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Martingale approach to first passage problems of Lévy processes over one-side moving boundaries

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Abstract

In this note, we study first passage problems for Lévy processes over one-side moving boundaries. We see that martingale approaches are useful for this problem.

1 Introduction

For a one-dimensional Lévy process L_t with $L_0 = 0$ and a function f : $[0,\infty) \to \mathbf{R}$ with $f(0) > 0$, the so called one-side moving boundary, set a first passage time

$$\tau_f = \inf\{t > 0 | L_t > f(t)\}.$$

The classical problem for this time is to know the probability $P(\tau_f > t)$ as $t \to \infty$.

In the case when L_t is a standard Brownian motion, there are many works on the probability $P(\tau_f > t)$. In particular, in Gärtner[3] and Uchiyama[11] the last probability is motivated by a study of the Kolmogorov-Petrovskii-Piskunov nonlinear parabolic equation. When $f(t)$ is bounded, we can easily find that

$$P(\tau_f > t) \asymp \frac{1}{\sqrt{t}},$$

where the notation $x(t) \asymp y(t)$ means that there are positive constants C_1 and C_2 such that $C_1 y(t) \leq x(t) \leq C_2 y(t)$ for all sufficiently large t. For unbounded $f(t)$, Uchiyama[11] obtained the interesting result that the last tail estimate is valid if and only if

$$\int_1^\infty |f(t)| t^{-\frac{3}{2}}\, dt < \infty,$$

provided $f(t)$ is increasing concave. By anther way Novikov[8] obtained that under the assumption $f(t)$ is nonincrasing convex or nondecreasing the last integral condition is valid if and only if the expectation $E[L_{\tau_f}]$ satisfies

$$0 < E[L_{\tau_f}] < \infty,$$

and then

$$\lim_{t\to\infty} \sqrt{t} P(\tau_f > t) = \sqrt{\frac{2}{\pi}} E[L_{\tau_f}].$$

In this proof he uses the tail estimates of quadratic variations of continuous local martingales, for which there are anther works and applications(see Elworthy, Li, and Yor[1], [2] and Takaoka[10]). On the other hand, in the general case when positive jumps of L_t are bounded and $-L_t$ has an exponential moment, Novikov[9] also proved the same result as the above by Novikov[8] under the additional assumption of the concave property of the function $f(t)$ in case of nondecreasing function.

In this note we consider the general case when positive jumps of L_t are bounded. Our question is what is a necessary and sufficient condition for that the asymptotic behavior of $P(\tau_f > t)$ as $t \to \infty$ is the order $\frac{1}{\sqrt{t}}$. Recently, the tail estimate of quadratic variations of càdlàg local martingales was established by Lipster and Novikov[7] and Kaji[4], [5], [6]. So, by using the improved martingale approach (see Appendix) we will provide the answer for the last question and extend the previous works Novikov[8], [9] in the main theorem and its corollary.

2 Notation and Main result

On a probability space $(\Omega, \mathcal{F}, P)$ we set a one-dimensinal Lévy process $\{L_t\}_{t\in[0,\infty)}$ with $L_0 = 0$ and a filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ by the completions of the σ-algebras generated by $\{L_t\}_{t\in[0,\infty)}$. Throughout this note, all martingales are considered with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ and we assume that the drift of $\{L_t\}_{t\in[0,\infty)}$ is zero and the Lévy measure $\nu(dz)$ satisfies for some $K > 0$

$$\nu((K,\infty)) = 0 \ and \int_{(-\infty,K]} z^2\nu(dz) < \infty. \tag{1}$$

So, $\{L_t\}_{t\in[0,\infty)}$ can be represented by

$$L_t = \sigma W_t + \int_{(0,t]\times(-\infty,K]} z\{N(dsdz) - ds\nu(dz)\},\ t \in [0,\infty),$$

where $\sigma \geq 0$, $\{W_t\}_{t\in[0,\infty)}$ is a standard Brownian motion with $W_0 = 0$, and $N(dsdz)$ is a Poisson random measure on $(0,\infty)\times\mathbf{R}$ with compensator $ds\nu(dz)$. Then, $\{L_t\}_{t\in[0,\infty)}$ obeys

$$E[e^{\lambda L_t}] = e^{\Phi(\lambda)t} < \infty,\ \ \forall\lambda > 0,$$

where $\Phi(\lambda) = \frac{\sigma^2}{2}\lambda^2 + \int_{(-\infty,K]} (e^{\lambda z} - 1 - \lambda z)\nu(dz)$.

For a continuous function $f : [0,\infty) \to \mathbf{R}$ with $f(0) > 0$ we set a first passage time

$$\tau_f = \begin{cases} \inf\{t > 0 | L_t > f(t)\} & if \ \ \{\} \neq \emptyset, \\ \infty & if \ \ \{\} = \emptyset. \end{cases}$$

Theorem 2.1 *Under the assumption that $f(t)$ is nondecreasing,*

$$\int_1^\infty f(t)t^{-\frac{3}{2}}dt < \infty \tag{2}$$

holds if and only if

$$P(\tau_f < \infty) = 1 \text{ and } (0 <)E[L_{\tau_f}] < \infty. \tag{3}$$

Then,

$$\lim_{t\to\infty} \sqrt{t}P(\tau_f > t) = \sqrt{\frac{2}{\pi\rho^2}}E[L_{\tau_f}], \tag{4}$$

where $\rho^2 = \sigma^2 + \int_{(-\infty,K]} z^2\nu(dz)$.

Corollary 2.1 *If $f(t)$ is nondecreasing, then* (2) *holds if and only if*

$$\limsup_{t\to\infty} \sqrt{t}P(\tau_f > t) < \infty. \tag{5}$$

Remark 2.1 *In the case when L_t is a standard Brownian motion Novikov*[8] *proved that under the assumption $f(t)$ is nondecreasing* (2) *is equivalent to* (3) *and then* (4) *holds* (*see Theorem2 in Novikov*[8]). *In the general case the equivalence of* (2) *and* (3) *is proved in Novikov*[9] *under the additional assumptions of the concave property of $f(t)$ and $E[e^{\lambda_* L_1}] < \infty$ for some $\lambda_* < 0$* (*see Theorem3 in Novikov*[9]).

3 Proof of Theorem 2.1

Lemma 3.1 *If* (2) *holds, then there exists the positive constant C_0 such that*

$$\lambda L_{\tau_f\wedge t} - \Phi(\lambda)(\tau_f \wedge t) \le C_0 \quad \textit{for all } t \in [0,\infty),\ 0 < \lambda < 1.$$

proof; First, the assumptions (2), the nondecreasing property of $f(t)$, and $f(0) > 0$ provides

$$\begin{aligned}(\infty >)\int_1^\infty f(s)s^{-\frac{3}{2}}ds &\ge \int_t^\infty f(s)s^{-\frac{3}{2}}ds \\ &\ge f(t)\int_t^\infty s^{-\frac{3}{2}}ds \\ &= f(t)\cdot\frac{2}{\sqrt{t}}, \quad t \in [1,\infty)\end{aligned}$$

and

$$f(t) \le f(1), \quad t \in [0,1),$$

and so

$$f(t) \le c\sqrt{t} + f(1), \quad t \in [0,\infty),$$

where $c = \frac{1}{2}\int_1^\infty f(s)s^{-\frac{3}{2}}ds \in (0,\infty)$ and $f(1) > 0$. The last fact and (1) imply

$$L_{\tau_f \wedge t} \leq f(\tau_f \wedge t) + K \leq c\sqrt{\tau_f \wedge t} + f(1) + K, \quad t \in [0,\infty).$$

On the other hand, by (1) there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 \lambda^2 \leq \Phi(\lambda), \quad 0 < \forall \lambda < 1.$$

The last two inequalities imply that for all $t \in [0,\infty)$ and $0 < \lambda < 1$

$$\begin{aligned} \lambda L_{\tau_f \wedge t} - \Phi(\lambda)(\tau_f \wedge t) &\leq \lambda c\sqrt{\tau_f \wedge t} + \lambda(f(1) + K) - \epsilon_0\lambda^2(\tau_f \wedge t) \\ &\leq \frac{c^2}{4\epsilon_0} + f(1) + K, \end{aligned}$$

which gives the desired result.

Lemma 3.2

$$E[f(\tau_f)] < \infty \tag{6}$$
$$\Longleftrightarrow (2)$$

proof; First, suppose (6). For any integer n integration by parts provides

$$\int_0^n P(\tau_f > t)df(t) = P(\tau_f > n)f(n) - P(\tau_f > 0)f(0) - \int_0^n f(t)dP(\tau_f > t),$$

and moreover the right hand side of the last equality is

$$= P(\tau_f > n)f(n) - P(\tau_f > 0)f(0) + E[f(\tau_f); \tau_f \leq n].$$

Therefore, for any integer n we have

$$\begin{aligned} \int_0^n P(\tau_f > t)df(t) &= P(\tau_f > n)f(n) - P(\tau_f > 0)f(0) + E[f(\tau_f); \tau_f \leq n] \\ &\leq E[f(\tau_f)] + E[f(\tau_f); \tau_f \leq n] \\ &\leq 2E[f(\tau_f)] < \infty, \end{aligned} \tag{7}$$

where the second line of the last inequality holds from the nondecreasing property of $f(t)$ and $f(0) > 0$.

On the other hand, according to Theorem6 in Kaji[],

$$\lim_{t\to\infty} \sqrt{t}P(\tau_{f(0)} > t) = \sqrt{\frac{2}{\pi\rho^2}}E[L_{\tau_{f(0)}}](\geq \sqrt{\frac{2}{\pi\rho^2}}f(0) > 0)$$

holds, and so for some $\epsilon_1 > 0$ there exists $\delta > 0$ such that

$$\sqrt{t}P(\tau_{f(0)} > t) \geq \epsilon_1 \quad for\ all\ t \geq \delta.$$

The last inequality implies from the nondecreasing property of $f(t)$ and $f(0) > 0$ that for all sufficiently large integer n

$$\begin{aligned}\frac{1}{\epsilon_1}\int_\delta^n P(\tau_f > t)df(t) &\geq \frac{1}{\epsilon_1}\int_\delta^n P(\tau_{f(0)} > t)df(t)\\ &\geq \int_\delta^n \frac{1}{\sqrt{t}}df(t)\\ &= \frac{1}{\sqrt{n}}f(n) - \frac{1}{\sqrt{\delta}}f(\delta) - \int_\delta^n f(t)(-\frac{1}{2}t^{-\frac{3}{2}})dt\\ &\geq -\frac{1}{\sqrt{\delta}}f(\delta) + \frac{1}{2}\int_\delta^n f(t)t^{-\frac{3}{2}}dt, \end{aligned} \tag{8}$$

where the third line of the last inequality holds by integrating by parts. Hence, (7) and (8) imply (2).

Conversely, suppose (2). Then we note that we can use the inequality in Lemma3.1. First, pick C_0 in view of Lemma3.1 and fix $A > 0$ as below. Applying the Chebyshev inequality we have

$$\begin{aligned}P(\tau_f \wedge A > t) &= P(1 - e^{-\Phi(\lambda)(\tau_f\wedge A)} > 1 - e^{-\Phi(\lambda)t})\\ &\leq \frac{1}{1 - e^{-\Phi(\lambda)t}}E[1 - e^{-\Phi(\lambda)(\tau_f\wedge A)}] \end{aligned} \tag{9}$$

for all $0 < \lambda < 1$ and $t > 0$. On the other hand, since by Lemma3.1 a process $\{e^{\lambda L_{\tau_f\wedge t} - \Phi(\lambda)(\tau_f\wedge t)}\}_{t\in[0,\infty)}$ is a uniformly integrable martingale for any $0 < \lambda < 1$,

$$\begin{aligned}&E[1 - e^{-\Phi(\lambda)(\tau_f\wedge A)}]\\ =\ & E[e^{\lambda L_{\tau_f\wedge A} - \Phi(\lambda)(\tau_f\wedge A)} - e^{-\Phi(\lambda)(\tau_f\wedge A)}]\\ =\ & E[(e^{\lambda L_{\tau_f\wedge A}} - 1)e^{-\Phi(\lambda)(\tau_f\wedge A)}]\\ \leq\ & E[\max\{0, \lambda L_{\tau_f\wedge A}\}e^{\lambda L_{\tau_f\wedge A} - \Phi(\lambda)(\tau_f\wedge A)}], \end{aligned} \tag{10}$$

where the last line of the last inequality holds by the inequality $e^x - 1 \leq \max\{0, x\}e^x$. It follows from (9) and (10) that for all $0 < \lambda < 1$ and $t > 0$

$$\begin{aligned}P(\tau_f \wedge A > t) &\leq \frac{1}{1 - e^{-\Phi(\lambda)t}}E[\max\{0, \lambda L_{\tau_f\wedge A}\}e^{\lambda L_{\tau_f\wedge A} - \Phi(\lambda)(\tau_f\wedge A)}]\\ &\leq \frac{1}{1 - e^{-\Phi(\lambda)t}} \cdot \lambda e^{C_0}(E[f(\tau_f \wedge A)] + K), \end{aligned}$$

where the last line of the last inequality holds from Lemma3.1 and (1). Setting $\lambda = \Phi^{-1}(\frac{1}{t})$, where we note that $\Phi(\lambda)$ is increasing on $(0, \infty)$, we see that for all sufficiently large t

$$P(\tau_f \wedge A > t) \leq \frac{e^{C_0}}{1 - e^{-1}}\Phi^{-1}(\frac{1}{t})(E[f(\tau_f \wedge A)] + K).$$

On the other hand, from (1) we easily see that $\lim_{\lambda\downarrow 0}\frac{\Phi(\lambda)}{\lambda^2} = \frac{\rho^2}{2}$ is valid, and so for all sufficiently large t

$$\Phi^{-1}(\frac{1}{t}) \leq \sqrt{\frac{4}{\rho^2}}t^{-\frac{1}{2}}.$$

The last two inequalities imply that there is $t_0 > 0$ such that for all $t \geq t_0$

$$P(\tau_f \wedge A > t) \quad \leq \quad \frac{e^{C_0}}{1-e^{-1}}\sqrt{\frac{4}{\rho^2}}t^{-\frac{1}{2}}(E[f(\tau_f \wedge A)] + K). \qquad (11)$$

To end this proof, set a continuous nondecreasing function $g(t)$ on $[0,\infty)$ with $g(0) > 0$ such that

$$g(t) = f(t) \cdot (\int_t^\infty f(s)s^{-\frac{3}{2}}ds)^{-\frac{1}{2}}, \quad t \geq t_0,$$

where $g(t)$ is well-defined by (2). Then, we can easily see

$$\int_{t_0}^\infty g(t)t^{-\frac{3}{2}}dt < \infty,$$

which implies from the nondecreasing property of $g(t)$ and integration by parts

$$\int_{t_0}^\infty t^{-\frac{1}{2}}dg(t) < \infty, \qquad (12)$$

and for any $\epsilon > 0$ there is the positive constant C_ϵ such that

$$f(t) - \epsilon g(t) \leq C_\epsilon, \ \forall t \in [0,\infty). \qquad (13)$$

By integrating by parts, for all sufficiently large integer n we have

$$\begin{aligned}\int_0^n P(\tau_f \wedge A > t)dg(t) \quad &= \quad P(\tau_f \wedge A > n)g(n) - P(\tau_f \wedge A > 0)f(0) \\ &\qquad - \int_0^n g(t)dP(\tau_f \wedge A > t) \\ &\geq \quad -P(\tau_f \wedge A > 0)g(0) + E[g(\tau_f \wedge A); \tau_f \wedge A \leq n],\end{aligned}$$

and so

$$\begin{aligned}E[g(\tau_f \wedge A); \tau_f \wedge A \leq n] \quad &\leq \quad P(\tau_f \wedge A > 0)g(0) + \int_0^n P(\tau_f \wedge A > t)dg(t) \\ &\leq \quad g(0) + \int_{t_0}^n P(\tau_f \wedge A > t)dg(t) + \int_0^{t_0} P(\tau_f \wedge A > t)dg(t) \\ &\leq \quad g(0) + \frac{e^{C_0}}{1-e^{-1}}\sqrt{\frac{4}{\rho^2}}\int_{t_0}^n t^{-\frac{1}{2}}dg(t) \cdot \{E[f(\tau_f \wedge A)] + K\} \\ &\qquad + g(t_0) - g(0),\end{aligned}$$

where the last line of the last inequality holds from (11). Going now to the limit as $n \to \infty$, by (12) we have

$$E[g(\tau_f \wedge A)] \quad \leq \quad C_1 E[f(\tau_f \wedge A)] + C_2, \qquad (14)$$

where $C_1 = \dfrac{e^{C_0}}{1-e^{-1}}\sqrt{\dfrac{4}{\rho^2}}\displaystyle\int_{t_0}^{\infty} t^{-\frac{1}{2}}dg(t) < \infty$ and $C_2 = C_1K + g(t_0)$. Choosing $\epsilon < \frac{1}{C_1}$ in view of (13), it follows from (13) and (14) that

$$\begin{aligned} E[f(\tau_f \wedge A)] &\leq C_\epsilon + \epsilon E[g(\tau_f \wedge A)] \\ &\leq C_\epsilon + \epsilon C_2 + \epsilon C_1 E[f(\tau_f \wedge A)], \end{aligned}$$

and so

$$E[f(\tau_f \wedge A)] \leq \frac{C_\epsilon + \epsilon C_2}{1-\epsilon C_1} < \infty \quad for\ any\ A > 0.$$

Hence, by noting the nondecreasing property of $f(t)$ and $f(0) > 0$, the last inequality provides from the Fatou lemma that (6) is valid.

proof of Theorem2.1.; According to Lemma3.3, to end the proof of the equivalence of (2) and (3), it is enought to show that under (6)

$$P(\tau_f < \infty) = 1 \tag{15}$$

holds. So, under (6) assume $P(\tau_f = \infty) > 0$. Then, this assumption implies from the nondecreasing property of $f(t)$ and $f(0) > 0$ that

$$\begin{aligned} E[f(\tau_f)] &= \int_0^\infty f(t)P(\tau_f > t)dt \\ &\geq \int_0^\infty f(0)P(\tau_f = \infty)dt = \infty, \end{aligned}$$

where contradicts to (6). Hence, (15) is valid.

Finally, suppose (3), which is equivalent to (2). Setting $M_t = -L_{\tau_f \wedge t}$, $t \in [0,\infty)$, $\{M_t\}_{t\in[0,\infty)}$ is a locally square integrable martingale with a quadratic variation $\langle M\rangle_\infty < \infty$ and $\{\max\{0, -M_\tau\}\}_{\tau\in\mathcal{T}}$ is uniformly integrable, where $\mathcal{T}$ is the set of all stopping times , since (1) and (3) are assumed. Considering that a counting measure μ of $\{M_t\}_{t\in[0,\infty)}$ is

$$\mu((0,t]\times U) = \int_{(0,\tau_f\wedge t]\times(\mathbf{R}\setminus\{0\})} I_U(-z)N(dsdz),$$

where $t > 0$, U is the Borel subset of $\mathbf{R}\setminus\{0\}$, and $I_U(x)$ is the characteristic function of U, its predictable compensator $\widehat{\mu}$ satisfies

$$\widehat{\mu}((0,t]\times U) = \int_0^{\tau_f\wedge t}\int_{\mathbf{R}\setminus\{0\}} I_U(-z)ds\nu(dz).$$

So, by using (1) we can check

$$\int_{(0,\infty)\times(-\infty,-K)} e^{-z}\widehat{\mu}(dsdz) = \int_0^{\tau_f}\int_K^\infty e^z ds\nu(dz) = 0.$$

Applying Theorem4.1 for $\{\langle M\rangle_t\}_{t\in[0,\infty)}$, Lemma3.1 and the last result imply the desired convergence.

4 Appendix

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ be a filtered probability space with usual conditions and a càdlàg process $\{M_t\}_{t\in[0,\infty)}$ is a locally square integrable martingale with $M_0 = 0$ defined on it. We know a decomposition

$$M_t = M_t^c + M_t^d, \quad t \in [0,\infty),$$

where locally square integrable martingales $\{M_t^c\}_{t\in[0,\infty)}$ and $\{M_t^d\}_{t\in[0,\infty)}$ are continuous and purely discontinuous, respectively.

Define a measure μ on $[0,\infty)\times(\mathbf{R}\backslash\{0\})$ by

$$\mu((0,t]\times U) = \sum_{0<s\leq t} I_U(\Delta M_s),$$

for $t\in[0,\infty)$ and Borel subsets U of $\mathbf{R}\backslash\{0\}$, where $\Delta M_t = M_t - M_{t-}$. Then, we denote by $\widehat{\mu}$ its predictable compensator and by $\widehat{\mu}^c$ the measure on $\mathbf{R}\backslash\{0\}$ such that

$$\widehat{\mu}^c(dsdx) = 1_{\{\widehat{\mu}(\{s\}\times(\mathbf{R}\backslash\{0\}))=0\}}\widehat{\mu}(dsdx).$$

Moreover, for any predictable function $\alpha(t,x)$ we write a stochastic integral $\{(\alpha*\xi)_t\}_{t\in[0,\infty)}$based on $\xi = \widehat{\mu}$ or $\widehat{\mu}^c$ by

$$(\alpha*\xi)_t = \int_{(0,t]\times(\mathbf{R}\backslash\{0\})} \alpha(s,x)\xi(dsdx)$$

if $\alpha(s,x)$ is integrable on $(0,t]\times(\mathbf{R}\backslash\{0\})$.

For any $\lambda > 0$ set

$$\psi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x$$

and

$$\mathcal{E}(\lambda)_t = \exp\{-\lambda M_t - \frac{\lambda^2}{2}\langle M^c\rangle_t - (\psi_\lambda*\widehat{\mu}^c)_t - \sum\nolimits_{0<s\leq t}\log(1+\Delta(\psi_\lambda*\widehat{\mu})_s)\}.$$

Assume that and $\{\max\{0, -M_\tau\}\}_{\tau\in\mathcal{T}}$ is uniformly integrable, where $\mathcal{T}$ is the set of all stopping times. The assumption $\langle M\rangle_\infty < \infty$ a.s. implies $M_\infty < \infty$ a.s. and provides that for any $\lambda > 0$

$$\langle M^c\rangle_\infty < \infty \ a.s., \ (|\psi_\lambda|*\widehat{\mu})_\infty < \infty \ a.s., \ and \ \mathcal{E}(\lambda)_\infty < \infty \ a.s.$$

hold(The detail can be found in section 2 of Kaji[6]).

Theorem 4.1 *Suppose*

$$E\left[\left(e^{-\lambda_0 x} I_{(-\infty,-K)}(x) * \widehat{\mu}\right)_\infty\right] < \infty$$

for some $\lambda_0, K > 0$ *and, moreover, there exists the nonnegative integrable random variable* $\mathcal{E}_+$ *such that*

$$\mathcal{E}(\lambda)_\infty \leq \mathcal{E}_+.$$

Then, it holds that

$$-\infty < E[M_\infty] \leq 0$$

and

$$\lim_{\lambda\to\infty} \lambda P\left(\sqrt{\langle M\rangle_\infty} > \lambda\right) = -\sqrt{\frac{2}{\pi}}\, E[M_\infty].$$

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