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Representations of fractional moments and their truncations

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Abstract

Several results of [3] are summarized, which are thought to be useful in applications. New expressions for fractional moments of both positive and negative parts are given in terms of characteristic function (ch.f.). Additionally, formulae to calculate truncated fractional moments by ch.f. are studied, from which conditional fractional moments are directly calculated. Moreover, formulae for truncated fractional moments for any real number are supplied based on [5]. Note that in this report, we give only results and omit proofs which are given in the former version of [3]. After the probability symposium on Dec.17~20, 2013 in Kyoto we obtained more general results which unifies our presented results in a better way. Therefore, this report serves as a prompt report.

1 Positive and negative parts moments

Notations and preliminary. Let $p \in (0,\infty)\setminus\mathbb{N}$, $k := \lfloor p \rfloor$ and $\lambda := p-k$, so that $\lambda \in (0,1)$. For a complex-valued function f, its fractional derivative of order $p = k+\lambda$ is given by

$$(\mathrm{D}^p f)(t) := (\mathrm{D}^\lambda f^{(k)})(t) := \frac{\lambda}{\Gamma(1-\lambda)} \int_{-\infty}^{t} \frac{f^{(k)}(t)-f^{(k)}(u)}{(t-u)^{1+\lambda}}\,\mathrm{d}u$$

for $t \in \mathbb{R}$ (see e.g. Eq. (2.1) of [1]).

Another tool is the following expression of x_+^p by Corollary 2 in [4] where $x_+ := x \vee 0$, which is obtained by the Cauchy integral theorem. Let $m \in \mathbb{Z}_+ := \{0\}\cup\mathbb{N}$ and $z \in \mathbb{C}$, $e_m(z)$ denotes the remainder of Maclaurin expansion of e^z, $e_m(z) := e^z - \sum_{j=0}^{m} \frac{z^j}{j!}$. Then

$$x_+^p = \frac{x^{\lfloor p \rfloor}}{2}\mathbf{1}_{\{p\in\mathbb{N}\}} + \frac{\Gamma(p+1)}{\pi}\int_0^\infty \mathfrak{Re}\,\frac{e_{\lceil p-1\rceil}(itx)}{(it)^{p+1}}\mathrm{d}t,$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are floor and ceiling functions respectively. Throughout we let f be a ch.f. of arbitrary random variable X, $f(t) = \mathsf{E}[e^{itX}]$ for $t \in \mathbb{R}$.

Main Results. In this report, we give only results and omit proofs which are given in the former version of [3] (available from the author).

In the next two theorems and one proposition, we will observe that various representations are possible for both positive and negative part of fractional moments and symmetric moments.

Theorem 1.1 *Let $p \in (0,\infty) \setminus \mathbb{N}$ and assume that $\mathsf{E}|X|^p < \infty$. Then for $p \in (0,\infty)$*

$$\mathsf{E}X_+^p = \frac{(-1)^{k+1}}{\sin \pi\lambda}\, \mathfrak{Re}\left(\mathrm{D}^p(i^{-p-1}\bar{f})\right)(0),$$
$$\mathsf{E}X_-^p = \frac{(-1)^{k+1}}{\sin \pi\lambda}\, \mathfrak{Re}\left(\mathrm{D}^p(i^{-p-1}f)\right)(0),$$
$$\mathsf{E}|X|^p = 2\frac{(-1)^{k+1}}{\sin \pi\lambda}\, \mathfrak{Re}(i^{-p-1})\, \mathfrak{Re}(\mathrm{D}^p f)(0),$$

where, as usual, $X_- := (-X)_+$.

Theorem 1.2 *Let $p \in (0,\infty)\setminus\mathbb{N}$, $k = \lfloor p \rfloor$ and $\lambda = p - k$ so that $0 < \lambda < 1$. Assume $\mathsf{E}|X|^p < \infty$, then $\mathsf{E}X^{[p]} = E[X_+^p - X_-^p]$ is given by*

$$\mathsf{E}X^{[p]} = \begin{cases} \dfrac{2i^k}{\sin \pi\lambda}\, \mathfrak{Re}\, i^{-\lambda}\, \mathfrak{Im}\,(\mathrm{D}^p f)(0) & \text{if } k \text{ is even,} \\ \dfrac{2i^{k+1}}{\sin \pi\lambda}\, \mathfrak{Im}\, i^{-\lambda}\, \mathfrak{Im}\,(\mathrm{D}^p f)(0) & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 1.1 *Let $p > 0$ and $0 \le m \le \lfloor p \rfloor$. Assume that $\mathsf{E}|X|^p < \infty$, then*

$$\mathsf{E}X_+^p = \frac{f^{\lfloor p \rfloor}(0)}{2i^{\lfloor p \rfloor}}\mathbf{1}_{\{p\in\mathbb{N}\}} + \frac{\Gamma(p-m+1)}{\pi}\, \mathfrak{Re} \int_0^\infty \frac{\frac{d^m}{dt^m}\left(f(t) - \sum_{j=0}^{\lceil p-1 \rceil} f^{(j)}(0)\frac{t^j}{j!}\right)}{i^{p+1} t^{p-m+1}}\mathrm{d}t,$$
$$\mathsf{E}X_-^p = \frac{f^{\lfloor p \rfloor}(0)}{2i^{\lfloor p \rfloor}}\mathbf{1}_{\{p\in\mathbb{N}\}} + \frac{\Gamma(p-m+1)}{\pi}\, \mathfrak{Re} \int_0^\infty \frac{\frac{d^m}{dt^m}\left(\bar{f}(t) - \sum_{j=0}^{\lceil p-1 \rceil} \bar{f}^{(j)}(0)\frac{t^j}{j!}\right)}{i^{p+1} t^{p-m+1}}\mathrm{d}t,$$
$$\mathsf{E}|X|^p = \frac{f^{\lfloor p \rfloor}(0)}{i^{\lfloor p \rfloor}}\mathbf{1}_{\{p\in\mathbb{N}\}} + 2\frac{\Gamma(p-m+1)}{\pi}\, \mathfrak{Re}\, i^{-p-1}\, \mathfrak{Re} \int_0^\infty \frac{\frac{d^m}{dt^m}\left(f(t) - \sum_{j=0}^{\lceil p-1 \rceil} f^{(j)}(0)\frac{t^j}{j!}\right)}{t^{p-m+1}}\mathrm{d}t.$$

We see in the following theorem that the information of negative part moments or equivalently that of negative tail probability is needed for representations of positive part moments with its ch.f.

Theorem 1.3 *Take any $p \in (0,\infty) \setminus \mathbb{N}$ with $k := \lfloor p \rfloor$ and any r.v. X with $\mathsf{E}X_+^p < \infty$. Then the following three conditions are equivalent to each other:*

(I) $\mathsf{E}|X|^k < \infty$ *and* $\mathsf{E}X_+^p = \frac{\Gamma(p+1)}{\pi}\int_{0+}^{\infty} \Re\mathfrak{e}\, \frac{\mathsf{E}e_k(itX)}{(it)^{p+1}}\,\mathrm{d}t;$

(I') $f^{(k)}(0)$ *exists in* $\mathbb{R}$ *and* $\mathsf{E}X_+^p = \frac{\Gamma(p+1)}{\pi}\int_{0+}^{\infty} \Re\mathfrak{e}\, \frac{(Rf)_k(t)}{(it)^{p+1}}\,\mathrm{d}t,$ *where*

$$(Rf)_k(t) := f(t) - \sum_{j=0}^{k} f^{(j)}(0)\frac{t^j}{j!}.$$

(II) $\mathsf{P}(X_- > x) = o(1/x^p)$ *as* $x \to \infty$.

2 Truncated fractional moments

We present truncated moments of any intervals via the corresponding ch.f. The Fourier transform of pth moments $\mathsf{E}X_\pm^p e^{iuX}$ are calculated from the original ch.f. Then by applying the generalized inversion formula to these ch.f. we obtain truncated moments $\mathsf{E}X_\pm^p \mathbf{1}_{\{X \geq x\}}$ for $x \in \mathbb{R}$.
Notations. Define a transform which generalizes the inversion formula;

$$\mathfrak{G}(f)(x) := \frac{i}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty} e^{-itx} f(t)\frac{\mathrm{d}t}{t},$$

where p.v. denotes "principal value", so that p.v. $\int_{-\infty}^{\infty} := \lim_{\{\varepsilon\downarrow 0, A\uparrow\infty\}}\left(\int_{-A}^{-\varepsilon} + \int_{\varepsilon}^{A}\right)$.

Main Results. Again we only give results and omit proofs which are given in the former version of [3] (available from the author).

Theorem 2.1 *Let X be r.v. with dist. $F(x)$ and ch.f. f. Define a measure $\mu_p(\mathrm{d}x) = x^p \mathrm{d}F(x)$ where $x^p = x_+^p + (-1)^p x_-^p$ for $p \in (0,\infty)\setminus\mathbb{N}$ and denote the positive and negative part measures respectively by $\mu_{p,+}(\mathrm{d}x) = x_+^p\mathrm{d}F(x)$ and $\mu_{p,-}(\mathrm{d}x) = x_-^p\mathrm{d}F(x)$. Further we define $|\mu_p|(\mathrm{d}x) = |x|^p\mathrm{d}F(x)$. Assume $\mathsf{E}|X|^p < \infty$.*
(1) *Fourier transforms of $\mu_{p,+}(\mathrm{d}x)$ and $\mu_{p,-}(\mathrm{d}x)$ respectively have the following expressions. For $k := \lfloor p \rfloor$ and $\lambda := p - k$,*

$$g_{\mu_{p,+}}(u) := \mathsf{E}[X_+^p e^{iuX}] = \frac{1}{2i^{k+1}\sin\pi\lambda}\left\{i^{\lambda}(\mathrm{D}^p f)(u) - (-1)^k i^{-\lambda}(\mathrm{D}^p \bar{f})(-u)\right\}$$

and

$$g_{\mu_{p,-}}(u) := \mathsf{E}[(-X)_+^p e^{iuX}] = \frac{1}{2i^{k+1}\sin\pi\lambda}\left\{i^{\lambda}(\mathrm{D}^p \bar{f})(-u) - (-1)^k i^{-\lambda}(\mathrm{D}^p f)(u)\right\}.$$

Accordingly, $g_{\mu_p}(u) := \mathsf{E}[X^p e^{iuX}] = g_{\mu_p,+}(u) + (-1)^p g_{\mu_p,-}(u)$ *and* $g_{|\mu_p|}(u) := \mathsf{E}[|X|^p e^{iuX}] = g_{\mu_p,+}(u) + g_{\mu_p,-}(u)$.
(2) *Moreover, for* $x \in \mathbb{R}$

$$\mathsf{E}[X^p \mathbf{1}_{\{X \geq x\}}] = \frac{1}{2}\mathsf{E}X^p - \mathfrak{G}(g_{\mu_p})(x) + \frac{1}{2}\mu_p(\{x\}),$$
$$\mathsf{E}[|X|^p \mathbf{1}_{\{X \geq x\}}] = \frac{1}{2}\mathsf{E}|X|^p - \mathfrak{G}(g_{|\mu_p|})(x) + \frac{1}{2}|\mu_p|(\{x\}).$$

If we evaluate the ch.f. $g_{\mu_p,\pm}(u)$ at the origin in Theorem 2.1, we obtain other expressions for both positive and negative part of fractional moments.

Corollary 2.1 *Under the same conditions of Theorem 2.1, we have*

$$\mathsf{E}X_+^p = g_{\mu_p,+}(0) = \frac{1}{2i^{k+1}\sin\pi\lambda}\Big\{i^\lambda(\mathrm{D}^p f)(0) - (-1)^k i^{-\lambda}(\mathrm{D}^p \bar{f})(0)\Big\},$$
$$\mathsf{E}X_-^p = g_{\mu_p,-}(0) = \frac{1}{2i^{k+1}\sin\pi\lambda}\Big\{i^\lambda(\mathrm{D}^p \bar{f})(0) - (-1)^k i^{-\lambda}(\mathrm{D}^p f)(0)\Big\}.$$

For $p \in \mathbb{N}$ the counterpart of Theorem 2.1 is as follows, where we do not need the fractional derivative operator D^p.

Theorem 2.2 *Let* $p \in \mathbb{N}$. *Define a measure* $\mu_p(\mathrm{d}x) := x^p \mathrm{d}F(x)$ *where* $x_+^p + (-1)^p x_-^p$ *and define measures* $\mu_{p,+}(\mathrm{d}x) := x_+^p \mathrm{d}F(x)$ *and* $\mu_{p,-}(\mathrm{d}x) := x_-^p \mathrm{d}F(x)$. *Further we define* $|\mu_p|(\mathrm{d}x) = |x|^p \mathrm{d}F(x)$. *Assume* $\mathsf{E}|X|^p < \infty$. *Then Fourier transforms of* $\mu_{p,\pm}(\mathrm{d}x)$ *respectively have*

$$g_{\mu_p,+}(u) = \mathsf{E}[X_+^p e^{iuX}] = \frac{f^{(p)}(u)}{2i^p} + \frac{1}{2\pi}\int_0^\infty \frac{f^{(p)}(t+u) - f^{(p)}(u-t)}{i^{p+1}t}\mathrm{d}t,$$
$$g_{\mu_p,-}(u) = \mathsf{E}[X_-^p e^{iuX}] = \frac{\bar{f}^{(p)}(-u)}{2i^p} + \frac{1}{2\pi}\int_0^\infty \frac{\bar{f}^{(p)}(t-u) - \bar{f}^{(p)}(-u-t)}{i^{p+1}t}\mathrm{d}t.$$

Accordingly, $g_{\mu_p}(u) := g_{\mu_p,+}(u) + (-1)^p g_{\mu_p,-}(u) = f^{(p)}(u)/i^p$ *and* $g_{|\mu_p|}(u) := g_{\mu_p,+}(u) + g_{\mu_p,-}(u)$. *Moreover, for* $x \in \mathbb{R}$

$$\mathsf{E}[X^p \mathbf{1}_{\{X \geq x\}}] = \frac{1}{2}\mathsf{E}X^p - \mathfrak{G}(g_{\mu_p})(x) + \frac{1}{2}\mu_p(\{x\}),$$
$$\mathsf{E}[|X|^p \mathbf{1}_{\{X \geq x\}}] = \frac{1}{2}\mathsf{E}|X|^p - \mathfrak{G}(g_{|\mu_p|})(x) + \frac{1}{2}|\mu_p|(\{x\}).$$

Similarly as before, we evaluate the ch.f. $g_{\mu_p,\pm}(u)$ at the origin in Theorem 2.2, we obtain other expressions for both positive and negative part of fractional moments.

Corollary 2.2 *Under the same assumptions of Theorem 2.2,*

$$\mathsf{E}X_+^p = \frac{f^{(p)}(0)}{2i^p} + \frac{1}{\pi}\int_0^\infty \frac{f^{(p)}(t) - f^{(p)}(-t)}{2i^{p+1}t}\mathrm{d}t,$$
$$\mathsf{E}X_-^p = \frac{\bar{f}^{(p)}(0)}{2i^p} + \frac{1}{\pi}\int_0^\infty \frac{\bar{f}^{(p)}(t) - \bar{f}^{(p)}(-t)}{2i^{p+1}t}\mathrm{d}t.$$

Notice that the inversion for ch.f. of integer valued moments in Theorem 2.2 is particularly useful since the expression is simple and does not require a double integral even for the general case. For convenience we further formulate this.

Corollary 2.3 *Let X be r.v. with dist. F. Under the same assumptions of Theorem 2.2,*

$$\mathsf{E}[X^p \mathbf{1}_{\{X\geq x\}}] = \frac{f^{(p)}(0)}{2i^p} + \frac{\mu_p(\{x\})}{2} - \frac{i}{2\pi}\int_{-\infty}^\infty \frac{e^{-itx} f^p(t)}{i^p t}\mathrm{d}t,$$
$$\mathsf{E}[X^p \mathbf{1}_{\{X<x\}}] = \frac{f^{(p)}(0)}{2i^p} - \frac{\mu_p(\{x\})}{2} + \frac{i}{2\pi}\int_{-\infty}^\infty \frac{e^{-itx} f^p(t)}{i^p t}\mathrm{d}t.$$

3 Fractional moments of whole real line

We extend the fractional derivative operator D^p for all $p \in \mathbb{R}$ by introducing a fractional integral of the Riemann-Liouville type as in [5], which gives both positive and negative parts moments of negative order and their truncation, additionally. Although only expressions of $\mathsf{E}|X|^p$ for $p \in \mathbb{R}$ have been investigated in [5], we find that the tool used there is applicable to our present concern.

Let D^p for $p \in (-\infty, 0] \cup \mathbb{N}$ denote the operations,
(1) for $p \in \mathbb{N}$, $\mathrm{D}^n f = f^{(n)}$ and $\mathrm{D}^0 f = f$.
(2) for $p \in (-1, 0)$

$$\mathrm{D}^p f(t) := \frac{1}{\Gamma(-p)}\int_{-\infty}^t \frac{f(u)}{(t-u)^{1+p}}\mathrm{d}u.$$

(3) for $p \in \mathbb{N}_-$, $\mathrm{D}^p f(t) := \lim_{q\downarrow p} \mathrm{D}^q f(t)$.
(4) for $p \in (-\infty, 0) \setminus \mathbb{N}_-$ with $\ell = \lceil p \rceil$,

$$\mathrm{D}^p f(t) := \mathrm{D}^{p-\ell}(\mathrm{D}^\ell f)(t).$$

Definitions of D^p for $p < 0$ bellow are called the Riemann-Liouville integral with the end point $(-\infty)$. Notice that for $p \in (0, \infty) \setminus \mathbb{N}$, the definition of D^p in [5] is

$$\mathrm{D}^p f(t) := \frac{d^k}{dt^k}\mathrm{D}^{p-k} f(t), \quad k = \lfloor p \rfloor,$$

which is different from ours in the order of the derivative and the integration. However, under $\mathsf{E}|X|^p < \infty$, two operations are commutable.

Our main tool here is the following identity by the complex integration, for $p \in \mathbb{R}$

$$\mathrm{D}^p f(t) = i^p \int_{-\infty}^{\infty} \overline{x^p} e^{itx} \mathrm{d}F(x),$$

which was obtained in [5].

Now fractional moments and their truncations for the $p \in \mathbb{R}$ are given.

Proposition 3.1 *Define $\mu_{p,+}$ and $\mu_{p,-}$ as those in Theorem* (2.1) *for any $p \in \mathbb{R}$. Assume $\mathsf{E}|X|^p < \infty$. Then for $u \in \mathbb{R}$,*

$$g_{\mu_{p,+}}(u) = \mathsf{E}[X_+^p e^{iXu}] = \frac{\mathrm{D}^p f(u)}{2i^p} + \frac{1}{2\pi i^{p+1}} \int_0^{\infty} \frac{\mathrm{D}^p f(u+t) - \mathrm{D}^p f(u-t)}{it} \mathrm{d}t,$$

$$g_{\mu_{p,-}}(u) = \mathsf{E}[X_-^p e^{iXu}] = \frac{\mathrm{D}^p \bar{f}(-u)}{2i^p} + \frac{1}{2\pi i^p} \int_0^{\infty} \frac{\mathrm{D}^p \bar{f}(-u+t) - \mathrm{D}^p \bar{f}(-u-t)}{it} \mathrm{d}t.$$

Accordingly,

$$EX_+^p = \frac{\mathrm{D}^p f(0)}{2i^p} + \frac{1}{\pi i^p} \int_0^{\infty} \frac{\mathrm{D}^p f(t) - \mathrm{D}^p f(-t)}{it} \mathrm{d}t,$$

$$EX_-^p = \frac{\mathrm{D}^p f(0)}{2i^p} + \frac{1}{2\pi i^p} \int_0^{\infty} \frac{\mathrm{D}^p \bar{f}(t) - \mathrm{D}^p \bar{f}(-t)}{it} \mathrm{d}t.$$

Proposition 3.2 *Let $p \in \mathbb{R}$ and assume $\mathsf{E}|X|^p < \infty$, then for $y < 0 < x$,*

$$\mathsf{E}X_+^p \mathbf{1}_{\{X \geq x\}} = \frac{(\mathrm{D}^p f)(0)}{2i^p} + \frac{x^p}{2} + \frac{1}{2\pi i^{p+1}} \int_0^{\infty} \frac{\mathfrak{Re}\, e^{ixu} (\mathrm{D}^p \bar{f})(u)}{u} \mathrm{d}u,$$

$$\mathsf{E}X_-^p \mathbf{1}_{\{X \geq y\}} = \frac{(\mathrm{D}^p f)(0)}{2i^p} + \frac{(-y)^p}{2} - \frac{1}{2\pi i^{p+1}} \int_0^{\infty} \frac{\mathfrak{Re}\, e^{-iyu} (\mathrm{D}^p f)(u)}{u} \mathrm{d}u.$$

Again we only give results and omit proofs which are given in the former version of [3] (available from the author).

Note that after the probability symposium on Dec.17~20, 2013 in Kyoto, we obtain new results which unifies and generalizes obtained results.

Therefore, our stated results of this report are included in the new one which will appear in [3]. Related examples for mostly heavy tailed cases have been intensively studied in [2].

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