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On instability of global path properties of symmetric Markov processes under Mosco convergence

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Abstract

In this report, we consider the following:

(1) Sufficient conditions for the Mosco convergence of symmetric Markov processes in two cases: **(a)** Lévy processes **(b)** uniformly elliptic diffusions;

(2) Instability of global path properties under the Mosco convergence such as recurrence/transience and conservativeness/explosion.

In the study of **(1)**, we obtain that the Mosco convergence follows from L^1**-local convergence** of the corresponding coefficients (e.g., Lévy exponents in the case **(a)** and diffusion coefficients in the case **(b)**). This result means that the Mosco convergence follows from a very weak convergence of the corresponding coefficients. In the study of **(2)**, we give several examples whose global path properties are not preserved under the Mosco convergence.

1 Introduction

In [M67], Umberto Mosco introduced the notion of a convergence of bilinear forms, now called *Mosco convergence.* For a closed form $(\mathcal{E},\mathcal{F})$ on a Hilbert space $\mathcal{H}$ (not necessarily densely defined), we let $\mathcal{E}(u,u)=\infty$ for every $u\in \mathcal{H}\setminus \mathcal{F}$. Then the Mosco convergence is defined as follows:

Definition 1 A sequence of closed forms $\mathcal{E}^{n}$ on a Hilbert space $\mathcal{H}$ is said to be convergent to $\mathcal{E}$ in the sense of Mosco if the following two conditions are satisfied:

(M.1) for every u and every sequence $\{u_{n}\}$ converging to u *weakly* in $\mathcal{H}$,

$$\liminf_{n\to\infty}\mathcal{E}^{n}(u_{n}, u_{n})\geq \mathcal{E}(u, u);$$

(M.2) for every u there exists a sequence $\{u_{n}\}$ converging to u in $\mathcal{H}$ so that

$$\limsup_{n\to\infty}\mathcal{E}^{n}(u_{n}, u_{n})\leq \mathcal{E}(u, u).$$

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In [M94], Mosco showed that a sequence of closed forms $\mathcal{E}^n$ on $\mathcal{H}$ is converging to $\mathcal{E}$ in the sense of Mosco if and only if the resolvents associated with $\mathcal{E}^n$ converges to the one associated with $\mathcal{E}$ *strongly* on $\mathcal{H}$, and hence the semigroups associated with $\mathcal{E}^n$ converges to the one associated with $\mathcal{E}$ *strongly* on $\mathcal{H}$ too. The strong convergence of semigroups gives us the convergence of finite-dimensional distributions of the corresponding Markov processes. This is one reason why the Mosco convergence is used in the study of stochastic processes. In fact, in Kuwae-Uemura[KU97a, KU97b], [Su98], Kim[Km06], they used the Mosco convergence to show the weak convergences of Markov processes corresponding to their respective Dirichlet forms (see also [Kol06]). In [H98], Hino also introduced a non-symmetric version of Mosco convergence and Kuwae and Shioya generalized the convergence of the case where the basic L^2-space changes in [KS03].

In this report, however, we only consider symmetric Dirichlet forms and our state spaces or L^2-spaces do not move. In this setting, we study

(1) Sufficient conditions for the Mosco convergence of symmetric Markov processes in two cases: **(a)** uniformly elliptic diffusions, **(b)** Lévy processes;

(2) Instability of global path properties under the Mosco convergence such as recurrence or transience, and conservativeness or explosion.

In the study of **(1)**, we obtain that the Mosco convergence follows from **L^1-local convergence**, which is quite weak one, of the corresponding coefficients (e.g. diffusion coefficients in the case **(a)** and Lévy exponents in the case **(b)**). We should note that **L^1-local convergence** is one of the weakest convergence in our settings and our results mean that the Mosco convergence follows from the very weak convergence of the corresponding coefficients.

In general, the global path properties such as recurrence/transience and conservativeness/explosion are not preserved under the Mosco convergence. It seems, however, that few people have studied how to construct such examples concretely. In the study of **(2)**, we give several examples whose global path properties are not preserved under the Mosco convergence.

2 Settings and results

We first consider the diffusion case of **(1)**. Let $A_n(x) = (a^n_{ij}(x))$ and $A(x) = (a_{ij}(x))$ be $d \times d$ symmetric matrix valued functions on $\mathbb{R}^d$ satisfying the following conditions:

Assumption (A)

(A1) For any compact set $K \subset \mathbb{R}^d$, there exists a constant $\lambda = \lambda(K) > 0$ so that, for any $n \geq 1$ and $\xi \in \mathbb{R}^d$,

$$0 < \frac{1}{\lambda}|\xi|^2 \leq (A_n(x)\xi, \xi) \leq \lambda|\xi|^2, \quad dx\text{-}a.e. \text{ on } K;$$

(A2) (**L^1-local convergence**) For any compact set K,

$$\int_K \|A_n(x) - A(x)\| dx \to 0 \quad (n \to \infty),$$

where $\|A(x)\| = \sqrt{\sum_{i=1}^d \sum_{j=1}^d a^2_{ij}(x)}$.

Consider quadratic forms on $L^2(\mathbb{R}^d)$ as follows:

$$\mathcal{E}^n(u,u)=\frac{1}{2}\int_{\mathbb{R}^d}(A_n(x)\nabla u,\nabla u)dx,\quad \mathcal{E}(u,u)=\frac{1}{2}\int_{\mathbb{R}^d}(A(x)\nabla u,\nabla u)dx.$$

for $n\in\mathbb{N}$ and appropriate functions u. Under the assumption **(A)**, it is then known that $(\mathcal{E}^n, C_0^\infty(\mathbb{R}^d))$ and $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ are Markovian closable forms on $L^2(\mathbb{R}^d)$. So they become regular symmetric Dirichlet forms $(\mathcal{E}^n,\mathcal{F}^n)$ and $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{R}^d)$. We then obtain:

Theorem 1 *Assume that* **(A)** *holds. Then the Dirichlet forms* $(\mathcal{E}^n,\mathcal{F}^n)$ *converges to the Dirichlet form corresponding to* $(\mathcal{E},\mathcal{F})$ *in the sense of Mosco.*

Remark 1 In [M94], Mosco investigated similar sufficient conditions for the Mosco convergence, but it seems to be difficult to check *the compactly injected condition* he imposed there.

We now consider the Lévy case of **(1)**. Let $\{\varphi_n\}$ be a sequence of the characteristic functions defined by symmetric convolution semigroups $\{\nu_t^n, t>0\}_{n\in\mathbb{N}}$:

$$e^{-t\varphi_n(x)}:=\widehat{\nu_t^n}(x)\Big(=\int_{\mathbb{R}^d}e^{i\langle x,y\rangle}\nu_t^n(dy)\Big),\quad x\in\mathbb{R}^d.$$

Let φ be also a characteristic function defined by a symmetric convolution semigroup $\{\nu_t, t>0\}$. The Dirichlet forms corresponding ν_t^n are defined by

$$\begin{cases}\mathcal{E}^n(u,v) &= \displaystyle\int_{\mathbb{R}^d}\widehat{u}(x)\overline{\widehat{v}}(x)\varphi_n(x)dx,\\ \mathcal{D}[\mathcal{E}^n] &= \Big\{u\in L^2(\mathbb{R}^d): \displaystyle\int_{\mathbb{R}^d}\left|\widehat{u}(x)\right|^2\varphi_n(x)dx<\infty\Big\}.\end{cases}$$

We assume that for each n, $\mathcal{E}^n(u,u)=\infty$ if $u\in L^2(\mathbb{R}^d)\setminus\mathcal{D}[\mathcal{E}^n]$. We assume the following condition on $\{\varphi_n\}$:

(B) φ_n converges to a function φ **locally in** $L^1(\mathbb{R}^d)$**.**

Then we show the following theorem:

Theorem 2 *Assume that* **(B)** *holds. Then the Dirichlet forms* $(\mathcal{E}^n,\mathcal{D}[\mathcal{E}^n])$ *converges to the Dirichlet form corresponding to* φ *in the sense of Mosco.*

In the rest of the report, we consider **(2)**. We first consider the instability of recurrence/transience in Lévy cases. Let α and α_n be continuous functions on $[0,\infty)$ satisfying that there exist positive constants $\underline{\alpha}$ and $\overline{\alpha}$ so that

$$0<\underline{\alpha}\le\alpha_n(t)\le\overline{\alpha}<2,\quad t\in[0,\infty)$$

and define Lévy measures on $\mathbb{R}^d$ as follows:

$$n_n(dx)=|x|^{-1-\alpha_n(|x|)}dx,\quad n(dx)=|x|^{-1-\alpha(|x|)}dx.$$

Then the corresponding characterisitc (Lévy) exponents and Dirichlet forms are given by

$$\varphi_n(x) = \int_{\mathbb{R}^d} \Big(1 - \cos(x\xi)\Big) n_n(d\xi), \quad \varphi(x) = \int_{\mathbb{R}^d} \Big(1 - \cos(x\xi)\Big) n(d\xi),$$

and

$$\mathcal{E}^n(u,u) = \int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \varphi_n(\xi) d\xi \ = \ \iint_{x \neq y} \big(u(x+h) - u(x)\big)^2 n(dh) dx,$$
$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \varphi(\xi) d\xi \ = \ \iint_{x \neq y} \big(u(x+h) - u(x)\big)^2 n(dh) dx,$$

respectively.

Now we give examples whose transience/recurrence are not preserved under the Mosco convergence.

Proposition 1 *Assume $d = 1$. Then the following convergence results hold.*

(i) **(recurrent ones $\Longrightarrow$ transient one)** *If we set*

$$\alpha_n(u) = 1 + 1/n - \Big(\log(u + e^2)\Big)^{-1/2}, \quad \alpha(u) = 1 - \Big(\log(u + e^2)\Big)^{-1/2}$$

for $u \geq 0$ and $n > 1$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is recurrent for any n and converges to the transient Dirichlet form $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco.

(ii) **(transient ones $\Longrightarrow$ recurrent one)** *If we set*

$$\alpha_n(u) = 1 - \Big(\log\big(u + e^2\big)\Big)^{-(1-1/n)}, \quad \alpha(u) = 1 - \Big(\log\big(u + e^2\big)\Big)^{-1}$$

for $u \geq 0$ and $n > 1$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is transient for any n and converges to the recurrent Dirichlet form $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco.

We consider the instability of conservetiveness/explosion in the diffusive cases. Let $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}, \mathcal{F})$ be the following Dirichlet forms on $L^2(\mathbb{R}^d; dx)$:

$$\begin{cases} \mathcal{E}^n(u,v) &= \displaystyle\int_{\mathbb{R}^d} \alpha_n(x) \langle \nabla u, \nabla v \rangle dx, \\ \mathcal{F}^n &= \overline{C_0^\infty(\mathbb{R}^d)}^{\mathcal{E}_1^n}, \end{cases}$$
$$\begin{cases} \mathcal{E}(u,v) &= \displaystyle\int_{\mathbb{R}^d} \alpha(x) \langle \nabla u, \nabla v \rangle dx, \\ \mathcal{F} &= \overline{C_0^\infty(\mathbb{R}^d)}^{\mathcal{E}_1}. \end{cases}$$

Proposition 2 *Then the following convergence results hold.*

(i) **(explosive ones $\Longrightarrow$ conservative one)** *If we set*

$$\alpha_n(x) = (2 + |x|)^2 \Big(\log(2 + |x|)\Big)^{1+1/n}, \quad \alpha(x) = (2 + |x|)^2 \Big(\log(2 + |x|)\Big)$$

for $n \in \mathbb{N}$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is explosive for any n and converges to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ which is conservative in the sense of Mosco.

(ii) (**conservative ones $\Longrightarrow$ explosive one**) *If we set*

$$\alpha_n(x) = (2+|x|)^{2-1/n}\Big(\log(2+|x|)\Big)^2, \quad \alpha(x) = (2+|x|)^2\Big(\log(2+|x|)\Big)^2$$

for $n > 1$, then $(\mathcal{E}^n, \mathcal{F}^n)$ is conservative for any n and converges to the explosive Dirichlet form $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco.

References

[FOT11] FUKUSHIMA, M., OSHIMA, Y. and M. TAKEDA, *Dirichlet forms and symmetric Markov processes*, 2nd revised and extended edition, de Gruyter, 2011

[H98] HINO, M., Convergence of non-symmetric forms, *Journal of Math. Kyoro Univ.*, **38** (1998), 329-341

[Km06] KIM, P., Weak convergence of censored and reflected stable processes, *Stoch. Proc. Appl.*, **116** (2006), 1792-1814

[Kol06] A.V. KOLESNIKOV, Mosco convergence of Dirichlet forms in infinite dimensions with changing reference measures, *J. Func. Anal.*, **230** (2006), 382-418

[KS03] KUWAE, K. and T. SHIOYA, Convergence of Spectral Stuctures, *Commun. Anal. Geom.*, **11** (2003), 599-673

[KU97a] KUWAE, K. and T. UEMURA, Weak convergence of symmetric diffusion processes II, *in Proc. of the seventh Japan-Russia Symp. Probab. Th. Math. Stat.*, 266-275, World Scientific, Singapore, 1996

[KU97b] KUWAE, K. and T. UEMURA, Weak convergence of symmetric diffusion processes, *Probability Theory and Related Fields*, **109** (1997), 159-182

[M67] MOSCO, U., Approximation of the solutions of some variational inequalities, *Ann. Scuola Normale Sup., Pisa*, **21** (1967), 373-394

[M94] MOSCO, U., Composite media and asymptotic Dirichlet forms, *Journal of Functional Analysis*, **123** (1994), 368-421

[S99] SATO, K., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999

[Su98] SUN, W., Weak convergence of Dirichlet forms, *Sci. China, A*, **41** (1998), 8-21

[U04] UEMURA, T., On symmetric stable-like processes: some path properties and generators, *Journal of Theoretical Probability*, **17** (2004), 541-555.