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The entropic curvature dimension condition and Bochner's inequality

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1 Introduction

This note is a short review of the paper [9] which is written by the author, M. Erbar and K.-Th. Sturm.

There are several different ways to characterize "Ric $\geq K$ and $\dim X \leq N$" on a Riemannian manifold X, where $K \in \mathbb{R}$ and $N \in (0, \infty)$. Among them, the curvature-dimension condition introduced by Sturm [24], Lott and Villani [15] works well even in the framework of abstract metric measure spaces. It is described in terms of optimal transportation and it possesses many nice geometric stability properties. On the other hand, Bochner's inequality introduced by Bakry and Émery is formulated for an abstract diffusion generator. As Bochner's formula has played significant roles in Riemannian geometry, Bochner's inequality provides enormous important functional inequalities in geometric analysis. The purpose of the paper [9] is to unify these two concepts by introducing new conditions equivalent to either (and hence both) of them on metric measure spaces. When $N = \infty$, this program was essentially finished by Ambrosio, Gigli, Savaré and their collaborators [1–4] and our main focus is in the case $N < \infty$.

2 Framework and main results

Let (X, d, m) be a Polish geodesic metric measure space, where the measure m is locally finite and σ-finite. Here "geodesic space" means that the distance coincides with the infimum of the length over all curves with fixed endpoints and a minimizing curve exists (We call it geodesic). Suppose $\operatorname{supp} m = X$ for simplicity. Fix $K \in \mathbb{R}$ and $N \in (0, \infty)$. Let us introduce comparison functions: for $\kappa \in \mathbb{R}$ and $\kappa\theta^2 \leq \pi^2$,

$$\mathfrak{s}_\kappa(\theta) := \frac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}}, \quad \sigma_\kappa^{(t)}(\theta) := \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)}.$$

We call a function V on a metric space (Y, d_Y) *(K, N)-convex* if for each $x, y \in Y$ there is a constant speed geodesic $\gamma : [0, 1] \to Y$ from x to y such that the following holds:

$$V_N(\gamma_t) \geq \sigma_{K/N}^{(1-t)}(d_Y(x, y))V_N(\gamma_0) + \sigma_{K/N}^{(t)}(d_Y(x, y))V_N(\gamma_1), \quad \text{where } V_N := \exp\left(-\frac{1}{N}V\right).$$

We call V *strongly (K, N)-convex* if the last inequality holds for each (and at least one) geodesic γ. This is an integral formulation of the following inequality in the distributional sense:

$$\partial_t^2 V_N(\gamma_t) \leq -\frac{K}{N}d(x, y)^2 V_N(\gamma_t).$$

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If V is C^2-function on a Riemannian manifold, then V is (K,N)-convex if and only if

$$\mathrm{Hess}\, V - \frac{1}{N}\nabla V \otimes \nabla V \geq K.$$

Let $\mathscr{P}_2(X)$ be the L^2-Wasserstein space, consisting of probability measures on X with finite second moment, equipped with the *L^2-Wasserstein distance* W_2 given by

$$W_2(\mu,\nu) := \inf\left\{ \|d\|_{L^2(q)} \;\middle|\; q\text{: a coupling of } \mu \text{ and } \nu \right\}.$$

Note that $(\mathscr{P}_2(X), W_2)$ is also a Polish geodesic metric space. Moreover, for each $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, we can always find a probability measure π on the space of constant speed geodesics $\mathrm{Geo}(X)$ parametrized by $[0,1]$ whose projections consist of a W_2-geodesic in $\mathscr{P}_2(X)$. To state it more precisely, we denote the evaluation map $\mathrm{Geo}(X) \to X$ by e_t, that is, $e_t(\gamma) := \gamma_t$ for $\gamma \in \mathrm{Geo}(X)$ and $t \in [0,1]$. We also denote the push-forward of a measure by e_t by $e_t^\sharp$. Then we call π a dynamic optimal coupling if $\pi \in \mathscr{P}(\mathrm{Geo}(X))$ such that $e_i^\sharp \pi = \mu_i$ for $i = 0,1$, $(e_t^\sharp \pi)_{t\in[0,1]}$ is a W_2-geodesic and $(e_t^\sharp \pi, e_s^\sharp \pi)$ is a optimal coupling of $e_t^\sharp \pi$ and $e_s^\sharp \pi$ for each $s,t \in [0,1]$.

We denote the *relative entropy* by Ent: For $\mu \in \mathscr{P}(X)$,

$$\mathrm{Ent}(\mu) := \begin{cases} \displaystyle\int_X \rho \log \rho \, dm & \text{if } \mu = \rho m \text{ with } (\rho\log\rho)_+ \in L^1(X,m), \\ \infty & \text{otherwise.} \end{cases}$$

We say that (X,d,m) satisfies the (strong) *entropic curvature dimension condition* with parameters K and N ($\mathsf{CD}^e(K,N)$ in short) if Ent is (strongly) (K,N)-convex on $\mathscr{P}_2(X)$ respectively.

Let Ch be *Cheeger's L^2-energy functional* given by a relaxation of the energy functional associated with local Lipschitz constants. That is,

$$\mathsf{Ch}(f) := \frac{1}{2} \liminf_{\substack{f_n:\text{ Lipschitz} \\ f_n \to f \text{ in } L^2(m)}} \liminf_{n\to\infty} \int_X |\nabla f_n|^2 \, dm,$$

where $|\nabla f_n|$ is the local Lipschitz constant of f_n. It can be written as an energy integral in terms of the *weak upper gradient* $|\nabla f|_w$, i.e.

$$\mathsf{Ch}(f) = \frac{1}{2}\int_X |\nabla f|_w^2 \, dm$$

(see [3]). We say (X,d,m) *infinitesimally Hilbertian* if Ch coincides with a closed symmetric bilinear form $\mathcal{E}$: $2\mathsf{Ch}(f) = \mathcal{E}(f,f)$. In this case $\mathcal{E}(f,g)$ has a density denoted by $\langle \nabla f, \nabla g\rangle$ and in particular $|\nabla f|_w^2 = \langle \nabla f, \nabla f \rangle$ m-a.e. (see [4]). Let Δ be the associated generator of $\mathcal{E}$ and T_t a Markov semigroup generated by Δ. Note that (X,d,m) need not be infinitesimally Hilbertian in order to define T_t or Δ (see [3]).

Example 2.1 *Let (X,d,m) be an N-dimensional complete connected Riemannian manifold, $\partial X = \emptyset$, equipped with the Riemannian distance d and the Riemannian volume measure m. Suppose* $\mathrm{Ric} \geq K$. *Let V be a (K',N')-convex function on (X,d). Then $(X,d,\mathrm{e}^{-V}m)$ satisfies $\mathsf{CD}^e(K+K',N+N')$. In this framework, Ch coincides with the usual Dirichlet energy (with respect to $\mathrm{e}^{-V}m$ instead of m) and hence $(X,d,\mathrm{e}^{-V}m)$ is infinitesimally Hilbertian.*

To derive a nice geometric properties, the curvature dimension condition $\mathsf{CD}(K,N)$ introduced first by Sturm [24] (Lott and Villani [15] also, when "$K = 0$ and $N < \infty$" or $N = \infty$) is modified to a reduced one (we denote it by $\mathsf{CD}^*(K,N)$) by Bacher and Sturm [6]. We say (X,d,m) satisfies $\mathsf{CD}^*(K,N)$ if, for $\mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathscr{P}(X)$ with bounded supports, there exists an optimal coupling q of them and a geodesic $\mu_t = \rho_t m \in \mathscr{P}_2(X)$ with bounded supports such that for all $t \in [0,1]$ and $N' \geq N$:

$$\int_X \rho_t^{-1/N'} d\mu_t \geq \int_{X\times X} \big[\sigma_{K/N'}^{(1-t)}(d(x_0,x_1))\rho_0(x_0)^{-1/N'} \\ + \sigma_{K/N'}^{(t)}(d(x_0,x_1))\rho_1(x_1)^{-1/N'}\big] q(dx_0,dx_1).$$

The strong $\mathsf{CD}^*(K,N)$ can be defined analogously. Note that $\mathsf{CD}^*(K,N)$ is a priori weaker than $\mathsf{CD}(K,N)$ and it is really weaker (see [17]). In what follows, we sometimes require the following assumption. We will mention it explicitly when they are required.

Assumption 1

(a) *There exists* $c > 0$ *such that* $\int_X \exp\left(-cd(x,x_0)^2\right)\, dm < \infty$ *for some* $x_0 \in X$.

(b) (X,d,m) *is infinitesimally Hilbertian.*

(c) *Every* $f \in L^2(m)$ *with* $\mathsf{Ch}(f) < \infty$ *and* $|\nabla f|_w \leq 1$ *m-a.e. has a 1-Lipschitz representative.*

We now turn to state our first main theorem, which extends the main theorem in [1,4] to the case $N < \infty$.

Theorem 2.2 *The following are equivalent:*

(i) *Assumption 1 (b) and* $\mathsf{CD}^*(K,N)$ *holds.*

(ii) *Assumption 1 (b) and* $\mathsf{CD}^e(K,N)$ *holds.*

(iii) *Assumption 1 (a) holds, and for each* $\mu \in \mathscr{P}(X)$ *with* $\mathrm{Ent}(\mu) < \infty$ *there exists a solution* $(\mu_t)_{t\geq 0}$ *to the* (K,N)-evolution variational inequality ($\mathsf{EVI}_{K,N}$ *in short) with* $\mu_0 = \mu$. *That is,* $(\mu_t)_{t\geq 0}$ *is a locally absolutely continuous curve in* $\mathscr{P}_2(X)$ *and, for each* $\sigma \in \mathscr{P}_2(X)$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\sigma)}{2}\right) + K\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\sigma)}{2}\right) \\ \leq \frac{N}{2}\left(1 - \exp\left(-\frac{1}{N}(\mathrm{Ent}(\sigma) - \mathrm{Ent}(\mu_t))\right)\right)$$

Note that $\mathsf{CD}^e(K,N)$ implies Assumption 1 (a). Moreover, the condition (ii) implies Assumption 1 (c). Since Assumption 1 (b) is included in the condition (i) or (ii), Assumption 1 is satisfied if either one of (i)–(iii) is satisfied.

In the condition (iii), the solution μ_t to $\mathsf{EVI}_{K,N}$ can be regarded as a gradient flow of Ent (in a stronger sense). It was (at least heuristically) known that the gradient flow of Ent coincides with the heat distribution. We can verify it in this framework (see [3]) and this fact together with Theorem 2.2 connects the curvature dimension condition in terms of the optimal transportation with analysis of the heat semigroup T_t. This connection was hidden in $\mathsf{CD}^*(K,N)$ when $N < \infty$ since there appears no Ent while $\mathsf{CD}(K,\infty)$ is written in terms of Ent. Thus, by introducing the new condition $\mathsf{CD}^e(K,N)$, we succeed in keeping this connection even when $N < \infty$.

We call that (X,d,m) satisfies $\mathsf{RCD}^*(K,N)$ (Riemannian curvature-dimension condition) if one of the conditions (i)–(iii) is satisfied. Next we will state the connection between $\mathsf{RCD}^*(K,N)$ and the behavior of heat distributions or Bochner's inequality.

Theorem 2.3 *If (X,d,m) satisfies $\mathsf{RCD}^*(K,N)$, then the following holds:*

(iv) [Space-time W_2-control] *For $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ and $t, s \geq 0$,*

$$\mathfrak{s}_{K/N}^2\left(\frac{W_2(T_t\mu_0, T_s\mu_1)}{2}\right) \leq e^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_0,\mu_1)}{2}\right) + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}\left(\sqrt{t}-\sqrt{s}\right)^2.$$

(v) [Bakry-Ledoux gradient estimate] *For $f \in D(Ch)$ and $t > 0$,*

$$|\nabla T_t f|_w^2 + \frac{2tC(t)}{N}|\Delta T_t f|^2 \leq e^{-2Kt}T_t(|\nabla f|_w^2) \quad m\text{-a.e.},$$

where $C(t) > 0$ is a function satisfying $C(t) = 1 + O(t)$ as $t \to 0$.

(vi) [(weak) Bochner's inequality] *For $f \in D(\Delta)$ with $\Delta f \in D(Ch)$ and all $g \in D(\Delta) \cap L^\infty(X,m)$ with $g \geq 0$ and $\Delta g \in L^\infty(X,m)$,*

$$\frac{1}{2}\int_X \Delta g |\nabla f|_w^2\, dm - \int_X g\langle \nabla f, \nabla \Delta f\rangle\, dm \geq K\int_X g|\nabla f|_w^2\, dm + \frac{1}{N}\int_X g(\Delta f)^2\, dm.$$

Conversely, if Assumption 1 holds, then one of (iv)–(vi) *implies* (i)–(iii) *and hence* (i)–(vi) *are all equivalent.*

Note that we can extend the heat semigroup T_t to a linear operator on the space of probability measures when Assumption 1 holds (see [2–4]). We should interpret T_t in (iv) in this sense. The constant $C(t)$ in (v) can be explicit, but it becomes different if we obtain it from (iv) or from (vi). However, the exact value of $C(t)$ is irrelevant to the implications from (v). The reason why we call (vi) *weak* is in the fact that we formulate the condition in integral form by using a test function g. All the conditions (i)–(vi) becomes weaker as K decreases and N increases. In particular, by taking $N \to \infty$, in a suitable way, we can recover the corresponding conditions for $N = \infty$.

As a review, we mention an overview of the proof of Theorem 2.2 and Theorem 2.3. First of all, we remark that the essence of the proof is mostly similar to the one in the case

$N = \infty$ studied in [1,2,4], although they are technically more involved and require some new idea in many cases. Possibly, the most difficult part of the proof of the equivalence is to *find* the conditions (ii) and (iv). Actually, the conditions (i), (v) and (vi) are already known and (iii) can be found from (ii). Implications dealt in the proof of Theorem 2.2 and Theorem 2.3 are listed as follows:

- (i) and (ii) are equivalent.
- (ii) and (iii) are equivalent.
- (iii) implies (iv) and Assumption 1.
- (iv) and Assumption 1 implies (v).
- Under Assumption 1 (b), (v) is equivalent to (vi).
- Under Assumption 1, (v) implies (ii).

Among them, we discuss something more on the equivalence between (i) and (ii) because we require an additional argument which does not appear in the case $N = \infty$. Indeed, as $N \to \infty$, $\mathsf{CD}^*(K,N)$ and $\mathsf{CD}^e(K,N)$ yield the same condition (so-called $\mathsf{CD}(K,\infty)$). A key observation is that we can *localize* $\mathsf{CD}^*(K,N)$ *along each geodesic* in the following sense: If $\mathsf{CD}^*(K,N)$ holds and (X,d) admits no branching geodesics, then for $\mu_0, \mu_1 \in D(\mathrm{Ent})$ with bounded support, there exists a dynamic optimal coupling π of μ_0 and μ_1 such that, $e_t^\sharp \pi \ll m$ (we denote $e_t^\sharp \pi = \rho_t m$) for each $t \in [0,1]$ and

$$\rho_t(\gamma_t)^{-1/N} \geq \sigma_{K/N}^{(1-t)}(d(\gamma_0,\gamma_1))\rho_0(\gamma_0)^{-1/N} + \sigma_{K/N}^{(t)}(d(\gamma_0,\gamma_1))\rho_1(\gamma_1)^{-1/N} \tag{2.1}$$

for π-a.e. $\gamma \in \mathrm{Geo}(X)$. We can recover $\mathsf{CD}^*(K,N)$ from (2.1) by integrating it by π and hence (2.1) is equivalent to $\mathsf{CD}^*(K,N)$ under the "non-branching" assumption. On the other hand, by taking a logarithm on the both hand side of (2.1) and integrating it by π together with the Jensen inequality, we can obtain $\mathsf{CD}^e(K,N)$. In addition, we can also localize $\mathsf{CD}^e(K,N)$ to derive (2.1) and hence $\mathsf{CD}^e(K,N)$ is equivalent to (2.1) under the "non-branching" assumption again. Thus the equivalence holds under the "non-branching" assumption. Under the condition (i) or (ii), we can employ the result in [19] and it follows that geodesics in (X,d,m) are *essentially non-branching.* It is weaker than the "non-branching" assumption but it is sufficient to make the same argument as above valid. Hence the equivalence of (i) and (ii) follows. Note that, as a by-product of the proof, strong $\mathsf{CD}^*(K,N)$ or strong $\mathsf{CD}^e(K,N)$ holds if $\mathsf{RCD}^*(K,N)$ holds.

3 Properties, applications and related results

First we review some properties of $\mathsf{RCD}^*(K,N)$. From geometric point of view, this condition behaves well under deformations. For instance, $\mathsf{RCD}^*(K,N)$ is stable under the convergence of metric measure spaces: If a sequence of metric measure spaces satisfying $\mathsf{RCD}^*(K,N)$ with a universal K and N converges in the measured Gromov-Hausdorff topology or $\mathbb{D}$-topology introduced in [24], then the metric measure space in the limit enjoys the same condition (See [11] in the case that X is not compact). $\mathsf{RCD}^*(K,N)$ is also stable under tensorization: If we take a product of two metric measure spaces satisfying the Riemannian curvature dimension condition (with possibly different parameter), then the product metric measure space again satisfies the condition as it does for Riemannian manifolds. In addition, $\mathsf{RCD}^*(K,N)$ enjoys a local-to-global property. Roughly speaking, if $\mathsf{RCD}^*(K,N)$ holds on (possibly small) open sets which covers the whole space with the same parameter K and N, then the whole space satisfies the $\mathsf{RCD}^*(K,N)$. Stability under taking a cone is also proved [13]. Note that, as a consequence of Theorem 2.3, all the same stability holds for (iv)–(vi) if it is combined with Assumption 1.

As geometric applications, it is known that $\mathsf{CD}^*(K,N)$ produces several sharp comparison theorems in Riemannian geometry. For example, $\mathsf{CD}^*(K,N)$ yields the measure contraction property $\mathsf{MCP}(K,N)$ [8]. As a result, the Bishop-Gromov volume comparison theorem, the Bonnet-Myers diameter bound etc. hold with a sharp constant. In particular, the local uniform volume doubling property and the local uniform Poincaré inequality holds [18, 20]. In other direction, a natural extension of the maximal diameter theorem holds under $\mathsf{RCD}^*(K,N)$ [13]. It describes what happens if the equality in the Bonnet-Myers diameter bound is attained, and the result is as optimal as we can expect. Note that the proof of this theorem in [13] requires (vi), and hence Theorem 2.3.

The curvature-dimension condition has a strong connection with several functional inequalities. In particular, when $K>0$ and $N=\infty$, it is well known that $\mathsf{CD}(K,\infty)$ yields the so-called HWI inequality and it produces the logarithmic Sobolev inequality, and Talagrand's transport inequality (see e.g. [25]). By a similar argument, $\mathsf{CD}^e(K,N)$ with $K>0$ and $N<\infty$ produces the following analogous inequalities:

- [N-HWI inequality] For $\mu_0,\mu_1\in\mathscr{P}_2(X)$ with $\mu_0=\rho m$,

$$\exp\left(\frac{1}{N}(\mathrm{Ent}(\mu_0)-\mathrm{Ent}(\mu_1))\right) \le \mathfrak{s}'_{K/N}(W_2(\mu_0,\mu_1))+\frac{1}{N}\mathfrak{s}_{K/N}(W_2(\mu_0,\mu_1))\sqrt{\int_X\frac{|\nabla\rho|_w^2}{\rho}\,dm}$$

- [N-log Sobolev inequality] Suppose $m\in\mathscr{P}_2(X)$. Then for $\mu\in\mathscr{P}_2(X)$,

$$KN\left(\exp\left(\frac{2}{N}\,\mathrm{Ent}(\mu)\right)-1\right)\le\int_X\frac{|\nabla\rho|_w^2}{\rho}\,dm.$$

- [N-Talagrand inequality] Suppose $m\in\mathscr{P}_2(X)$. Then for $\mu\in\mathscr{P}_2(X)$, we have $W_2(\mu,m)\le\pi\sqrt{N/4K}$ and

$$W_2(\mu,m)\le\sqrt{\frac{N}{K}}\arccos\left(\exp\left(-\frac{1}{N}\,\mathrm{Ent}(\mu)\right)\right).$$

Note that the N-Sobolev inequality yields the global Sobolev inequality (with a possibly non-optimal constant). See [7, Proposition 6.2.3]. By other means, we can obtain a sharp global Poincaré or the spectral gap inequality involving N and K on spaces satisfying $\mathsf{RCD}^*(K,N)$. It immediately yields a lower bound of the first nonzero eigenvalue of $-\Delta$ (Note that $\mathsf{RCD}^*(K,N)$ with $K > 0$ and $N < \infty$ implies the compactness of X). We do not know what happens if the equality holds on $\mathsf{RCD}^*(K,N)$ spaces.

The $\mathsf{RCD}^*(K,N)$ condition also ensures some sort of regularity of the solution to the heat equation, or the heat semigroup T_t. First of all, on spaces satisfying $\mathsf{RCD}^*(K,N)$, the heat semigroup T_t is associated with a heat kernel density with respect to m which enjoys the two-sided Gaussian bound since the local Poincaré inequality and the volume doubling property hold (See [23]). Note that the absolute continuity also follows from the fact $T_t\mu$, $\mu \in \mathscr{P}(X)$ coincides with the gradient flow of Ent since $\mathrm{Ent}(T_t\mu) < \infty$ implies $T_t\mu \ll m$. In addition, $\mathsf{RCD}(K,\infty)$ ensures the Lipschitz continuity of $T_t f$ ($f \in L^2(m)$), the heat kernel and in particular eigenfunctions [1, 4]. More precisely, we can obtain the following quantitative Lipschitz regularization bound for T_t ([4, Proposition 6.9] or [1, Theorem 7.3]):

$$|\nabla T_t f| \le \sqrt{\frac{K}{e^{2Kt}-1}}\|f\|_\infty.$$

Note that this estimate is related with Assumption 1 (c) (See [1, 2, 4]). By a potential theoretic approach based on the parabolic Harnack inequality, it is known that the two-sided Gaussian bound implies the Hölder continuity of the heat kernel. We can improve it if (X,d,m) satisfies the stronger assumption $\mathsf{RCD}^*(K,N)$.

Finally we exhibit related results appeared after [9]. Some of them are already mentioned at the end of the second version of [9] and hence we treat what is not mentioned there. The list is probably far from being complete but the author hopes it is helpful for readers. First, F.-Y. Wang's dimension-free Harnack inequality is extended to $\mathsf{RCD}(K,\infty)$ spaces [14], with the aid of a self-improvement of the gradient estimate in [21]. The localized version of the Bochner's inequality (vi) and its relation with (vi) are studied in [5]. The behavior of Bochner's inequality under transformations in Riemannian geometry and in the theory of Dirichlet forms is discussed in [22]. The (K,N)-convexity for $N < 0$ is considered in [16]. Even in that case, many results still hold true but some do not. Especially the connection between (K,N)-convexity of the relative entropy and behavior of heat distributions does not seem to be completely understood. The question on the existence and the uniqueness of the optimal transport map on $\mathsf{RCD}^*(K,N)$ spaces and its relation with an extension of the exponential map on those spaces are discussed in [12]. We will close this exhibition by remarking that there are ongoing extensive studies on geometric structure of $\mathsf{RCD}^*(K,N)$ spaces. For instance, see [10] and references therein.

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