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A Discrete-Time Clark-Ocone Formula for Poisson Functionals

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Abstract

We discretize the Clark-Ocone formula along the n-equidistant partition of a given time interval $[0,T]$. Then we discuss the error caused by the discretization procedure. Our main achievements are: (i) multi-level central limit theorem for the errors, (ii) strong $O(n^{-1/2})$-convergence for the first order errors, and (iii) successful in proving that the Sobolev differentiability index is the rate of convergence.

1 A Discrete-Time Clark-Ocone Formula

Let $p=(p,\mathbf{D}_p)$ be a stationary Poisson point process on a given σ-finite measure space $(\mathbf{X},\mathscr{B}_{\mathbf{X}},\boldsymbol{n})$. Throughout this article (except for §**3.2**), we fix a bounded measurable function $f:\mathbf{X}\to\mathbb{R}$ and we consider a Lévy process $L=(L_t)_{0\leq t\leq T}$ given by

$$L_t:=\int_0^{t+}\int_{\mathbf{X}}f(x)N_p(\mathrm{d}s\mathrm{d}x).$$

We denote by $(\mathcal{F}_t^p)_{0\leq t\leq T}$ the filtration generated by the point process p. Along an n-equidistant partition $\{t_l:=:t_l^{(n)}:=lT/n\}_{l=0}^n$, we define a discretized filtration $(\mathcal{F}_l^{(n)})_{l=0}^n$ by setting $\mathcal{F}_0^{(n)}$ to be the trivial σ-field and

$$\mathcal{F}_l^{(n)}:=\sigma(N_p((t_{k-1},t_k],\mathrm{d}x):k=1,2,\cdots,l) \tag{1}$$

for $l=1,2,\cdots,n$.

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From the Clark-Ocone formula for Poisson functionals, we can deduce the following presumption: For a $\sigma(\Delta L_1, \cdots, \Delta L_n)$-measurable Poisson functional F, we would find that

$$F \sim \mathbf{E}[F] + \sum_{l=1}^{n} \int_{\mathbf{X}} \mathbf{E}[\vartheta_{(l,f(x))} F | \mathcal{F}_{l-1}^{(n)}] \widetilde{N}_p((t_{l-1}, t_l], \mathrm{d}x), \tag{2}$$

where $\vartheta_{(l,\nu)}$ is defined in the following sense: by assumption, we can write $F = F(\Delta L_1, \cdots, \Delta L_n)$. Then $\vartheta_{(l,\nu)} F$ is

$$\vartheta_{(l,\nu)} F := F(\Delta L_1, \cdots, \Delta L_l + \nu, \cdots, \Delta L_n) - F,$$

that is, $\vartheta_{(l,\nu)}$ is the forward difference operator in the "variable" ΔL_l by step ν. However, the filtration $(\mathcal{F}_l^{(n)})_{l=0}^n$ does not have the martingale representation theorem with respect to the (measure-valued) discrete-time martingale $(\sum_{k \le l} \widetilde{N}_p((t_{k-1}, t_k], \mathrm{d}x))_{l=0}^n$ (though $(\mathcal{F}_t)_{0 \le t \le T}$ does with respect to $(\widetilde{N}_p((0,t], \mathrm{d}x))_{0 \le t \le T}$), which implies that the both sides in (2) could not be equal for generic F.

This motivates us to ask how much is the difference of both sides in (2). Our discrete-time Clark-Ocone formula is one of the answers:

Theorem 1 (Discrete-Time Clark-Ocone Formula, [3])
For $F \in L^2(\sigma(\Delta L_1, \cdots, \Delta L_n))$, we have the following L^2-convergent series expansion:

$$\begin{aligned} F &- \mathbf{E}[F] \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{n} \int_{\mathbf{X}^k} \mathbf{E}[\vartheta_{(l,f(x_1))} \cdots \vartheta_{(l,f(x_k))} F | \mathcal{F}_{l-1}^{(n)}] \int_{t_{l-1} < s_1 < \cdots < s_k \le t_l} \widetilde{N}_p^{\otimes k}(\mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x_1 \cdots \mathrm{d}x_k), \end{aligned} \tag{3}$$

where the conditional expectations are understood in generalized sense if necessary, and $\widetilde{N}_p^{\otimes k}(\mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x_1 \cdots \mathrm{d}x_k) := \bigotimes_{i=1}^k \widetilde{N}_p(\mathrm{d}s_i \mathrm{d}x_i)$.

Remark 1

(i) We notice that the first order ($k = 1$) term in (3) coincides with the last term in (2).

(ii) One can establish also a discrete Clark-Ocone formula for Wiener functionals (see [1]): Let $W = (W_t)_{0 \le t \le T}$ be the one-dimensional Brownian motion starting from zero. We define a filtration $(\mathcal{G}_l^{(n)})_{l=0}^n$ as in (1) by using the increments of W, instead of the Lévy process L. Then for $F \in L^2(\mathcal{G}_n^{(n)})$, it holds that

$$F - \mathbf{E}[F] = \sum_{m=1}^{\infty} \sum_{l=1}^{n} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}[\partial_l^m F | \mathcal{G}_{l-1}^{(n)}] H_m\Big(\frac{\Delta W_l}{\sqrt{\Delta t}}\Big) \tag{4}$$

where H_m is the m-th Hermite polynomial which is given by

$$H_m(x) = \frac{(-1)^m}{\sqrt{m!}} e^{x^2/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} e^{-x^2/2}$$

and ∂_l is the differential in the "variable" ΔW_l.

2 Generalizing The Malliavin Difference Operator to Higher Order via Consistency

We put this section to prepare some notations necessary to state a limit theorem in the next section.

The difference operator $\vartheta_{(l,\nu)}$ is merely a simple reduction on the class of $\sigma(\{\Delta L_l\}_{l=1}^n)$-measurable Poisson functionals, of a more general operator, that is, the Malliavin difference operator, as we shall see in the next proposition. Assume that our Poissonian framework is setup on canonical space (that is, our probability space is the space $\Pi_{\mathbf{X}}$ of all point functions on $\mathbf{X}$).

Proposition 2 (Consistency)
For $F \in \mathbb{D}_{2,1} \cap L^2(\sigma(\Delta L_1, \cdots, \Delta L_n))$,

$$(D_{(t,x)}F)(p) = \sum_{l=1}^{n} 1_{\{t_{l-1} \le t < t_l\}}(\vartheta_{(l,f(x))}F)(p)$$

for a.a. $(p, t, x) \in \Pi_{\mathbf{X}} \times [0, T] \times \mathbf{X}$.

If $F \in \mathbb{D}_{2,1}$ is a functional of $L = (L_t)_{0 \le t \le T}$, Proposition 2 implies that

$$(D_{(t,x)}F)(p) = \lim_{n\to\infty} \sum_{l=1}^{n} 1_{\{t_{l-1} \le t < t_l\}}(\vartheta_{(l,f(x))}\mathbf{E}[F|\Delta L_1, \cdots, \Delta L_n])(p) \tag{5}$$

for a.a. $(p, t, x) \in \Pi_{\mathbf{X}} \times [0, T] \times \mathbf{X}$. Note that in the Brownian motion case, the derivative D on the Wiener space is defined via a similar relation to (5) with $n = 2^m$ in [9]. Although the proof is obvious by definition, this relation motivates us to generalize to higher order operators. Following this approach in [9], we define, for functionals $F \in \mathbb{D}_{2,k}$ of L,

$$D^k_{(\cdot,x_1,\cdots,x_k)}F \in L^2[0, T] \tag{6}$$

as the L^2-limit of the sequence

$$\sum_{l=1}^{n} 1_{\{t_{l-1} \le t < t_l\}}\vartheta_{(l,f(x_1))} \cdots \vartheta_{(l,f(x_k))}\mathbf{E}[F|\Delta L_1, \cdots, \Delta L_n]$$

if it exists.

3 Asymptotic Analysis for Errors

Assume that we are given a sequence $F_* = (F_n)_{n=1}^{\infty}$ such that

$\triangleright$ $F_n \in L^2(\sigma(\Delta L_1, \cdots, \Delta L_n))$ for each $n = 1, 2, \cdots$.

Therefore we can apply to each F_n our discrete Clark-Ocone formula along the n-equidistant partition of $[0,T]$, and then we define for $m = 1,2,\cdots,$

$$\begin{aligned}
&\mathrm{Err}_n(F_*;m)\\
&:= F_n - \mathbf{E}[F_n]\\
&\quad - \sum_{k=1}^{m}\sum_{l=1}^{n}\int_{\mathbf{X}^k}\mathbf{E}[\vartheta_{(l,f(x_1))}\cdots\vartheta_{(l,f(x_k))}F|\mathcal{F}_{l-1}^{(n)}]\int_{t_{l-1}<s_1<\cdots<s_k\leq t_l}\widetilde{N}_p^{\otimes k}(\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x_1\cdots\mathrm{d}x_k),
\end{aligned}$$

which is the difference between F_n and what approximates F_n up to the m-th order with using the discrete Clark-Ocone formula (3). As a convention, we write $\mathrm{Err}_n(F_*;0) := F_n - \mathbf{E}[F_n]$. Note that the $m = 1$ case becomes

$$\mathrm{Err}_n(F_*;1) = F_n - \mathbf{E}[F_n] - \sum_{l=1}^{n}\int_{\mathbf{X}}\mathbf{E}[\vartheta_{(l,f(x))}F|\mathcal{F}_{l-1}^{(n)}]\widetilde{N}_p((t_{l-1},t_l],\mathrm{d}x),$$

which is the most interest for us, as explained in **§1**.

3.1 Preceding Literatures

Let us begin with the case of a one dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$ starting from zero. For a Wiener functional F, the Clark-Ocone formula enables us to write

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}[D_tF|\mathcal{F}_t^W]\mathrm{d}W_t,$$

where $(\mathcal{F}_t^W)_{0\leq t\leq T}$ is the filtration generated by W, so that a natural candidate approximating F would be the Riemann-sum approximation, and then the error is given by

$$\mathrm{Err}_n^{\mathrm{track}}(F) := F - \mathbf{E}[F] - \sum_{l=1}^{n}\mathbf{E}[D_{t_l^{(n)}}F|\mathcal{F}_{t_l^{(n)}}^W]\Delta W_l.$$

Although the Wiener functional F and the definition of the error may differ depending on the contexts, one has roughly the following results:

- Convergence in law of the normalized error:

$$\sqrt{n}\,\mathrm{Err}_n^{\mathrm{track}}(F) \overset{n\to\infty}{\to} \frac{1}{\sqrt{2}}\int_0^T \mathbf{E}[D_{t,t}^2F|\mathcal{F}_t^W]\mathrm{d}B_t \quad \text{in law} \tag{7}$$

  where $F = f(X_T)$ with $X = (X_t)_{0\leq t\leq T}$ denoting a diffusion defined via a stochastic differential equation driven by the Brownian motion W (Bertimas-Kogan-Lo [4]). The process $B = (B_t)_{0\leq t\leq T}$ is a Brownian motion independent of W. For general Itô processes, see Hayashi-Mykland [8].

- L^2-convergence of the error:

$$\|\mathrm{Err}_n^{\mathrm{track}}(F)\|_{L^2} \to 0 \quad \text{as } n \to \infty \tag{8}$$

with the order

— $O(n^{-1/2})$ when $F = \max\{X_T - K, 0\}$, $\max\{K - X_T, 0\}$ or $F = f(X_T)$ where f is *absolutely continuous* with polynomial growth, and where $X = (X_t)_{0 \le t \le T}$ is a diffusion process defined by a stochastic differential equation (Zhang [10]).

— $O(n^{-1/4})$ when $F = 1_{[K,+\infty)}(X_T)$. Which is more irregular than the above, is shown by Gobet-Temam [7].

— $O(n^{\theta-1/2})$ when $F = f(X_T)$ with f belonging to a fractional Sobolev-type space indexed by $\theta \in [0, 1/2)$ (Geiss-Geiss [5]). They revealed the reason why the absolute continuity assumption on f was needed to get the $O(n^{-1/2})$-convergence (which is the best possible) with equidistant time partitions.

— $O(n^{-s/2})$ when $F = f(W_T)$ with $f \in (\mathbb{D}_{2,1}(\gamma), L^2(\gamma))_{1-s,W}$ (Geiss-Hujo [6]). Where $(\mathbb{D}_{2,1}(\gamma), L^2(\gamma))_{1-s,W}$ is an interpolation space between $\mathbb{D}_{2,1}(\gamma) = \mathbb{D}_{2,1}^{(1)}$ and $L^2(\gamma) = \mathbb{D}_{2,0}^{(1)}$ which are discrete (or finite dimensional) versions of Malliavin-Watanabe Sobolev spaces $\mathbb{D}_{2,1}$ and $\mathbb{D}_{2,0}$, respectively.

— $O(n^{-s/2})$ when $F = f(W_T) \in \mathbb{D}_{2,s}$ for $0 \le s \le 1$ (Akahori-Amaba-Okuma [1]). Furthermore, they deduced an inequality

$$\|\mathrm{Err}_n^{\mathrm{track}}(f(W_T))\|_{L^2} \le n^{-s/2} \|f(W_T)\|_{\mathbb{D}_{2,s}}$$

for every $0 \le s \le 1$ and obtained an implication: $f(W_T) \in \mathbb{D}_{2,s} \Rightarrow f \in (\mathbb{D}_{2,1}(\gamma), L^2(\gamma))_{1-s,W}$. This convergence result is extended to "*stationary*" sequences in [1].

— $O\big((m/n)^{-s/2}\big)$ when $F = f(\Delta^{(m)} W_1, \cdots, \Delta^{(m)} W_m) \in \mathbb{D}_{2,s}$ for $0 \le s \le 1$ (Akahori-Amaba-Okuma [1]). Where $\Delta^{(m)}$ is a fixed m-equidistant partition $\{t_k^{(m)}\}_{k=0}^m$ of $[0, T]$. In fact, it holds that

$$\|\mathrm{Err}_n^{\mathrm{track}}(F)\|_{L^2} \le \left(\frac{m}{n}\right)^{-s/2} \|F\|_{\mathbb{D}_{2,s}}$$

for every $0 \le s \le 1$. It might be interesting that this inequality alludes roughly that the convergence rate would get worse when $m = n \to \infty$, that is, when we consider the approximations of generic truely infinite dimensional Wiener functionals.

3.2 A Central Limit Theorem for the Errors

Let us back to our settings. In this subsection, we assume that $f = 1_U$ for some $U \in \mathscr{B}_{\mathbf{X}}$ with $n(U) =: \lambda < +\infty$. Therefore, the Lévy process $L = (L_t)_{0 \le t \le T}$ is just a

Poisson process $N = (N_t)_{0\leq t\leq T}$ with intensity λ, given by

$$N_t := L_t = N_p((0,t] \times U).$$

Then by setting $\mathbf{D}_{p'} := \{t \in [0,T] : N_{t-} \neq N_t\}$ and $p'(t) := N_t - N_{t-}$ for $t \in \mathbf{D}_{p'}$, the point process $p' = (p', \mathbf{D}_{p'})$ is a stationary Poisson point process on the singleton $\{1\}$. In the following, we denote $D_t^k := D^k_{(t, \underbrace{1,\cdots,1}_{k\text{-members}})}$ which was defined in (6).

Theorem 3 (see [2, Theorem 4.1])
Let $m \in \mathbb{N}$. Suppose that $F_n \in \mathbb{D}_{2,m+2}$ for each $n = 1, 2, \cdots$ and for some $F \in \mathbb{D}_{2,m+1}$, we have

- ▷ $F_n \to F$ *in* $L^2(\mathbf{P})$,
- ▷ $D_t^{k+1}F$ *exists for a.a.* $t \in [0,T]$ *and* $\int_0^T \|D_t^{k+1}F_n - D_t^{k+1}F\|^2_{L^2}\mathrm{d}t \to 0$ *as* $n \to \infty$ *for each* $k = 0, 1, \cdots, m$ *and*
- ▷ $\sup_n \int_0^T \|D_t^{m+2}F_n\|^2_{L^2}\mathrm{d}t < +\infty$.

Then we have

$$\begin{pmatrix} \mathrm{Err}_n(F_*;0) \\ (\Delta t)^{-1/2}\mathrm{Err}_n(F_*;1) \\ \vdots \\ (\Delta t)^{-m/2}\mathrm{Err}_n(F_*;m) \end{pmatrix} \to \begin{pmatrix} \int_0^T \mathbf{E}[D_tF|\mathcal{F}^{p'}_{t-}]\mathrm{d}\widetilde{N}_t \\ \frac{\lambda^{1/2}}{\sqrt{2}}\int_0^T \mathbf{E}[D_t^2F|\mathcal{F}^{p'}_{t-}]\mathrm{d}B_t^1 \\ \vdots \\ \frac{\lambda^{m/2}}{\sqrt{(m+1)!}}\int_0^T \mathbf{E}[D_t^mF|\mathcal{F}^{p'}_{t-}]\mathrm{d}B_t^m \end{pmatrix}$$

in probability on an extended probability space as $n \to \infty$, where $(B^1, \cdots, B^m)$ is an m-dimensional Brownian motion.

Remark 2

(i) Due to the formula (4), one can discuss the corresponding result in the Brownian case, which is already established in [1].

(ii) This theorem appears restrictive in its setting, since we take only $f = 1_U$. This should hold for more large class of f. For a generic f, we naturally expect that

$$\begin{pmatrix} \mathrm{Err}_n(F_*;0) \\ (\Delta t)^{-1/2}\mathrm{Err}_n(F_*;1) \\ \vdots \\ (\Delta t)^{-m/2}\mathrm{Err}_n(F_*;m) \end{pmatrix}$$

would converge to

$$\begin{pmatrix} \int_0^{T+}\int_{\mathbf{X}} \mathbf{E}[D_{(t,x)}F|\mathcal{F}^p_{t-}]\widetilde{N}_p(\mathrm{d}t\mathrm{d}x) \\ \frac{1}{\sqrt{2}}\int_0^T\int_{X^2} \mathbf{E}[D^2_{(t,x_1,x_2)}F|\mathcal{F}^p_{t-}]W_1(\mathrm{d}t\mathrm{d}x_1\mathrm{d}x_2) \\ \vdots \\ \frac{1}{\sqrt{(m+1)!}}\int_0^T\int_{\mathbf{X}^m} \mathbf{E}[D^m_{(t,x_1,\cdots,x_m)}F|\mathcal{F}^p_{t-}]W_m(\mathrm{d}t\mathrm{d}x_1\cdots\mathrm{d}x_m) \end{pmatrix}$$

in law as $n \to \infty$, where $W_k(\mathrm{d}t\mathrm{d}x_1\cdots\mathrm{d}x_k)$ are independent Gaussian measures on $[0,T]\times\mathbf{X}^k$, and which is now in progress in [3].

As a corollary of Theorem 3, we can obtain

Corollary 4
If $\sup_n \int_0^T \|D_t^2 F_n\|^2_{L^2}\mathrm{d}t < +\infty$ then $\|\mathrm{Err}_n(F_*;1)\|_{L^2} = O(n^{-1/2})$.

Roughly speaking, this corollary says that if $F_* = (F_n)_{n=1}^\infty$ is smooth enough and their second differences are L^2-bounded then the convergence rate is always assured to be $O(n^{-1/2})$. Therefore, in the next subsection, we are interested in the case where F_* has only low regularities.

3.3 Rate of Convergence under Only Low Regularity

Let $\varphi:\mathbb{R}\to\mathbb{R}$ be Borel measurable and we set $F_n \equiv \varphi(L_T)$. Although it is an abuse of a notation, we write $F_* = \varphi(L_T)$.

Theorem 5 (Sobolev differentiability index is the rate of convergence, [3])
For every $0\le s\le 1$, we have

$$\|\mathrm{Err}_n(\varphi(L_T);1)\|_{L^2} \le n^{-s/2}\|\varphi(L_T)\|_{\mathbb{D}_{2,s}}.$$

Therefore, if $\varphi(L_T)\in\mathbb{D}_{2,s}$ for some $0\le s\le 1$, we have $\|\mathrm{Err}_n(\varphi(L_T);1)\|_{L^2} = O(n^{-s/2})$.

Remark 3
One may naturally expect a similar result in the case where F_* is coming from the Euler-Maruyama approximation of a stochastic differential equation defining a diffusion. However, the corresponding result has not been obtained yet. A key to obtain such results might be a derivation of a series of derivative estimates of the transition density associated to the diffusion, in which the dependence on the order of derivative is described explicitly.

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