数理解析研究所講究録
第 1903 巻 2014 年 205-208

Large deviation principles from hydrodynamic limits for asymptotically degenerate systems

Kenkichi Tsunoda[1]

The University of Tokyo

Abstract

In [1], Gonçalves et al. considered a certain class of particle systems which derive the porous medium equation as a consequence of a hydrodynamic limit:

$$\begin{cases} \partial_t \rho = \Delta d(\rho), \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases}$$

where $d(\alpha) = \alpha^2$. Since the porous medium equation is degenerate in the sense that

$$D(\alpha) := d'(\alpha) = 2\alpha \to 0, (\alpha \to 0),$$

the equation looses its parabolic character. From this fact, the analysis of the macroscopic equation and the microscopic particle system becomes difficult. In this paper, we report on the large deviation principle from the hydrodynamic limit derived in [1].

1 Particle system

Let us formulate our dynamics precisely. Let $\mathbb{T}_N^d$ be the d-dimensional discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$. Denote by χ_N^d a configuration space $\{0,1\}^{\mathbb{T}_N^d}$. A generic element of χ_N^d will be denoted by the Greek letter η. The dynamics is described by the generator $L_N = L_P + N^{-\theta} L_S$ and

$$(L_P f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{|e|=1} (\eta(x-e) + \eta(x+2e))\eta(x)(1-\eta(x+e))(f(\eta^{x,x+e}) - f(\eta)),$$

$$(L_S f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{|e|=1} \eta(x)(1-\eta(x+e))(f(\eta^{x,x+e}) - f(\eta)),$$

where $0 < \theta < 2$, $|x| = \sum_{1\le i\le d} |x_i|$ is the sum norm in $\mathbb{R}^d$, f is a local function and $\eta^{x,y}$ is defined as

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{if } z \ne x, y, \end{cases}$$

[1]e-mail: tsunoda@ms.u-tokyo.ac.jp

for $x, y \in \mathbb{T}_N^d$. Notice that $L_N f$ can be rewritten as

$$(L_N f)(\eta) = \sum_{x \in \mathbb{T}_N^d} \sum_{|e|=1} c_N(x, x+e, \eta)(f(\eta^{x,x+e}) - f(\eta)),$$

where we set $c_N(x, x+e, \eta) = (\eta(x-e) + \eta(x+2e) + N^{-\theta})\eta(x)(1-\eta(x+e))$.

We now recall the result of [1]. To see this, we need to set some notations. Define the empirical measure by

$$\pi^N(du) = \pi^N(\eta, du) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\delta_{\frac{x}{N}}(du),$$

where δ_u denotes the Dirac measure at u. Let $\mathbb{T}^d$ be the d-dimensional torus. Denote by $\mathbb{P}^N$ the probability measure on the space $D([0,T], \chi_N^d)$ induced by the Markov process with the generator $N^2 L_N$ and an initial measure μ^N. Let Q^N be the distribution of the empirical measure process π^N under the probability $\mathbb{P}^N$.

Gonçalves et al. showed the hydrodynamic limit for the empirical measure process in [1].

Theorem 1.1. *Let $\rho_0 : \mathbb{T}^d \to [0,1]$ and $(\mu^N)_N$ be a sequence of probability measures on χ_N^d associated to the profile ρ_0. Then, the sequence of probabilities Q^N converges in distribution to the probability measure concentrated on the absolutely continuous path $\pi_t(du) = \rho(t,u)du$ whose density $\rho(t,u)$ is the unique weak solution of the Cauchy problem*

$$\begin{cases} \partial_t \rho = \Delta d(\rho), \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases}$$

2 Main result

Let us introduce our main result. To mention about it, we first describe the rate function of the large deviations.

Let $\mathcal{M}_+$ be the set of all nonnegative measures on $\mathbb{T}^d$. For each continuous function $\gamma : \mathbb{T}^d \to (0,1)$, define the functions $h_\gamma : \mathcal{M}_+ \to \mathbb{R}$ and $I_{ini} : \mathcal{M}_+ \to \bar{\mathbb{R}}_+$ by

$$\begin{aligned} h_\gamma(\omega) &= \langle \omega, \log \frac{\gamma(1-\rho_0)}{(1-\gamma)\rho_0} \rangle + \langle \lambda, \log \frac{1-\gamma}{1-\rho_0} \rangle, \\ I_{ini} &= \sup_\gamma h_\gamma(\omega), \end{aligned}$$

where λ stands the Lebesgue measure on $\mathbb{T}^d$. It is easy to see that I_{ini} is convex and lower semicontinuous.

Denote by $\mathcal{M}_+^\circ$ the closed subset of $\mathcal{M}_+$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_+^\circ = \{\omega \in \mathcal{M}_+ \mid \omega(du) = \theta(u)du, \ 0 \leq \theta(u) \leq 1 \ a.e.\}.$$

For each $\pi \in D([0,T], \mathcal{M}_+^0)$, we define the energy $\mathcal{Q}(\pi) = \sum_{i=1}^d \mathcal{Q}_i(\pi)$ and $\mathcal{Q}_i(\pi)$ by

$$\mathcal{Q}_i(\pi) = \sup_G \{2 \int_0^T dt \int_{\mathbb{T}^d} du d(\rho(t,u)) \partial_i G(t,u) - \int_0^T dt \int_{\mathbb{T}^d} du \sigma(\rho(t,u)) G^2(t,u)\},$$

where the supremum is over all functions in $C^{0,1}([0,T] \times \mathbb{T}^d)$ and the functions d and σ are defined by $d(\alpha) = \alpha^2$, $\sigma(\alpha) = \chi(\alpha) D(\alpha)$, $\chi(\alpha) = \alpha(1-\alpha)$ and $D(\alpha) = d'(\alpha) = 2\alpha$. For each smooth function G in $C^{1,2}([0,T] \times \mathbb{T}^d)$, define the functional $\bar{J}_G : D([0,T], \mathcal{M}_+^0) \to \mathbb{R}$ by

$$\begin{aligned}\bar{J}_G(\pi) = \langle \pi_T, G_T \rangle - \langle \pi_0, G_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t G_t \rangle - \int_0^T dt \int_{\mathbb{T}^d} du d(\rho(t,u)) \Delta G(t,u) \\ - \int_0^T dt \int_{\mathbb{T}^d} du \sigma(\rho(t,u)) \|\nabla G(t,u)\|^2,\end{aligned}$$

where $\|x\|^2 = \sum_{1 \le i \le d} x_i^2$ is the Euclidean norm in $\mathbb{R}^d$ and ∇G stands for the gradient of G: $\nabla G = (\partial_1 G, \cdots, \partial_d G)$. Let $J_G : D([0,T], \mathcal{M}_+) \to \bar{\mathbb{R}}$ be the functional defined by

$$J_G(\pi) = \begin{cases} \bar{J}_G(\pi) & \text{if } \pi \in D([0,T], \mathcal{M}_+^0), \\ \infty & \text{otherwise,} \end{cases}$$

and $I_{dyn} : D([0,T], \mathcal{M}_+) \to \bar{\mathbb{R}}$ be the functional defined by

$$I_{dyn}(\pi) = \begin{cases} \bar{J}_G(\pi) & \text{if } \mathcal{Q}(\pi) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Finally, we define the functional $I : D([0,T], \mathcal{M}_+) \to \bar{\mathbb{R}}$ by $I = I_{ini} + I_{dyn}$. We can prove that the rate function I is lower semicontinuous.

To state our main result, assume that the initial distributions $(\mu^N)_N$ are given by the product measure $\nu_{\rho_0(\cdot)}^N$ and the initial profile $\rho_0 : \mathbb{T}^d \to [0,1]$ is continuous.

Theorem 2.1. *(i) For each closed subset $\mathcal{K}$ of $D([0,T], \mathcal{M}_+)$,*

$$\limsup_{N \to \infty} N^{-d} \log Q^N[\mathcal{K}] \le - \inf_{\pi \in \mathcal{K}} I(\pi).$$

(ii) Assume that there exists a positive constant ε_0 such that $\varepsilon_0 \le \rho_0(u) \le 1 - \varepsilon_0$. Then, for each open subset $\mathcal{O}$ of $D([0,T], \mathcal{M}_+)$,

$$\liminf_{N \to \infty} N^{-d} \log Q^N[\mathcal{O}] \ge - \inf_{\pi \in \mathcal{O}} I(\pi).$$

The key elements for the proof of the above theorem are the following:

1. The super exponential estimate to assure the local ergodicity.

2. Energy estimates at the macroscopic and microscopic view.

3. The approximation lemma to show the full lower large deviation principle.

These steps are fundamental ingredients to prove the large deviation principle. It becomes entirely non-trivial because our jump rates are asymptotically degenerate. After overcoming this difficulty, we obtain the large deviation principle.

References

[1] P. Gonçalves, C. Landim, C. toninelli, *Hydrodynamic limit for a particle system with degenerate rates*, Ann. Inst. H. Poincaré, Probab. Statis., **45** (2009), 887-909.