u_0\|_\infty, m\}$$

for $(x,t) \in \Omega \times [0,T)$. Hence we can conclude $T = \infty$ and obtain a required estimate. □

Remark 2.1. *Since we have established the global existence results for* (P), *our next task would be to study asymptotic behaviors of global solutions as $t \to \infty$. However, there are two difficulties in the analysis of nonlocal problems:*
(i) *lack of the comparison theorem,*
(ii) *construction of suitable Lyapunov functions,*
which are useful tools in the study of dynamics of solutions for local problems. So it is still an open problem to get precise information on the asymptotic behavior of global solutions of (P) *as*

3 Analysis of stationary problems –elementary approach–

In this section we will study stationary problem (SP) associated with (P). For semilinear elliptic equations with nonlocal terms, there are lots of works (see, e.g., [1], [2], [3], [6], [9], [11], [18]). In most papers, existence results of positive solutions have been established with use of bifurcation theory or Leray-Schauder degree theory. Recently, Corrêa, Delgado and Suárez [3] have shown the existence of positive solutions for a certain class of nonlocal problems by an elementary method. Inspired by their work, we will exhibit a very elementary and constructive method to look for positive solutions of (SP) in case

$$\text{(A2)} \qquad k(x,y) = p(x)q(y)$$

where $p, q \in C(\overline{\Omega})$ are nonnegative functions.

When k satisfies (A.2), one can write (SP) as follows:

$$\text{(SP.1)} \qquad \begin{cases} d\Delta u + u\left(a - f(u) - p(x)\displaystyle\int_\Omega q(y)g(u(y))dy\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

Our strategy is to rewrite (SP.1) as a boundary value problem for a usual diffusive logistic equation:

$$\begin{cases} -d\Delta u + \alpha p(x)u = a(a - f(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

with

$$\alpha = \int_\Omega q(y)g(u(y))dy. \tag{3.2}$$

Our procedure to solve (P.1) consists of two steps as follows:

1. For each $\alpha \geq 0$, find a solution $\theta(x, \alpha p)$ of (3.1).
2. After substitution of $\theta(x, \alpha p)$ into (3.2), look for $\alpha = \alpha^*$ satisfying

$$\alpha^* = \int_\Omega q(y)g(\theta(y, \alpha^* p))dy. \tag{3.3}$$

Clearly, $\theta(x, \alpha^* p)$ becomes a solution of (SP-1).

In order to accomplish the above procedure, we will give some preliminary results. Let $c : \overline{\Omega} \to R$ be a continuous function and consider the following eigenvalue problem

$$-d\Delta u + c(x)u = \lambda u \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

We denote by $\lambda_1(c)$ the principal eigenvalue of (3.4). It is well known that $\lambda_1(c)$ can be expressed by the following variational characterization:

$$\lambda_1(c) = \inf\left\{\int_\Omega \{d|\nabla u|^2 + c(x)u^2\}dx; u \in H_0^1(\Omega) \text{ and } \|u\|_2 = 1\right\}. \tag{3.5}$$

For any $c \in C(\overline{\Omega})$, consider the following boundary value problem for a diffusive logistic equation

$$\begin{cases} -d\Delta u + c(x)u = u(a - f(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{3.6}$$

where a, d are positive constants and f satisfies (A.1). Then we have the following result.

Proposition 3.1. *Let c be a nonnegative continuous function in $\overline{\Omega}$. Then there exists a unique solution $\theta(x; c)$ of* (3.6) *if and only if $a > \lambda_1(c)$. Moreover, $\theta(x; c)$ has the following properties:*
(i) *a mapping $c \to \theta(\cdot, c)$ is continuous from $C(\overline{\Omega})$ to itself,*
(ii) *if $c_1 \geq c_2$ $(c_1 \not\equiv c_2)$, then $\theta(x; c_2) > \theta(x; c_1)$ for $x \in \Omega$.*

Proof. Since $\lambda_1(c)$ is the principal eigenvalue, one can choose a positive eigenfunction $\varphi(x; c)$ corresponding to $\lambda_1(c)$ such that

$$\max_{x \in \Omega} \varphi(x; c) = 1 \quad \text{and} \quad \varphi(x; c) > 0 \quad \text{in } \Omega.$$

If we set $u^*(x) = m_1$ with positive constant m_1 satisfying $f(m_1) \geq \max\{a - c(x);\ x \in \overline{\Omega}\}$, then we see that u^* is a supersolution of (3.6).

We next take
$$v_*(x) = \varepsilon\varphi(x;c) \quad \text{with positive number } \varepsilon.$$
Then
$$-d\Delta v_* + v_*(c(x) - a + f(v_*)) = \varepsilon\varphi(x;c)(\lambda_1(c) - a + f(\varepsilon\varphi(x;c))).$$
Hence, if $a > \lambda_1(c)$, then one can take a sufficiently small $\varepsilon > 0$ such that $f(\varepsilon) \le a - \lambda_1(c)$. In this case,
$$-d\Delta v_* + v_*(c(x) - a + f(v_*)) \le 0;$$
which means that v_* is a subsolution of (3.6). Thus we can construct a supersolution u^* and a subsolution v_* satisfying $u^* \ge v_*$. Hence it follows from the result of Sattinger [17] that (3.6) has a positive solution.

The proofs of the necessity part, the uniqueness of positive solutions and the assertion (i) are standard; so we omit them.

Finally, we will prove the order preserving property. Let $c_1 \ge c_2$; then it can be seen that $\theta(x;c_2)$ is a supersolution of (3.6) with $c = c_1$. Therefore, by virtue of the uniqueness of a positive solution of (3.6),
$$\theta(x;c_2) \ge \theta(x;c_1) \quad \text{in } \Omega.$$
Moreover, if we set $w(x) = \theta(x;c_2) - \theta(x;c_1)$, then w satisfies
$$\begin{cases} -d\Delta w + c_2 w + w\{f(\theta(x;c_2)) + \theta(x;c_1)h(x) - a\} \ge 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$
where
$$h(x) = \int_0^1 f'(\sigma\theta(x;c_2) + (1-\sigma)\theta(x;c_1))d\sigma.$$
Therefore, the strong maximum principle ([15]) enables us to conclude $w > 0$ in Ω. □

We are ready to study (3.1). It follows from Proposition 3.1 that (3.1) has a unique solution $\theta(x;\alpha p)$ if and only if
$$a > \lambda_1(\alpha p). \tag{3.7}$$
Here it should be noted that a mapping $\alpha \to \lambda_1(\alpha p)$ has the following properties.

Lemma 3.1. *Let $p(\not\equiv 0)$ be a nonnegative continuous function in $\overline{\Omega}$ and assume that $\Omega_0 := \mathrm{Int}\{x \in \Omega;\ p(x) = 0\}$ is connected. Then the following properties hold true:*
(i) *The mapping $\alpha \to \lambda_1(\alpha p)$ is continuous and strictly increasing for $\alpha \ge 0$.*
(ii) $\lim_{\alpha\to 0} \lambda_1(\alpha p) = \lambda_1(0) = d\lambda_1(\Omega)$.
(iii) $\lim_{\alpha\to\infty} \lambda_1(\alpha p) = \begin{cases} \infty & \textit{in case } \Omega_0 = \emptyset, \\ d\lambda_1(\Omega_0) & \textit{in case } \Omega_0 \ne \emptyset, \end{cases}$
where $\lambda_1(D)$ denotes the principal eigenvalue of
$$-\Delta v = \lambda v \quad \textit{in } D \quad \textit{and} \quad v = 0 \quad \textit{on } \partial D.$$

Proof. Assertions (i) and (ii) come from (3.5). For the proof of (iii), see López-Gómez [13]. □

In what follows, assume

$$a > d\lambda_1(\Omega). \tag{3.8}$$

On account of Lemma 3.1 one can find a unique $\bar{\alpha} > 0$ satisfying $a = \lambda_1(\bar{\alpha}p)$ in case $\Omega_0 = \emptyset$. In case $\Omega_0 \neq \emptyset$, if we additionally assume $a < d\lambda_1(\Omega_0)$; then it is also possible to find $\bar{\alpha}$ which satisfies $a = \lambda_1(\bar{\alpha}p)$. When a satisfies $a \geq d\lambda_1(\Omega_0)$ in case $\Omega_0 \neq \emptyset$, Lemma 3.1 implies that $a > \lambda_1(\alpha p)$ for all $\alpha \geq 0$; so we define $\bar{\alpha} = \infty$ in this case. Then we see that (3.7) is equivalent to

$$0 \leq \alpha < \bar{\alpha} \tag{3.9}$$

and that (3.1) has a unique positive solution $\theta(x; \alpha p)$ if and only if α satisfies (3.9).

Furthermore, we can show that $\theta(x; \alpha p)$ has the following properties.

Lemma 3.2. *Let $\theta(\cdot; \alpha p)$ be a unique solution of* (3.1) *for $\alpha \in [0, \bar{\alpha})$. Then the mapping $\alpha \to \theta(x; \alpha p)$ is of class C^1 from $[0, \bar{\alpha})$ to $C(\overline{\Omega})$ and strictly decreasing. Moreover, it satisfies the following properties:*

(i) $\lim_{\alpha\to 0} \theta(\cdot; \alpha p) = \theta_0$ *uniformly in Ω, where $v = \theta_0$ is a unique positive solution of*

$$d\Delta v + v(a - f(v)) = 0 \quad \textit{in } \Omega \quad \textit{and } v = 0 \quad \textit{on } \partial\Omega.$$

(ii) $$\lim_{\alpha\to\bar{\alpha}} \theta(\cdot; \alpha p) = \begin{cases} 0 & \textit{uniformly in } \Omega \quad \textit{if } \bar{\alpha} < \infty, \\ \theta_\infty & \textit{uniformly in } \Omega \quad \textit{if } \bar{\alpha} = \infty. \end{cases}$$

Here θ_∞ is a function satisfying $\theta_\infty \equiv 0$ in $\Omega \setminus \Omega_0$ and

$$\begin{cases} d\Delta\theta_\infty + \theta_\infty(a - f(\theta_\infty)) = 0 & \textit{in } \Omega_0, \\ \theta_\infty = 0 & \textit{on } \partial\Omega_0, \\ \theta_\infty > 0 & \textit{in } \Omega_0. \end{cases}$$

Before proving the proof of Lemma 3.2 we will give the following main result in this section.

Theorem 3.1. *Assume* (A.1) *and* (A.2). *Then* (SP.1) *admits a unique positive solution u^* if and only if $a > d\lambda_1(\Omega)$.*

Proof. Assume $a > d\lambda_1(\Omega)$; then (3.1) has a unique positive solution $\theta(x; \alpha p)$ for $0 \leq \alpha < \bar{\alpha}$. We should recall that $\theta(x; \alpha p)$ is a positive solution of (SP.1) if and only if α and $u = \theta(x; \alpha p)$ satisfy (3.2). Define

$$G(\alpha) = \int_\Omega q(x) g(\theta(x; \alpha p)) dx.$$

Since g is strictly increasing and continuous, it follows from Lemma 3.2 that $G(\alpha)$ is strictly decreasing for $\alpha \in [0, \bar{\alpha}]$ and satisfies

$$G(0) = \int_\Omega q(x) \theta_0(x) dx > 0$$

and

$$\begin{cases} G(\bar{\alpha}) = 0 & \text{in case } \bar{\alpha} < \infty, \\ \lim_{\alpha\to\infty} G(\alpha) = \int_{\Omega_0} q(x) g(\theta_\infty(x)) dx & \text{in case } \bar{\alpha} = \infty. \end{cases}$$

Therefore, it is easy to find a unique α^* satisfying $\alpha^* = G(\alpha^*)$ in both cases $\bar{\alpha} < \infty$ and $\bar{\alpha} = \infty$. Clearly, $\theta(x; \alpha^* p)$ becomes a unique positive solution of (SP.1).

It remains to prove the necessity part. If (SP.1) has a positive solution u^*, then u^* is a solution of (3.1) with

$$\alpha^* = \int_\Omega q(x)g(u^*(x))dx > 0.$$

Hence it follows from the existence result for logistic diffusion equations that a must satisfy

$$a > \lambda_1(\alpha^* p) > \lambda_1(0) = d\lambda_1(\Omega).$$

Thus we complete the proof. □

Proof of Lemma 3.2. Observe that $\theta = \theta(x; \alpha p)$ satisfies

$$-d\Delta\theta + \alpha p(x)\theta + \theta(f(\theta) - a) = 0 \quad \text{in } \Omega$$

with $\theta(x; \alpha p) = 0$ on $\partial\Omega$. Differentiation of the above equation with respect to α leads to

$$-d\Delta w + \alpha p(x)w + (f(\theta) + \theta f'(\theta) - a)w = -p(x)\theta \quad \text{in } \Omega \quad \text{and} \quad w = 0 \quad \text{on } \partial\Omega$$

with $w(x) = (\partial/\partial\alpha)\theta(x; \alpha p)$. Recall that $-d\Delta + \alpha p(x) + f(\theta(x; \alpha p)) + \theta(x; \alpha p)f'(\theta(x; \alpha p)) - a$ is an invertible and order-preserving operator from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to $L^p(\Omega)$ (see, e.g., [19, Lemma 1.1]). Therefore, the implicit function theorem assures to show

$$\frac{\partial\theta(\alpha p)}{\partial\alpha} = w = -\{-d\Delta + \alpha p(x) + f(\theta(\alpha p)) + \theta(\alpha p)f'(\theta(\alpha p)) - a\}^{-1}(p(x)\theta(\alpha p)) < 0 \quad \text{in } \Omega.$$

Thus $\alpha \to \theta(x; \alpha p)$ is strictly decreasing.

It is easy to see $\theta(0) = \theta_0$ and $\theta(\bar{\alpha} p) = 0$ in case $\bar{\alpha} < \infty$.

It remains to study $\lim_{\alpha\to\infty} \theta(\alpha p)$ in case $\bar{\alpha} = \infty$. Since $\theta(\alpha p)$ is positive and strictly decreasing with respect to α, there exists a nonnegative function θ_∞ such that

$$\lim_{\alpha\to\infty} \theta(\alpha p) = \theta_\infty \quad \text{pointwise in } \Omega. \tag{3.10}$$

Take any $\varphi \in C_0^\infty(\Omega)$; then it holds that

$$-d\int_\Omega \theta(x; \alpha p)\Delta\varphi dx + \alpha\int_\Omega p(x)\theta(x; \alpha p)\varphi dx = \int_\Omega \theta(x; \alpha p)(a - f(\theta(x; \alpha p)))dx. \tag{3.11}$$

Since $p(x) = 0$ in Ω_0, we see from (3.11) that

$$\int_{\Omega\setminus\Omega_0} p(x)\theta(x; \alpha p)\varphi dx = \frac{1}{\alpha}\left\{d\int_\Omega \theta(x; \alpha p)\Delta\varphi dx + \int_\Omega \theta(x; \alpha p)(a - f(\theta(x; \alpha p)))dx\right\}. \tag{3.12}$$

Making use of the uniform boundedness of $\theta(x; \alpha p)$ for $\alpha \geq 0$ and letting $\alpha \to \infty$ in (3.12) one can find from (3.10) that

$$\int_{\Omega\setminus\Omega_0} p(x)\theta_\infty(x)\varphi dx = 0 \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Therefore, $\theta_\infty(x) = 0$ for $x \in \Omega \setminus \Omega_0$.

We next take any $\varphi \in C_0^\infty(\Omega_0)$ and define $\widetilde{\varphi} \in C_0^\infty(\Omega)$ by $\widetilde{\varphi}(x) = \varphi(x)$ if $x \in \Omega_0$ and $\widetilde{\varphi}(x) = 0$ if $x \in \Omega \setminus \Omega_0$. Setting $\varphi = \widetilde{\varphi}$ in (3.11) leads to

$$-d\int_{\Omega_0} \theta(x;\alpha p)\Delta\varphi dx = \int_{\Omega_0} \theta(x;\alpha p)(a - f(\theta(x;\alpha p)))\varphi dx.$$

Letting $\alpha \to \infty$ in the above identity and using (3.10) we get

$$-d\int_{\Omega_0} \theta_\infty \Delta\varphi dx = \int_{\Omega_0} \theta_\infty(a - f(\theta_\infty))\varphi dx;$$

which implies

$$\begin{cases} -d\Delta\theta_\infty = \theta_\infty(a - f(\theta_\infty)) & \text{in } \Omega, \\ \theta_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

It should be noted by elliptic regularity theory that θ_∞ becomes continuous in $\overline{\Omega}$. Therefore, one can conclude from Dini's theorem that the convergence in (3.10) is uniform. Thus the proof is complete. □

We have shown in Theorem 3.1 that (SP.1) has a unique positive solution u^*. Then it is very important to answer the following problem:

Problem

Is the unique positive solution u^* stable?

The spectral problem for the linearized operator around u^* is given by

$$\begin{cases} Lu := -d\Delta u + a_1(x)u + p(x)u^*(x)\int_\Omega q(y)g'(u^*(y))u(y)dy = \sigma u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where

$$a_1(x) = \{f(u^*(x)) + f'(u^*(x))u^*(x) - a\} + p(x)\int_\Omega q(y)g(u^*(y))dy.$$

The adjoint operator of L is given by

$$L^*v = -d\Delta v + a_1(x)v + q(x)g'(u^*(x))\int_\Omega p(y)u^*(y)v(y)dy \tag{3.14}$$

with $v = 0$ on $\partial\Omega$. Therefore, L is not self-adjoint; so that it is not easy to study the spectrum of L. For nonlocal Sturm-Liouville eigenvalue problems, there are important results due to Freitas [8, 10], who has obtained some sufficient conditions for real eigenvalues. However, it is difficult to check his conditions in our case.

We now put special assumptions in order to study the stability of the unique positive stationary solution u^*:

(A.3) $\qquad p(x) = q(x) \quad \text{and} \quad g(u) = bu^2 \text{ with } b > 0.$

In this case

$$Lu = L^*u = -d\Delta u + a_1u + 2bpu^*\int_\Omega p(y)u^*(y)u(y)dy,$$

where

$$a_1(x) = \{f(u^*(x)) + f'(u^*(x))u^*(x) - a\} + bp(x) \int_\Omega p(y)(u^*(y))^2\, dy.$$

Then we can show the following result:

Theorem 3.2. *Assume* (A.1), (A.2) *and* (A.3). *Then the unique solution* u^* *of* (SP-1) *is asymptotically stable.*

Proof. In the expression of Lu, the integral term is a bounded linear operator in $L^2(\Omega)$. Then we see that, for sufficiently large number $c > 0$, $L+c$ has a compact inverse operator in $L^2(\Omega)$. Therefore, the Riesz-Schauder theory, together with the fact that L is self-adjoint operator, implies that the spectrum of L consists of real eigenvalues.

We now use the positivity of u^*. Since u^* is a solution of (SP.1), one can see from the Krein-Rutman theory that $\lambda = 0$ is the principal eigenvalue of the following problem

$$\begin{cases} -d\Delta w + a_2(x)w = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$a_2(x) = f(u^*(x)) + bp(x) \int_\Omega p(y)(u^*(y))^2 dy - a.$$

By the variational characterization of the principal eigenvalue we see

$$d\|\nabla w\|_2^2 + (a_2 w, w)_2 \geq 0 \quad \text{for all } \; w \in H_0^1(\Omega), \tag{3.15}$$

where $(\cdot,\cdot)_2$ denote $L^2(\Omega)$-inner product. Since $a_1(x) = a_2(x) + f'(u^*(x))u^*(x)$, it follows from (3.15) that

$$\begin{aligned} (Lw, w)_2 &= d\|\nabla w\|_2^2 + (a_2 w, w)_2 + (f'(u^*)u^* w, w)_2 + 2b\left(\int_\Omega pu^* w\, dx\right)^2 \\ &\geq (f'(u^*)u^* w, w)_2 + 2b\left(\int_\Omega pu^* w\, dx\right)^2 > 0 \end{aligned} \tag{3.16}$$

for all $w\ (\neq 0) \in H_0^1(\Omega)$. Thus it is proved that all the eigenvalues of L are positive; so that u^* is asymptotically stable. □

4 Analysis of stationary problem–bifurcation approach–

In this section we will show the existence of solutions for (SP) by bifurcation approach and study their stability properties. For this purpose, it will be convenient to rewrite (SP). Recall that f is a strictly increasing function which satisfies (A.1); so that there exists a unique number m satisfying $a = f(m)$. We now set

$$u = m\tilde{u}, \quad \tilde{f}(\tilde{u}) = \frac{1}{a} f(m\tilde{u}), \quad \tilde{g}(\tilde{u}) = \frac{1}{a} g(m\tilde{u}),$$

then (SP) is rewritten as follows:

$$\begin{cases} d\Delta\tilde{u} + a\tilde{u}\left(1 - \tilde{f}(\tilde{u}) - \int_\Omega k(x,y)\tilde{g}(\tilde{u}(y))dy\right) = 0 & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \\ \tilde{u} > 0 & \text{in } \Omega. \end{cases}$$

In what follows, we will use $u, f(u)$ and $g(u)$ in place of $\tilde{u}, \tilde{f}(\tilde{u})$ and $\tilde{g}(\tilde{u})$ and study

$$\text{(SP.2)} \qquad \begin{cases} d\Delta u + au\left(1 - f(u) - \int_\Omega k(x,y)g(u(y))dy\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

Here it is assumed that

(A.1)' f and g are strictly increasing functions of class C^1 satisfying

$$f(0) = g(0) = 0 \quad \text{and} \quad f(1) = 1.$$

We will apply the local bifurcation theory due to Crandall and Rabinowitz [4] in order to study (SP.2). Regard a as a bifurcation parameter and set $a^* = d\lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition. Let φ be the positive eigenfunction corresponding to λ_1 such that $\|\varphi\|_2 = 1$. Define $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p > \max\{N/2, 1\}$ and $X_1 = \left\{w \in X;\ \int_\Omega w\varphi\, dx = 0\right\}$. Then it is possible to show the following result by the local bifurcation theory.

Theorem 4.1. *Assume* (A.1)'. *There exist a positive number* ϵ_0 *and continuously differentiable function* $\varepsilon \mapsto (a(\varepsilon), v(\varepsilon))$ *from* $[0, \varepsilon_0]$ *to* $R \times X_1$ *such that* (SP.2) *has a positive solution* $(u, a) = (u(\varepsilon), a(\varepsilon))$ *in the following form*

$$u(\varepsilon) = \varepsilon(\varphi + v(\varepsilon)), \quad a = a^* + b(\varepsilon),$$

where $v(\varepsilon)$ *and* $b(\varepsilon)$ *satisfy* $v(0) = 0, b(0) = 0$ *and*

$$b'(0) = f'(0)\int_\Omega \varphi^3(x)\, dx + g'(0)\iint_{\Omega\times\Omega} k(x,y)\varphi^2(x)\varphi(y)\, dxdy.$$

Remark 4.1. *Since* $f'(0) \geq 0$ *and* $g'(0) \geq 0$, *Theorem 4.1 implies that the bifurcation of positive solutions is supercritical at* $a = a^*$. *Moreover, if we apply the linearized stability result of Crandall and Rabinowitz [5], we can prove that, if* $b'(0) > 0$, *then bifurcating positive solutions for sufficiently small* $\varepsilon > 0$ *are asymptotically stable.*

Since we have established the local bifurcation theorem, we will next study the global structure of bifurcating positive solutions. We note that every positive solution u of (SP.2) satisfies

$$-d\Delta u = au(1 - f(u) - \int_\Omega k(x,y)g(u(y))dy) \leq au(1 - f(u)) \quad \text{in } \Omega; \tag{4.1}$$

so that it satisfies

$$0 < u(x) \leq 1 \quad \text{in } \Omega. \tag{4.2}$$

Moreover, if (SP.2) admits a positive solution u, then a must satisfy

$$a > d\lambda_1 = a^*. \tag{4.3}$$

Indeed, multiplying (4.1) by u and integrating the resulting expression over Ω we get

$$d\|\nabla u\|_2^2 \leq a(u(1 - f(u)), u) < a\|u\|_2^2.$$

Since $d\|\nabla u\|_2^2 \geq d\lambda_1\|u\|_2^2$, it is easy to see (4.3).

Thus we are ready to show the following result.

Theorem 4.2. *There exists a solution u^* of* (SP.2) *if and only if $a > a^*$.*

Proof. One can apply the global bifurcation theory of Rabinowitz [16]. Let $\mathcal{C} \subset \{(u^*, a^*) \in X \times R;\ u^*$ is a solution of (SP.2)$\}$ be a connected set such that $\mathcal{C}$ contains bifurcating positive solutions in Theorem 4.1. Then it can be shown that $\mathcal{C}$ is unbounded in $X \times R$. This fact, together with (4.2) and (4.3), implies that (P.2) has a positive solution if $a > a^*$. □

In order to study the stability of positive solutions of (SP.2), assume

(A.4) $$k(x,y) = k(y,x) \quad \text{for } x, y \in \Omega \quad \text{and} \quad g(u) = bu^2, \quad b > 0.$$

Let u^* be any positive solution of (SP.2). The linearized operator around $u = u^*$ is given by

$$L_2 v = -d\Delta v + a_3(x)v + 2bu^*(x)\int_\Omega k(x,y)u^*(y)v(y)dy$$

with $v = 0$ on $\partial\Omega$, where

$$a_3(x) = \{f(u^*(x)) + u^*(x)f'(u^*(x)) - a\} + b\int_\Omega k(x,y)u^*(y)^2\, dy.$$

Note $L_2 = L_2^*$. Moreover, one can show in the same way as (3.16) that

$$\begin{aligned}(L_2 v, v) \geq &\int_\Omega u^*(x)f'(u^*(x))v(x)^2 dx \\ &+ 2b\iint_{\Omega\times\Omega} k(x,y)u^*(x)v(x)u^*(y)v(y)dxdy.\end{aligned} \tag{4.4}$$

Here we introduce the notion of positive definite kernel and assume that

(A.5) k is a positive definite kernel; namely,

$$\iint_{\Omega\times\Omega} k(x,y)w(x)w(y)dxdy \geq 0 \quad \text{for all } w \in L^2(\Omega).$$

A typical example of a positive definite kernel is given by $k(x,y) = e^{-\alpha|x-y|^2}$ with $\alpha > 0$. Then making use of (4.4), (A.5) and repeating the arguments in the proof of theorem 3.2 we have

Proposition 4.1. *Assume* (A.1)', (A.4) *and* (A.5). *Then the spectrum of L^2 consists of positive eigenvalues.*

This proposition implies that every positive solution is non-degenerate. Therefore, we can apply the implicit function theorem at any point on a bifurcation branch of positive solutions to show that $\mathcal{C}$ is a smooth curve in $X \times R$ and that a positive solution is unique for each $a > a^*$. Thus we can prove the following result.

Theorem 4.3. *Assume* (A.1)', (A.4) *and* (A.5). *Then* (SP.2) *has a unique positive solution u^* if and only if $a > a^* = d\lambda_1$, Moreover, u^* is asymptotically stable.*

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