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On planar rank-based diffusions with skew-elastic collisions

TOMOYUKI ICHIBA
Department of Statistics & Applied Probability,
University of California Santa Barbara

Abstract

In this short note we shall discuss planar diffusions where each of its component particles behaves locally like Brownian motion and the local characteristics of these random motions are assigned by rank, in addition to name-based drifts. In the interest of concreteness and simplicity we shall look at the system of particles competing each other and colliding elastically.

The system we consider is a competing planar diffusion $(X_1(\cdot), X_2(\cdot))$, where the leader has drift $-h \le 0$ and dispersion $\rho \ge 0$, whereas the laggard has drift $g \ge 0$ and dispersion $\sigma \ge 0$, in addition to the name based drifts γ_i, $i = 1, 2$, with

$$\lambda := g + h > 0, \quad \rho^2 + \sigma^2 = 1$$

for simplicity. At the times $\{t : X_1(t) = X_2(t)\}$ of collisions of its component particles in $\mathbb{R}$, they interact through their left and right local times in a skew-elastic manner.

To be more precise, we shall construct and examine a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ endowed with a filtration $\mathbf{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$ that satisfies the "usual conditions" of right continuity and of augmentation by $\mathbb{P}$-negligible sets, and on it two pairs $(B_1(\cdot), B_2(\cdot))$ and $(X_1(\cdot), X_2(\cdot))$ of continuous, $\mathbf{F}$-adapted processes, such that $(B_1(\cdot), B_2(\cdot))$ is planar Brownian motion and $(X_1(\cdot), X_2(\cdot))$ is a continuous planar semimartingale that starts at some given site $(X_1(0), X_2(0)) = (x_1, x_2) \in \mathbb{R}^2$ on the plane and satisfies

$$\begin{aligned} \mathrm{d}X_1(t) &= \Big(\gamma_1 + g\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}} - h\,\mathbf{1}_{\{X_1(t) > X_2(t)\}}\Big)\,\mathrm{d}t \\ &\qquad + \Big(\rho\,\mathbf{1}_{\{X_1(t) > X_2(t)\}} + \sigma\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}}\Big)\,\mathrm{d}B_1(t) \\ &\qquad + \frac{1-\zeta_1}{2}\,\mathrm{d}L^{X_1 - X_2}(t) + \frac{1-\eta_1}{2}\,\mathrm{d}L^{X_2 - X_1}(t), \qquad (1) \\ \mathrm{d}X_2(t) &= \Big(\gamma_2 + g\,\mathbf{1}_{\{X_1(t) > X_2(t)\}} - h\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}}\Big)\,\mathrm{d}t \\ &\qquad + \Big(\rho\,\mathbf{1}_{\{X_1(t) \le X_2(t)\}} + \sigma\,\mathbf{1}_{\{X_1(t) > X_2(t)\}}\Big)\,\mathrm{d}B_2(t) \\ &\qquad + \frac{1-\zeta_2}{2}\,\mathrm{d}L^{X_1 - X_2}(t) + \frac{1-\eta_2}{2}\,\mathrm{d}L^{X_2 - X_1}(t). \qquad (2) \end{aligned}$$

Here and in the sequel we denote by $L^X(\cdot) \equiv L^X(\cdot\,; 0)$ the right-continuous local time accumulated at the origin by a generic continuous semimartingale $X(\cdot)$, i.e.,

$$L^X(t) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0 \le X(s) < \varepsilon\}} \mathrm{d}\langle X \rangle(s); \quad t \ge 0.$$

Let us also denote by $L^X_-(\cdot) \equiv L^{-X}(\cdot\,;0)$ its left-continuous version, and by $\widehat{L}^X(\cdot) = (L^X(\cdot) + L^X_-(\cdot))/2$ its symmetric version. The system (1)-(2) is an extension of the system considered in FERNHOLZ ET AL. (2013a), in the sense that the drift component contains the name-based drifts γ_i, $i = 1, 2$. For some interpretations and applications to Mathematical Finance problems we refer FERNHOLZ ET AL. (2013a).

In (1)-(2) with the notation $\zeta := 1+(\zeta_1-\zeta_2)/2$, $\eta := 1-(\eta_1-\eta_2)/2$, $\nu := g-h$, $y := x_1 - x_2$, $z := x_1 + x_2$, $r_1 := x_1 \vee x_2$, $r_2 := x_1 \wedge x_2$, we assume $\zeta + \eta \neq 0$,

$$0 \le \alpha := \frac{\eta}{\zeta + \eta} \le 1 . \tag{3}$$

The general theory of Martingale Problems developed by STROOCK & VARADHAN (2006), KRYLOV (1980) and BASS & PARDOUX (1987) tells us that the system has the weak unique solution $(\Omega, \mathcal{F}, \mathbb{P})$, $(X_1(\cdot), X_2(\cdot), B_1(\cdot), B_2(\cdot))$, $(\mathcal{F}_t)$, if it is *non-degenerate* $\rho\,\sigma \neq 0$. Here we shall study the *degenerate case* $\rho\,\sigma = 0$ as well.

Let us briefly look at the system (1)-(2). The difference and the sum of the two component process $Y(\cdot) := X_1(\cdot) - X_2(\cdot)$, $Z(\cdot) := X_1(\cdot) + X_2(\cdot)$ satisfy

$$Y(t) = y + \int_0^t (\gamma_1 - \gamma_2 - \lambda\,\mathrm{sgn}(Y(s)))\,\mathrm{d}s + (1-\zeta)\,L^Y(t) - (1-\eta)\,L^Y_-(t) + W(t), \tag{4}$$

$$Z(t) = z + (\nu + \gamma_1 + \gamma_2)\,t + V(t) + (1-\overline{\zeta}\,)\,L^Y(t) + (1-\overline{\eta}\,)\,L^Y_-(t); \qquad 0 \le t < \infty, \tag{5}$$

where $\mathrm{sgn}(\cdot) := \mathbf{1}_{\{\cdot>0\}} - \mathbf{1}_{\{\cdot\le 0\}}$, $\overline{\zeta} := (\zeta_1+\zeta_2)/2$, $\overline{\eta} := (\eta_1+\eta_2)/2$ and $V(\cdot)$, $W(\cdot)$ are standard Brownian motions defined by $W(\cdot) := \rho W_1(\cdot) + \sigma W_2(\cdot)$ and $V(\cdot) := \rho V_1(\cdot) + \sigma V_2(\cdot)$ with a planar Brownian motion

$$W_1(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\,\mathrm{d}B_1(t) - \int_0^\cdot \mathbf{1}_{\{Y(t)\le 0\}}\,\mathrm{d}B_2(t),$$
$$W_2(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)\le 0\}}\,\mathrm{d}B_1(t) - \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\,\mathrm{d}B_2(t)$$

and another planar Brownian motion

$$V_1(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\,\mathrm{d}B_1(t) + \int_0^\cdot \mathbf{1}_{\{Y(t)\le 0\}}\,\mathrm{d}B_2(t),$$
$$V_2(\cdot) := \int_0^\cdot \mathbf{1}_{\{Y(t)\le 0\}}\,\mathrm{d}B_1(t) + \int_0^\cdot \mathbf{1}_{\{Y(t)>0\}}\,\mathrm{d}B_2(t).$$

Because of ITÔ's isometry, we observe the amount of time that the process $Y(\cdot)$ stays at the origin is zero almost surely, i.e.,

$$\int_0^\infty \mathbf{1}_{\{Y(t)=0\}}\mathrm{d}t = \int_0^\infty \mathbf{1}_{\{Y(t)=0\}}\mathrm{d}\langle Y\rangle(t) = 0 .$$

Then using this fact, we obtain the relations between the left and right continuous local times

$$L^Y(\cdot) - L^Y_-(\cdot) = (1-\zeta)L^Y(\cdot) - (1-\eta)L^Y_-(\cdot),$$

or equivalently

$$\zeta L^Y(\cdot) \,=\, \eta L_-^Y(\cdot)\,,$$

and also for the symmetric $\widehat{L}^Y(\cdot)$ and for $L^{|Y|}(\cdot)$:

$$2\widehat{L}^Y(\cdot) \,=\, L^{|Y|}(\cdot)\,, \quad L^Y(\cdot) \,=\, \alpha L^{|Y|}(\cdot)\,, \quad L_-^Y(\cdot) \,=\, (1-\alpha)L^{|Y|}(\cdot)\,.$$

Rewriting the left continuous local time $L_-^Y(\cdot)$ in (4), in terms of the symmetric local time $\widehat{L}^Y(\cdot)$, we observe that as a special case of BASS & CHEN (2005), the equation (4) admits a pathwise unique strong solution for all values of skewness parameter $\alpha \in [0,1]$.

Here we may construct further the other Brownian motions $Q(\cdot)$ and $W^\flat(\cdot)$, $V^\flat(\cdot)$, $U^\flat(\cdot)$ as $Q(\cdot) := \sigma V_1(\cdot) + \rho V_2(\cdot)$, $W^\flat(\cdot) := \rho W_1(\cdot) - \sigma W_2(\cdot)$, $V^\flat(\cdot) := \rho V_1(\cdot) - \sigma V_2(\cdot)$, $U^\flat(\cdot) := \sigma W_1(\cdot) - \rho W_2(\cdot)$; we note the independence of $Q(\cdot)$ and $W(\cdot)$, the independence of $Q(\cdot)$ and $V^\flat(\cdot)$, and observe the intertwinements among these Brownian motions

$$V_j(\cdot) = (-1)^{j+1}\int_0^{\cdot} \mathrm{sgn}\big(Y(t)\big)\mathrm{d}W_j(t) \quad (j=1,2)\,, \qquad V^\flat(\cdot) = \int_0^{\cdot} \mathrm{sgn}\big(Y(t)\big)\mathrm{d}W(t)$$

and

$$V(\cdot) = \int_0^{\cdot} \mathrm{sgn}\big(Y(t)\big)\mathrm{d}W^\flat(t)\,, \qquad Q(\cdot) = \int_0^{\cdot} \mathrm{sgn}\big(Y(t)\big)\mathrm{d}U^\flat(t)\,.$$

Now let us construct the solution to the system (1)-(2) of stochastic differential equations by reverse-engineering. Given a planar Brownian motion $(W_1(\cdot), W_2(\cdot))$ on a filtered probability space, we define $W(\cdot) := W_1(\cdot) + W_2(\cdot)$ and then obtain the pathwise unique, strong solution $Y(\cdot)$ to (4), and then its local time $L^Y(\cdot)$ accumulated at the origin. From the initial values $(x_1, x_2) \in \mathbb{R}^2$ and the processes $(W_1(\cdot), W_2(\cdot), Y(\cdot), L^Y(\cdot))$ we shall construct $(X_1(\cdot), X_2(\cdot))$, $(B_1(\cdot), B_2(\cdot))$ as

$$\begin{aligned} X_1(t) \,:=\, x_1 &+ \int_0^t \big(\gamma_1 + g\,\mathbf{1}_{\{Y(s)\le 0\}} - h\,\mathbf{1}_{\{Y(s)>0\}}\big)\mathrm{d}s \\ &+ \int_0^t \big(\rho\,\mathbf{1}_{\{Y(s)>0\}}\mathrm{d}W_1(s) + \sigma\,\mathbf{1}_{\{Y(s)\le 0\}}\mathrm{d}W_2(s)\big) \\ &+ \frac{1-\zeta_1}{2}\,\mathrm{d}L^Y(t) + \frac{1-\eta_1}{2}\,\mathrm{d}L_-^Y(t)\,, \end{aligned} \tag{6}$$

$$\begin{aligned} X_2(t) \,:=\, x_2 &+ \int_0^t \big(\gamma_2 + g\,\mathbf{1}_{\{Y(s)>0\}} - h\,\mathbf{1}_{\{Y(s)\le 0\}}\big)\mathrm{d}s \\ &- \int_0^t \big(\rho\,\mathbf{1}_{\{Y(s)\le 0\}}\mathrm{d}W_1(s) + \sigma\,\mathbf{1}_{\{Y(s)>0\}}\mathrm{d}W_2(s)\big) \\ &+ \frac{1-\zeta_2}{2}\,\mathrm{d}L^Y(t) + \frac{1-\eta_2}{2}\,\mathrm{d}L_-^Y(t)\,, \end{aligned} \tag{7}$$

as well as

$$B_1(t) \,:=\, \int_0^t \big(\mathbf{1}_{\{Y(s)>0\}}\mathrm{d}W_1(s) + \mathbf{1}_{\{Y(s)\le 0\}}\mathrm{d}W_2(s)\big)\,,$$

$$B_2(t) := -\int_0^t \big(\mathbf{1}_{\{Y(s)\le 0\}}\mathrm{d}W_1(s) + \mathbf{1}_{\{Y(s)>0\}}\mathrm{d}W_2(s)\big)$$

for $0 \le t < \infty$. One can verify that $(X_1(\cdot), X_2(\cdot))$ and $(B_1(\cdot), B_2(\cdot))$ defined in (6)-(7), in fact, satisfy (1)-(2). Thus in this way we may construct a weak solution to (1)-(2).

By TANAKA-MEYER formula the ranked versions (the leader and laggard, respectively) $R_1(\cdot) := X_1(\cdot) \vee X_2(\cdot)$ and $R_2(\cdot) =: X_1(\cdot) \wedge X_2(\cdot)$ of components satisfy

$$\begin{aligned} R_1(t) &= r_1 + \int_0^t \big(-h + \gamma_1 \mathbf{1}_{\{Y(s)>0\}} + \gamma_2 \mathbf{1}_{\{Y(s)\le 0\}}\big)\mathrm{d}s \\ &\qquad + \rho V_1(t) + \big(1 - (\beta/2)\big) L^{R_1 - R_2}(t), \\ R_2(t) &= r_2 + \int_0^t \big(g + \gamma_2 \mathbf{1}_{\{Y(s)>0\}} + \gamma_1 \mathbf{1}_{\{Y(s)\le 0\}}\big)\mathrm{d}s \\ &\qquad + \sigma V_2(t) - (\beta/2) L^{R_1 - R_2}(t), \end{aligned}$$

for $0 \le t < \infty$, where $\beta := (\eta\overline{\zeta} + \zeta\overline{\eta}) / (\eta + \zeta)$. By the sum $R_1(\cdot) + R_2(\cdot) = X_1(\cdot) + X_2(\cdot)$ and the difference $Y(\cdot) = X_1(\cdot) - X_2(\cdot)$ we have the *skew* representation:

$$\begin{aligned} X_1(t) &= x_1 + \mu t + \rho^2(Y^+(t) - y^+) - \sigma^2(Y^-(t) - y^-) \\ &\quad - \frac{1}{2}(\rho^2 - \sigma^2)(\gamma_1 - \gamma_2)\Gamma^Y(t) + (1 - \beta - \rho^2 + \sigma^2)\widehat{L}^Y(t) + \rho\sigma Q(t), \quad (8) \\ X_2(t) &= x_2 + \mu t - \sigma^2(Y^+(t) - y^+) + \rho^2(Y^-(t) - y^-) \\ &\quad - \frac{1}{2}(\rho^2 - \sigma^2)(\gamma_1 - \gamma_2)\Gamma^Y(t) + (1 - \beta - \rho^2 + \sigma^2)\widehat{L}^Y(t) + \rho\sigma Q(t), \quad (9) \end{aligned}$$

where $\mu := g_2\rho^2 + g_1\sigma^2$ and $\Gamma^Y(t) := \int_0^t \mathrm{sgn}(Y(s))\mathrm{d}s$ for $0 \le t < \infty$. Since the joint distribution $(Y(t), \widehat{L}^Y(t), \Gamma^Y(t))$ is uniquely determined, and $Q(t)$ is independent of $(Y(t), L^Y(t), \Gamma^Y(t))$, the joint distribution of $(X_1(t), X_2(t))$ is uniquely determined.

Theorem 1. *The system of stochastic differential equations (1)-(2) is well-posed, that is, has a weak solution which is unique in the sense of the probability distribution.*

Let us denote the filtrations $\mathfrak{F}^X(t) := \sigma(X(s), 0 \le s \le t)$, $0 \le t < \infty$ generated by the generic semimartingale $X(\cdot)$. In the degenerate case $\sigma = 0$ and $\rho = 1$, we have the relations

$$\mathfrak{F}^{(R_1,R_2)}(t) = \mathfrak{F}^V(t) = \mathfrak{F}^{|X_1 - X_2|}(t) \subsetneq \mathfrak{F}^{X_1 - X_2}(t) = \mathfrak{F}^W(t) = \mathfrak{F}^{(X_1,X_2)}(t)$$

for every $0 < t < \infty$, where the inclusion is strict. In the special case $\beta = 1$ we have in addition $\sigma(V(t)) = \sigma(X_1(t) + X_2(t))$, thus also $\mathfrak{F}^V(t) = \mathfrak{F}^{X_1+X_2}(t)$, for every $0 \le t < \infty$. In the non-degenerate case $\rho\sigma > 0$, we have for every $0 < t < \infty$ the filtration relations

$$\begin{aligned} \mathfrak{F}^{(V_1,V_2)}(t) &= \mathfrak{F}^{(R_1,R_2)}(t) = \mathfrak{F}^{(|Y|,V)}(t) = \mathfrak{F}^{(|Y|,Q)}(t) \\ &\subsetneq \mathfrak{F}^{(Y,Q)}(t) = \mathfrak{F}^{(Y,V)}(t) = \mathfrak{F}^{(W_1,W_2)}(t) = \mathfrak{F}^{(X_1,X_2)}(t), \end{aligned}$$

where the inclusion is strict. These filtration equalities and inequalities can be verified in the same manner as in FERNHOLZ ET AL. (2013a). The key observation here (and also in FERNHOLZ ET AL. (2013a)) for the case of $\rho \neq \sigma$ is about pathwise uniqueness of the following extended skew TANAKA equation:

$$Y(t) = y + \frac{\rho - \sigma}{2} \int_0^t \overline{\mathrm{sgn}}(Y(s)) \mathrm{d}\beta(s) - \frac{\rho + \sigma}{\sqrt{2}} \vartheta(t) + 2(2\alpha - 1) \widehat{L}^Y(t) , \tag{10}$$

where $\overline{\mathrm{sgn}}(\cdot) := \mathbf{1}_{\{\cdot > 0\}} - \mathbf{1}_{\{\cdot < 0\}}$ and $(\beta(\cdot), \vartheta(\cdot))$ is planar Brownian motion. The original TANAKA equation driven by Brownian motion $\beta(\cdot)$:

$$Y(t) = y + \int_0^t \mathrm{sgn}(Y(s)) \mathrm{d}\beta(s) \tag{11}$$

does not admit pathwise unique, strong solution, however, its perturbed version (10) does (e.g., PROKAJ (2013), FERNHOLZ ET AL. (2013ab)). With these considerations we obtain the following.

Theorem 2. *The system of stochastic differential equations (1)-(2) admits a pathwise unique, strong solution. In particular, the filtration identity* $\mathfrak{F}^{(B_1, B_2)}(t) = \mathfrak{F}^{(X_1, X_2)}(t)$ *holds for* $t \geq 0$.

• Following the analysis of FERNHOLZ ET AL. (2013b) one can show that each of $B_1(\cdot)$ and $B_2(\cdot)$ is complementable by the other one in $\mathfrak{F}^{(W_1, W_2)}(\cdot)$, and so also maximal in the sense of BROSSARD & LEURIDAN (2008). Similarly, the pairs of $W(\cdot)$ and $U^{\flat}(\cdot)$, $U(\cdot)$ and $W^{\flat}$ are complement each other in $\mathfrak{F}^{(W_1, W_2)}(\cdot)$. $V_1(\cdot)$ is complementable by $W_2(\cdot)$ and $V_2(\cdot)$ is complemantable by $W_1(\cdot)$, however, $V_1(\cdot)$ is not complemented by $V_2(\cdot)$ in $\mathfrak{F}^{(W_1, W_2)}(\cdot)$.

• In this planar diffusion case we may compute explicitly the transition probability and time-reversal of the planar diffusions for (1)-(2) from the skew representation (8)-(9) and the properties of skew Brownian motion with bang-bang drifts. For instance, in the case of $\beta < 2$ and $\gamma_1 = \gamma_2 = 0$ with degeneracy $\rho = 0$, $\sigma = 1$, we obtain

$$\begin{aligned} &\mathbb{P}(X_1(t) \in \mathrm{d}\xi_1 , \ X_2(t) \in \mathrm{d}\xi_2) \\ &= (2\alpha) \cdot \frac{2}{2 - \beta} \cdot e^{-2\lambda(\xi_1 - \xi_2)} \cdot \frac{\mathfrak{c}_3}{\sqrt{2\pi t^3}} \exp\Big\{ - \frac{(\mathfrak{c}_3 - \lambda t)^2}{2t} \Big\} \mathrm{d}\xi_1 \mathrm{d}\xi_2 , \\ &\text{where} \quad \mathfrak{c}_3 := \Big(\frac{4 - \beta}{2 - \beta} \Big) \xi_1 - \xi_2 - \Big(\frac{\beta}{2 - \beta} \Big) x_1 - x_2 + \Big(\frac{4 - \beta}{2 - \beta} \Big) h t \end{aligned}$$

for $\xi_1 \geq \xi_2$ and $\xi_1 > x_1 - ht$, and

$$\begin{aligned} &\mathbb{P}(X_1(t) \in \mathrm{d}\xi_1 , \ X_2(t) \in \mathrm{d}\xi_2) \\ &= 2(1 - \alpha) \cdot \frac{2 e^{-2\lambda(\xi_2 - \xi_1)}}{2 - \beta} \cdot \frac{\mathfrak{c}_4}{\sqrt{2\pi t^3}} \exp\Big\{ - \frac{(\mathfrak{c}_4 - \lambda t)^2}{2t} \Big\} \mathrm{d}\xi_1 \mathrm{d}\xi_2 , \\ &\text{where} \quad \mathfrak{c}_4 := \Big(\frac{4 - \beta}{2 - \beta} \Big) \xi_2 - \xi_1 - \Big(\frac{\beta}{2 - \beta} \Big) x_1 - x_2 + \Big(\frac{4 - \beta}{2 - \beta} \Big) h t \end{aligned}$$

for $\xi_2 \geq \xi_1$ and $\xi_2 > x_1 - h\,t$. Furthermore, for the case $\xi_1 = x_1 - h\,t > \xi_2$, the local time $\widehat{L}^Y(\cdot)$ does not accumulate, that is, the transition density is

$$\begin{aligned}
&\mathbb{P}(X_1(t) = x_1 - h\,t\,,\, X_2(t) \in \mathrm{d}\xi_2) = \\
&\qquad = \frac{1}{\sqrt{2\pi t}}\Big(\exp\Big\{-\frac{(a - x_1 + x_2 + \lambda t)^2}{2t}\Big\} \\
&\qquad\qquad - e^{-2\lambda a}\exp\Big\{-\frac{(a + x_1 - x_2 + \lambda t)^2}{2t}\Big\}\Big)\Big|_{a = x_1 - \xi_2 - h\,t}\mathrm{d}\xi_1\,.
\end{aligned}$$

The transition densities for all the other cases are computable from the skew representations (8)-(9).

• For a fixed $T > 0$ what is the dynamics of its time reversal $\widetilde{X}_i(t) := X_i(T - t) - X_i(T)$? It follows from the skew representations (8)-(9) that with the backwards filtration $\widetilde{\mathfrak{F}}(t)$, $0 \leq t \leq T$ generated by $Y(T)$, $\widetilde{W}(\cdot) := W(T - \cdot) - W(t)$, $\widetilde{Q}(\cdot) := Q(T - \cdot) - Q(T)$ the time reversal $(\widetilde{X}_1(t), \widetilde{X}_2(t))$ for $0 \leq t \leq T$ is given by

$$\begin{aligned}
\widetilde{X}_1(t) &= -\mu\,t + \rho^2(\widehat{Y}^+(t) - \widehat{Y}^+(0)) - \sigma^2(\widehat{Y}^-(t) - \widehat{Y}^-(0)) \\
&\quad - \frac{1}{2}(\rho^2 - \sigma^2)(\gamma_1 - \gamma_2)\Gamma^{\widehat{Y}}(t) + (1 - \beta - \rho^2 + \sigma^2)\widehat{L}^{\widehat{Y}}(t) + \rho\,\sigma\,\widetilde{Q}(t)\,, \quad (12) \\
\widetilde{X}_2(t) &= -\mu\,t - \sigma^2(\widehat{Y}^+(t) - \widehat{Y}^+(0)) + \rho^2(\widehat{Y}^-(t) - \widehat{Y}^-(0)) \\
&\quad - \frac{1}{2}(\rho^2 - \sigma^2)(\gamma_1 - \gamma_2)\Gamma^{\widehat{Y}}(t) + (1 - \beta - \rho^2 + \sigma^2)\widehat{L}^{\widehat{Y}}(t) + \rho\,\sigma\,\widetilde{Q}(t)\,, \quad (13)
\end{aligned}$$

where $\widehat{Y}(t) := Y(T - t)$ for $0 \leq t \leq T$. The time-reversal process $(\widetilde{X}_1(\cdot), \widetilde{X}_2(\cdot))$ has some applications to the study of financial equity markets (e.g., FERNHOLZ ET AL. (2013)).

• What is the solvability of TANAKA equation (11) or the extended skew TANAKA equation (10) driven by general semimartingales (i.e., after replacing Brownian motion $(\beta(\cdot), \vartheta(\cdot))$ by general semimartingales)? This question is partially answered in ICHIBA & KARATZAS (2014) for the skew TANAKA equation. An interesting case is the TANAKA equation driven by OCONE martingales, in which the equation does not necessarily determine the probability distribution uniquely anymore. That is a contrast from the case of Brownian driven TANAKA equation (11).

• The study of skew TANAKA equation provides an excursion theoretic construction of the solution to (1)-(2) in the following way.

Given the planar diffusion $(X_1(\cdot), X_2(\cdot))$ without friction, i.e., $\eta_i = \zeta_i = 1$, $i = 1, 2$ on a probability space $(\Omega, \mathfrak{F}, \mathbb{P}, \mathbf{F})$ with $\rho\sigma > 0$, and given any (η_i^*, ζ_i^*), $i = 1, 2$ with the condition (3), there exists a planar diffusion $(X_1^*(\cdot), X_2^*(\cdot))$ with skew-elastic collisions of given parameter (η_i^*, ζ_i^*), $i = 1, 2$ on an enlargement $(\Omega^*, \mathfrak{F}^*, \mathbb{P}^*, \mathbf{F}^*)$ such that

$$(X_1(t) - X_2(t)) + \sup_{0 \leq s \leq t}(X_1(s) - X_2(s))^+ = |X_1^*(t) - X_2^*(t)|\,,$$

for $0 \le t < \infty$. For the details of construction we refer ICHIBA & KARATZAS (2014).
• When $\alpha = 1$, $\zeta = 0$, $\eta \neq 0$, that is, $\zeta_2 - \zeta_1 = 2 \neq \eta_1 - \eta_2$, collisions of particles occur with perfect reflections. Another perfect reflection is the case $\alpha = 0$. Those two cases correspond to one-dimensional reflected Brownian motion. When $(1-\zeta_1)\eta+(1-\eta_1)\zeta = 0$ (e.g., $\eta_i = 1$, $\zeta_i = 1$, $i = 1,2$), the local time components in (1)-(2) disappear, that is, there is no friction in the collisions of particles. Neither of those cases is of elastic collision. Another interesting case $\eta\overline{\zeta} + \zeta\overline{\eta} = 0$ is Brownian motion reflected on another independent Brownian motion studied by SOUCALIUC, TÓTH & WERNER (2000), BURDZY & NUALART (2002) and others.

• In general, we may consider multidimensional stochastic differential equations that involve local time supported on a smooth hyper surface starts with the work of ANULOVA (1978), PORTENKO (1979) and TOMISAKI (1980), followed by OSHIMA (1982), TAKANOBU (1987), SZNITMAN & VARADHAN (1986) and others. The recent work of KARATZAS ET AL. (2012) studies systems of the form

$$\mathrm{d}X_i(t) = \sum_{k=1}^{n} \mathbf{1}_{\{X_i(t)=X_{(k)}(t)\}}(\gamma_i + \delta_k \mathrm{d}t + \sigma_k \mathrm{d}B_i(t)) \tag{14}$$

$$+\sum_{k=1}^{n} \mathbf{1}_{\{X_i(t)=X_{(k)}(t)\}}\Big[\Big(q_k^- - \frac{1}{2}\Big)\mathrm{d}L^{X_{(k)}-X_{(k+1)}}(t) - \Big(q_k^+ - \frac{1}{2}\Big)\mathrm{d}L^{X_{(k-1)}-X_{(k)}}(t)\Big]$$

where $(X_{(1)}(\cdot), \ldots, X_{(n)}(\cdot))$ are the reverse order statistics, i.e., $X_{(1)}(\cdot) \ge \cdots \ge X_{(n)}(\cdot)$, and δ_k, σ_k, $q_k^{\pm}(\ge 0)$ are some constants that satisfy $q_k^- + q_{k+1}^+ = 0$, $k = 1, \ldots, n-1$, $i = 1, \ldots, n$ for $0 \le t < \infty$.

In the no friction case with $q_k^{\pm} = 1/2$ and $\sigma_k > 0$ in (14) the system admits the pathwise, strong solution up to the time τ of triple collision :

$$\tau := \inf\{s : X_i(s) = X_j(s) = X_k(s) \text{ for some different indices}(i,j,k)\}$$

(e.g., ICHIBA, KARATZAS & SHKOLNIKOV (2013)). Then strong solvability of the system (14) reduces to the problem of finding the triple collision probability $\mathbb{P}(\tau < \infty)$. For the recent development of this line of research we refer KARATZAS ET AL. (2012) and SARANSTEV (2013). This study is closely related to the theory of reflected diffusions in nonnegative orthants and more generally, in polyhedral domains.

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Department of Statistics and Applied Probability,
South Hall
University of California,
Santa Barbara, CA 93106
E-mail address: ichiba@pstat.ucsb.edu

カリフォルニア大学サンタバーバラ 一場 知之