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Weak Solution of Renormalization Group Equation

Ken-Ichi Aoki, Shin-Ichiro Kumamoto and Daisuke Sato*
Institute for Theoretical Physics, Kanazawa University

Abstract

In the approach of non-perturbative renormalization group (NPRG), the spontaneous chiral symmetry breaking induces a singularity in its solution, e.g., the flow of the 4-fermi coupling constant blows up at a critical renormalization group (RG) scale. Thus, as long as directly solving the NPRG equation as a partial differential equation, the RG flow cannot go beyond the critical scale to obtain infrared quantities such as the chiral condensates. In order to treat this singularity in a mathematically rigorous way, we introduce the notion of weak solution of the NPRG equation. The weak solution is found to give a unique global solution toward the infrared limit, and we can calculate infrared quantities without any ambiguities.

1 Introduction

The almost 100 percent of mass is originated from spontaneous chiral symmetry breaking (SχSB), while a few percent of mass is given by the so-called Higgs mechanism. The SχSB is included by the strong interaction between quarks at the low energy scale which is described by quantum chromodynamics (QCD). Because of the strong interaction, the perturbation theory does not work, and so we need non-perturbative methods such as the lattice simulation and the Schwinger-Dyson (SD) approach.

In this article, we use the approach of non-perturbative renormalization group (NPRG) that is originated from the Wilsonian idea. This approach does not have the sign problem at finite chemical potential just like the lattice simulation, and can improve the gauge dependence of physical quantities, which the SD approach suffers, in systematic approximations [1, 2].

For simplicity, we are limited to the analysis using the Nambu-Jona-Lasinio (NJL) model with a simplified discrete chiral symmetry, which is a low energy model of QCD explaining the SχSB. Its Lagrangian is given by

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi - \frac{G_0}{2}(\bar{\psi}\psi)^2, \tag{1}$$

where ψ and $\bar{\psi}$ is a quark field and an antiquark field, respectively. Here the discrete chiral symmetry is that the Lagrangian is invariant under the following discrete chiral transformation:

$$\psi \to \gamma_5\psi,\ \bar{\psi} \to -\bar{\psi}\gamma_5. \tag{2}$$

*e-mail address: `satodai@hep.s.kanazawa-u.ac.jp`

The chiral symmetry thus forbids the mass term $m\bar{\psi}\psi$ and the chiral condensates $\langle\bar{\psi}\psi\rangle$. However, when the 4-fermi coupling constant G_0 is larger than a critical coupling constant, its strong coupling induces the SχSB. If using the mean field approximation, the critical coupling constant G_c is $4\pi^2/\Lambda_0^2$, where Λ_0 is the ultraviolet cutoff scale. Note that the mean field approximation is equivalent to the self-consistency equation limited to the large-N leading, where N is the number of quark flavors.

Many NPRG analyses of SχSB have been performed by introducing the bosonization of the multi-fermi interactions [3–6], which is the so-called auxiliary field method or Hubbard-Stratonovich transformation. However the analysis without the bosonization is difficult because in the RG procedure the 4-fermi coupling constant blows up at a critical scale as a signal of SχSB [7, 8]. Consequently we cannot go beyond the critical scale to obtain infrared physical quantities such as the chiral condensates.

The goal of this article is to analyze SχSB in the NPRG approach without introducing the bosonization. For this goal we adopt the method of weak solution [9], which has firstly been introduced in the NPRG approach by the authors [10]. As the NPRG equation is given by a partial differential equation (PDE), the weak solution satisfies the integral-form (weak) equation of the PDE. Since the weak solution is globally defined, it can include singularities such as the explosive behavior of the 4-fermi coupling constant.

This article is organized as follows. In Sect. 2, we briefly explain the Wegner-Houghton equation that is a formulation of the NPRG and the difficulty of the NPRG analysis of SχSB without the bosonization. In Sect 3, the method of weak solution is adopted to overcome this difficulty. In Sect 4, the bare mass of quark is introduced to define the chiral order parameters. In Sect 5, the method of weak solution is applied to the first order phase transition at finite chemical potential, and the convexity of the effective potential given by the weak solution is discussed. Finally we summarize this article in Sect 6.

2 Non-perturbative renormalization group

In the NPRG approach, a central object is the Wilsonian effective action $S_{\rm eff}[\phi;\Lambda]$ defined by integrating the microscopic degrees of freedom $\phi_{\rm H}$ with momentums higher than the scale Λ:

$$\int \mathcal{D}\phi_{\rm H} e^{-S_0[\phi_{\rm L},\phi_{\rm H};\Lambda_0]} = e^{-S_{\rm eff}[\phi_{\rm L};\Lambda]}, \tag{3}$$

where $S_0[\phi;\Lambda_0]$ is a bare action with the ultraviolet cutoff scale Λ_0. Now we parametrize the cutoff scale Λ by a dimensionless scale t such that

$$\Lambda(t) = \Lambda_0 e^{-t}. \tag{4}$$

The t-dependence of the effective action $S_{\rm eff}[\phi;\Lambda]$ is given by a NPRG equation as the following functional partial differential equation:

$$\partial_t S_{\rm eff}[\phi;t] = \beta_{\rm WH}\left[\frac{\delta S_{\rm eff}}{\delta\phi}, \frac{\delta^2 S_{\rm eff}}{\delta\phi^2}; t\right], \tag{5}$$

which is called the Wegner-Houghton (WH) equation [11] (see Ref. [12] for the detail form of $\beta_{\rm WH}$). The WH equation is the exact equation that provides macroscopic informations, such as the chiral condensates and the effective quark mass, by setting the bare action S_0 to the initial condition at $t=0$ and solving it as a differential equation toward the infrared scale ($t \to \infty$). Of course, it cannot be solved exactly, but various non-perturbative approximation to solve it are available.

In this article, the WH equation is applied to the NJL model (3). As an approximation, we restrict the full interaction space of the effective action $S_{\text{eff}}[\psi, \bar{\psi}; t]$ to be the subspace relevant to SχSB as follows:

$$S_{\text{eff}}[\psi, \bar{\psi}; t] = \int d^4x \left\{ \bar{\psi} \partial\!\!\!/ \psi - V_W(x;t) \right\}, \tag{6}$$

where a scalar fermion-bilinear field, $x = \bar{\psi}\psi$, is introduced. The potential term $V_W(x;t)$ is called the fermion potential here, whose initial condition is set to $V_W(x; t=0) = (G_0/2)x^2$ according to the NJL Lagrangian (1).

In addition to the restriction of the interaction space, the large-N non-leading parts of the WH equation (3) are ignored. Then, the NPRG equation for the fermion potential in the large-N approximation is given by the following partial differential equation:

$$\partial_t V_W(x;t) = \frac{\Lambda^4}{4\pi^2} \log\left(1 + \frac{1}{\Lambda^2}(\partial_x V_W)^2\right) \equiv -F(\partial_x V_W; t). \tag{7}$$

Here the momentum cutoff Λ have been performed with respect to the length of four Euclidean momentum p_μ: $\sum_{\mu=1}^{4} p_\mu^2 \leq \Lambda$. Note that the approximation used here is equivalent to the mean field one.

Now we introduce the mass function, $M(x;t) = \partial_x V_W(x;t)$, to interpret the S$\chi$SB in this framework. The value of the mass function at the origin is the coefficient of mass term $\bar{\psi}\psi$ in the effective action as its name suggests. The chiral symmetry is realized by the invariance of the fermion potential under the chiral transformation, $x \to -x$, given by Eq. (2): $V(-x;t) = V(x;t)$, and then $M(-x;t) = -M(x;t)$. The NPRG equation (7) with the chiral-invariant fermion potential is also invariant under the chiral transformation, and thus its solution with the chiral-invariant initial condition maintain its chiral-invariant structure at all scales. If the mass function is analytic, its value at the origin vanishes since the mass function with the chiral invariance is odd with respect to x.

While the NPRG equation does not spontaneously break the chiral-invariant structure of the fermion potential, its second derivative at the origin that is the 4-fermi coupling constant $G(t) \equiv \partial^2 V(x;t)/\partial x^2|_{x=0}$ blows up at a critical scale t_c if its initial coupling constant G_0 is larger than the critical coupling constant G_c [7,8]. This explosive behavior is nothing but a signal of the SχSB, and suggests that the SχSB solution of the mass function after t_c has a finite jump at the origin with the chiral-invariant structure. Mathematically, such a singular solution of the PDE cannot be authorized. However the singular solution can be defined as a weak solution and predict the physical quantities as shown in the next section.

3 Method of weak solution

In this section, we define a weak solution [9] of the mass function and show how to construct it. Differentiating the PDE (7) with respect to x, we then obtain that for the mass function,

$$\partial_t M(x;t) = -\frac{\partial}{\partial x} F(M(x;t);t)$$
$$= -\frac{\partial}{\partial M} F(M;t) \cdot \frac{\partial M}{\partial x}. \tag{8}$$

This equation belongs to a class of conservation low type which includes the famous Bergers' equation without viscosity. To define the singular solution with finite jumps, we extend the PDE into the following weak equation:

$$\int_0^\infty dt \int_{-\infty}^\infty dx \left(M\frac{\partial \varphi}{\partial t} + F\frac{\partial \varphi}{\partial x} \right) + \int_{-\infty}^\infty dx \ \ M\,\varphi|_{t=0} = 0, \tag{9}$$

where $\varphi(x;t)$ is any smooth and bounded test function. Compared to the strong equation (8), the derivative of the mass function is gotten rid off in the weak equation (9), which can then have a singular solution with finite jumps. In general, weak solutions are not uniquely determined depending on the initial conditions. However the physical initial condition is expected to give the unique weak solution.

An important fact derived from the weak equation,(9) is the Rankine-Hugoniot (RH) condition,

$$(M_L - M_R)\, dS(t) = [F(M_L) - F(M_R)]\, dt, \tag{10}$$

where $S(t)$ is a jump position, and M_L and M_R are values of right and left limits of the mass function at $x = S(t)$. This RH condition will be used to construct the weak solution.

In the rest of this section, the method of characteristics to construct the weak solution is shown. We now consider a characteristic curve, $x = \bar{x}(t)$, and the mass function on it, $\bar{M}(t) = M(\bar{x};t)$, which satisfy the following coupled ordinary differential equations (ODEs):

$$\frac{d\bar{x}(t)}{dt} = \frac{\partial}{\partial \bar{M}} F(\bar{M};t), \tag{11}$$
$$\frac{d\bar{M}(t)}{dt} = \frac{\partial}{\partial \bar{x}} F(\bar{M};t) = 0. \tag{12}$$

Here the initial condition is given by

$$\bar{x}(t=0) = x_0, \tag{13}$$
$$\bar{M}(t=0) = \partial_x V_W(x;t)|_{x=x_0,t=0}\,. \tag{14}$$

We now emphasize that $\bar{M}(t)$ is the value of the "local" strong solution of Eq. (8) on the characteristic curve $\bar{x}(t)$. Thus the initial value problem of the PDE (8) is transformed to the partially equivalent one of the coupled ODEs (11), (12), although the set of "local" strong solutions is not necessarily the global solution of the original PDE as will be seen later. We can now easily construct the set of local strong solutions by varying the value of x_0 and solving the coupled ODEs. Moreover the value $\bar{V}_W$ of the fermion potential on $\bar{x}(t)$ is obtained by solving the following ODE with Eqs. (11), (12):

$$\frac{d\bar{V}_W(t)}{dt} = \bar{M}\frac{\partial F(\bar{M};t)}{\partial \bar{M}} - F(\bar{M};t). \tag{15}$$

Note that the PDE (7) is a Hamiltonian-Jacobi type equation well-known in the analytical mechanics, where t, x, $V_W(x;t)$, $M(x;t)$ and $F(M;t)$ correspond to the time, the coordinate, the action, the momentum and the time-dependent Hamiltonian, respectively. Then the coupled ODEs (11), (12) are nothing but the canonical equations of Hamiltonian.

The numerical solution of the characteristic curves and the set of local strong solutions $M(x;t)$ constructed by them are shown in Fig. 2 (a), (b). The characteristic curves $\bar{x}(t)$ can be regarded as the contour lines of $M(x;t)$ because the right-hand side of Eq. (12) vanishes. After t_c, the contour lines cross each other, and then the set of the local strong solutions $M(x;t)$ has a folding structure. Thus the set, which has the multi values, can no longer be the global solution of the PDE (8). On the other hand, the weak solution can be constructed by the patchwork of the local strong solutions, which is determined using the RH condition (10).

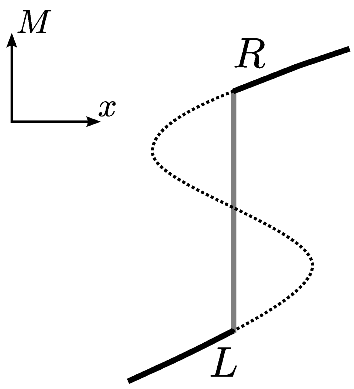

Figure 1: Equal (vanishing) area rule.

For the practical purpose, we here convert the RH condition to a geometric one equivalent to it as follows. The total derivatives of left and right limits of the fermion potential $V_W(x;t)$ on the jump position $S(t)$ is given by

$$\begin{aligned} dV_W^{\mathrm{L,R}} &= \left.\frac{\partial V}{\partial x}\right|_{\mathrm{L,R}} dS + \left.\frac{\partial V}{\partial t}\right|_{\mathrm{L,R}} dt \\ &= M|_{L,R} dS - F|_{\mathrm{L,R}} dt, \end{aligned} \tag{16}$$

The difference between the left and right limits of Eq. (16) vanishes because of the RH condition: $d(V_W^{\mathrm{L}} - V_W^{\mathrm{R}}) = 0$. Since no singular point doesn't exist at the initial condition, the fermion potential is entirely continuous even at the jump position $S(t)$:

$$V_W^{\mathrm{L}} = V_W^{\mathrm{R}}. \tag{17}$$

Next, we integrate the set of the local strong solutions of $M(x;t)$ as follows:

$$\begin{aligned} \int_{\mathrm{L}}^{\mathrm{R}} M dx &= \int_{\mathrm{L}}^{\mathrm{R}} \frac{\partial V_W}{\partial x} dx \\ &= V_W^{\mathrm{R}} - V_W^{\mathrm{L}} \\ &= 0. \end{aligned} \tag{18}$$

Thus the jump position can geometrically be determined by the equal (vanishing) area rule (Fig. 1). In Fig. 2, we now show the weak solution constructed from the local strong solutions at $G_0 = 1.005G_c$ using the equal area rule. The jump appears at the origin after the critical scale t_c. The uniqueness of our weak solution is proved because of the entropy condition, which guarantees the uniqueness of weak solution when the selected characteristic curves fill the x-t plane [9] as shown in Fig. 2 (a').

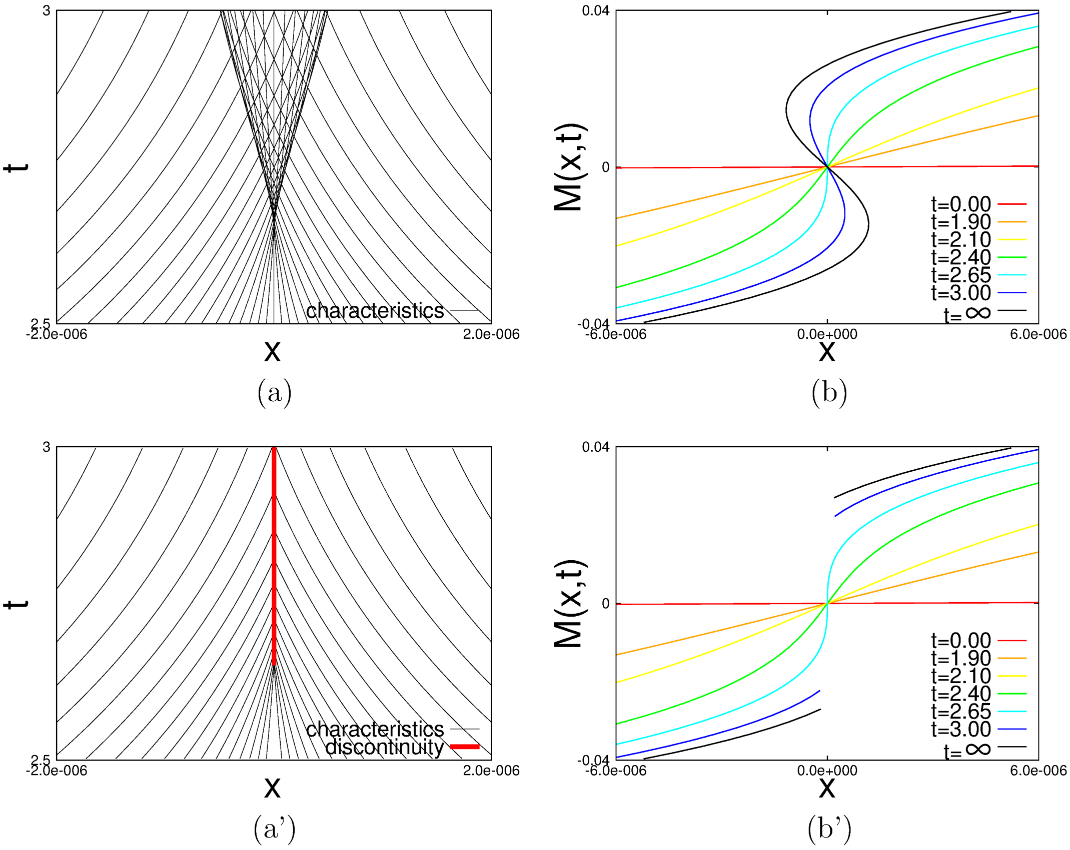

Figure 2: (a) Characteristics. (b) Set of local strong solutions of mass function given by the characteristics. (a') Characteristics selected by the RH condition and jump (discontinuity). (b') Weak solution of mass function. The jump position obey the equal (vanishing) area rule.

4 Bare mass

In the previous section, we have obtained the SχSB weak solution with a finite jump at the origin. However the physical mass of quark as an order parameter of chiral symmetry cannot be determined since the mass function at the origin is not defined. To define the order parameter, we introduce the bare mass term $m_0\bar{\psi}\psi$, which explicitly breaks the chiral symmetry, to the Lagrangian (1). The bare mass term modifies the initial condition of the PDE (8): $M(x; t = 0) = G_0 x + m_0$. Then, because of the translation invariance of the PDE with respect to x, the mass function at $m_0 \neq 0$ is given by the one at $m_0 = 0$ as follows:

$$M(x;t,m_0) = M(x + m_0/G; t, m_0 = 0). \tag{19}$$

Then the mass function at the origin is well defined because the jump appears not at the origin but at $x = -m_0/G$. Thus, taking the limit $m_0 \to +0$ (called the chiral limit), we can define the t-dependent effective mass $M_{\text{phs}}(t)$ as the order parameter:

$$M_{\text{phys}}(t) = \lim_{m_0 \to +0} M(x;t,m_0)|_{x=0}\,. \tag{20}$$

Fig. 3 shows the RG evolutions of the physical masses in the chiral limit and at the non-zero bare mass.[1] The physical mass in the chiral limit shows the second order phase transition due to the singular behavior of the mass function at the origin, while the physical mass at $m_0 \neq 0$ shows the cross over. The reader may think that the weak-solution method is not necessary if $m_0 \neq 0$. However global methods, such as the weak solution, is needed at the small bare mass compared to the physical mass since the mass function has the jump near the origin. Actually, Ref. [2] shows that the Taylor expansion to solve the PDE 7 does not work at the small bare mass.

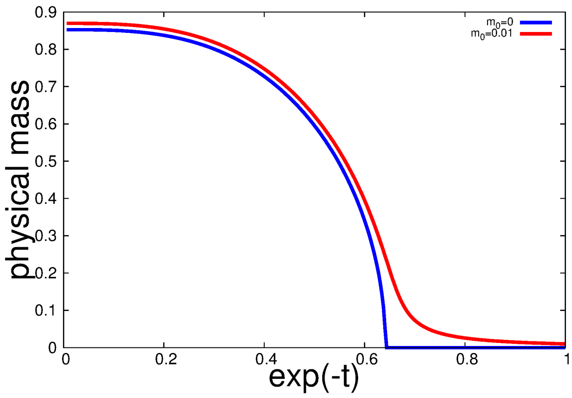

Figure 3: RG evolution of the physical masses in $m_0 \to 0$ and $m_0 = 0$. The NPRG equation given by Eq. (21) ($\mu = 0$, $G = 1.7G_c$) is used for evaluating the physical mass to compare the result at finite chemical potential $\mu \neq 0$.

5 First order phase transition at finite chemical potential

Let us consider the first order phase transition at finite chemical potential ($\mu \neq 0$) using the weak-solution method. The first order phase transition is more non-trivial than the second order phase transition because the RG evolution of the physical mass has a finite jump even at $m_0 \neq 0$ (as shown in Fig. 4). Moreover the non-uniqueness of weak solution is associated with the fact that the effective potential, which is non-convex, has the multi-local minima in the mean-field analysis. In this section, we show that the weak solution

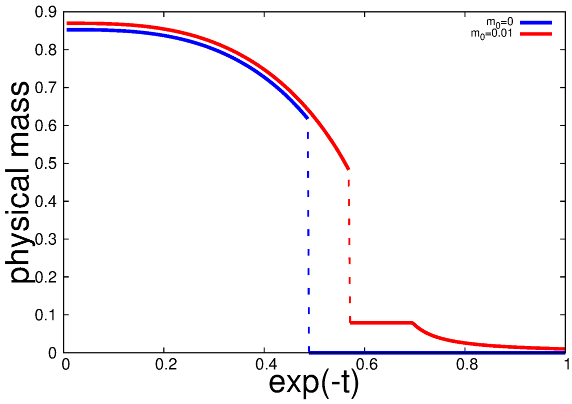

Figure 4: RG evolutions of the physical mass at finite density.

at finite chemical potential is uniquely determined, and the effective potential constructed by the weak solution is automatically "convexised" with only the one correct minimum.

The chemical potential μ is introduced by adding the term $\mu\bar{\psi}\gamma_0\psi$ to the Lagrangian (1). For the simple NPRG equation, we use the spacial momentum cutoff: $\sum_{i=1}^{3} p_i^2 \leq \Lambda^2$. Then the right-hand side of Eq. (8) changes to[2]

$$-F(x;t) = \frac{\Lambda^3}{\pi^2}\left[\sqrt{\Lambda^2+M^2} + \left(\mu - \sqrt{\Lambda^2+M^2}\right)\cdot\Theta\left(\mu - \sqrt{\Lambda^2+M^2}\right)\right], \tag{21}$$

where $\Theta(x)$ is the Heaviside step function. The characteristic curve is consequently given by the following ODE:

$$\frac{d\bar{x}(t)}{dt} = -\frac{\Lambda^3 M}{\pi^2\sqrt{\Lambda^2+M^2}}\,\Theta\left(\sqrt{\Lambda^2+M^2} - \mu\right). \tag{22}$$

In Fig. 5, we show the characteristic curves and those selected by the RH condition which are evaluated at $m_0 = 0$, $\mu = 0.7$. and $G_0 = 1.7G_c$. Fig. 5 (b) shows the uniqueness of our weak solution because the entropy condition is satisfied. Fig. 6 (a), (b) show the weak solutions of the mass function and the fermion potential at $m_0 = 0.01\Lambda_0$. These figures shows that in the RG procedure the two jumps simultaneously appear, move toward each other, and finally merge into one. Thus the RG evolution of the physical mass shows the first order phase transition as shown in Fig. 4.

In the rest of this section, we discuss the convexity of the Legendre effective potential constructed by the weak solution. At first, we define the free energy $W(j;t)$ by introducing the external source for the chiral condensates $\langle\bar{\psi}\psi\rangle$: its source term $j\bar{\psi}\psi$, which is distinguished from the mass term, is added to the Lagrangian (1). Then the initial condition of the fermion potential is

$$V_W(x;t=0,j) = m_0 x + \frac{G_0}{2}x^2 + jx. \tag{23}$$

[1]In the real world the bare mass has the non-zero value given by the Higgs mechanism.
[2]The critical coupling constant also changes: $G_c = 2\pi^2/\Lambda_0$.

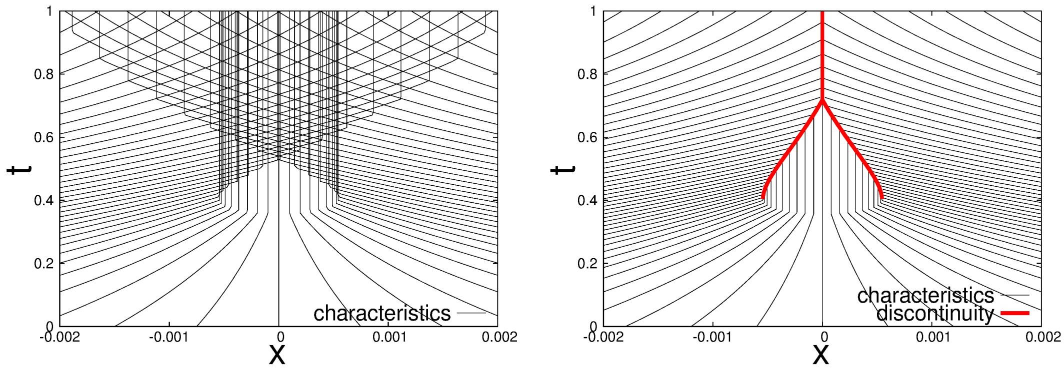

Figure 5: (a) Characteristics. (b) Characteristics selected by the RH condition and jump (discontinuity).

Now the free energy and the chiral condensates are given by

$$W(j;t) = V_W(x=0;t,j), \tag{24}$$

$$\phi(j;t) \equiv \langle \bar{\psi}\psi \rangle_j = \frac{\partial W(j;t)}{\partial j}, \tag{25}$$

respectively. We eventually define the Legendre effective potential of the chiral condensate as follows:

$$V_L(\phi;t) = j\phi(j;t) - W(j;t), \tag{26}$$

where $\partial V_L/\partial \phi = j$ is satisfied.

As seen in the previous section, because of the translation invariance of the PDE with respect to x, the fermion potential at $j \neq 0$ is given by the one at $j = 0$:

$$V_W(x;t,j) = V_W(x + j/G_0;t,j=0) - \frac{m_0 j}{G_0} - \frac{j^2}{2G_0}, \tag{27}$$

Thus the free energy and the chiral condensates are given by the quantities at $j = 0$ as follows:

$$W(j;t) = V_W(x=0;t,j) = V_W(j/G_0;t,j=0) - \frac{m_0 j}{G_0} - \frac{j^2}{2G_0}, \tag{28}$$

$$\phi(j;t) = \frac{1}{G_0}\left[M(j/G_0;t,j=0) - m_0 - j\right]. \tag{29}$$

Since the set of local strong solutions of the mass function $M(j/G_0;t,j=0)$ is multi-valued, that of $\phi(j;t)$ is so. Obeying the equal area rule, the weak solution of $\phi(j;t)$ is then constructed from its local strong solutions as well as the mass function.

The set of strong solutions of $V_L(\phi;t)$ is not convex and has multi local minima as shown Fig. 6 (c). On the other hand, we can prove that its weak solution is the "convexised" potential whose minimum agrees with the global minimum of the set of the local strong solution as follows. Because of the continuity of the fermion potential (17), the free energy is also continuous at the jump position j_s of the mass function $M(j/G_0; t, j=0)$:

$$W^L - W^R = V_W^L - V_W^R = 0. \tag{30}$$

Using this continuity of the free energy and Eq. (26), we obtain

$$\frac{V_L^L - V_L^R}{\phi^L - \phi^R} = j_s, \tag{31}$$

which means that the line connecting the Legendre effective potential $V_L(\phi; t)$ at the two positions ϕ^L, ϕ^R agrees with the envelope since $\partial V_L/\partial\phi|_{L,R} = j_s$. Thus the weak solution of the effective potential is automatically convexised and has the correct minimum which agrees with the global minimum of the local strong solutions as shown in Fig. 6.

6 Summary

In this article, we have introduced the weak solution to define the singular SχSB solution of NPRG equation that can predict physical quantities such as the physical quark mass and the chiral condensates. The weak solution satisfies the integral-form (weak) of the PDE. Specifically we have evaluated the weak solution of the large-N NPRG equation for the mass function which is the first derivative of the fermion potential with respect to the scalar bilinear-fermion field $\bar{\psi}\psi$.

We have constructed the weak solution by the method of characteristics. The set of local strong solutions given by the characteristics is multi-valued and thus no longer is the global solution of the PDE. The weak solution can geometrically be constructed by the patchwork of the local strong solutions using the equal area rule, which is derived by the Rankine-Hugoniot condition. The uniqueness of the weak solution has been guaranteed by the entropy condition. Then we have obtained the SχSB weak solution of the mass function with a finite jump at the origin.

The method of weak solution has also been applied to the first order phase transition at finite chemical potential. We have shown that in the RG procedure two jump appear simultaneously, move toward each other, and finally merge into one. This RG evolution is nothing but the first order phase transition. Finally we have discussed the convexity of the Legendre effective potential of the chiral condensates which is constructed by the weak solution of the fermion potential. The weak solution of the effective potential, which shows the first order phase transition in the RG procedure, is automatically "convexised" with only one correct minimum, while the effective potential obtained by the mean-field analysis has the multi local minima.

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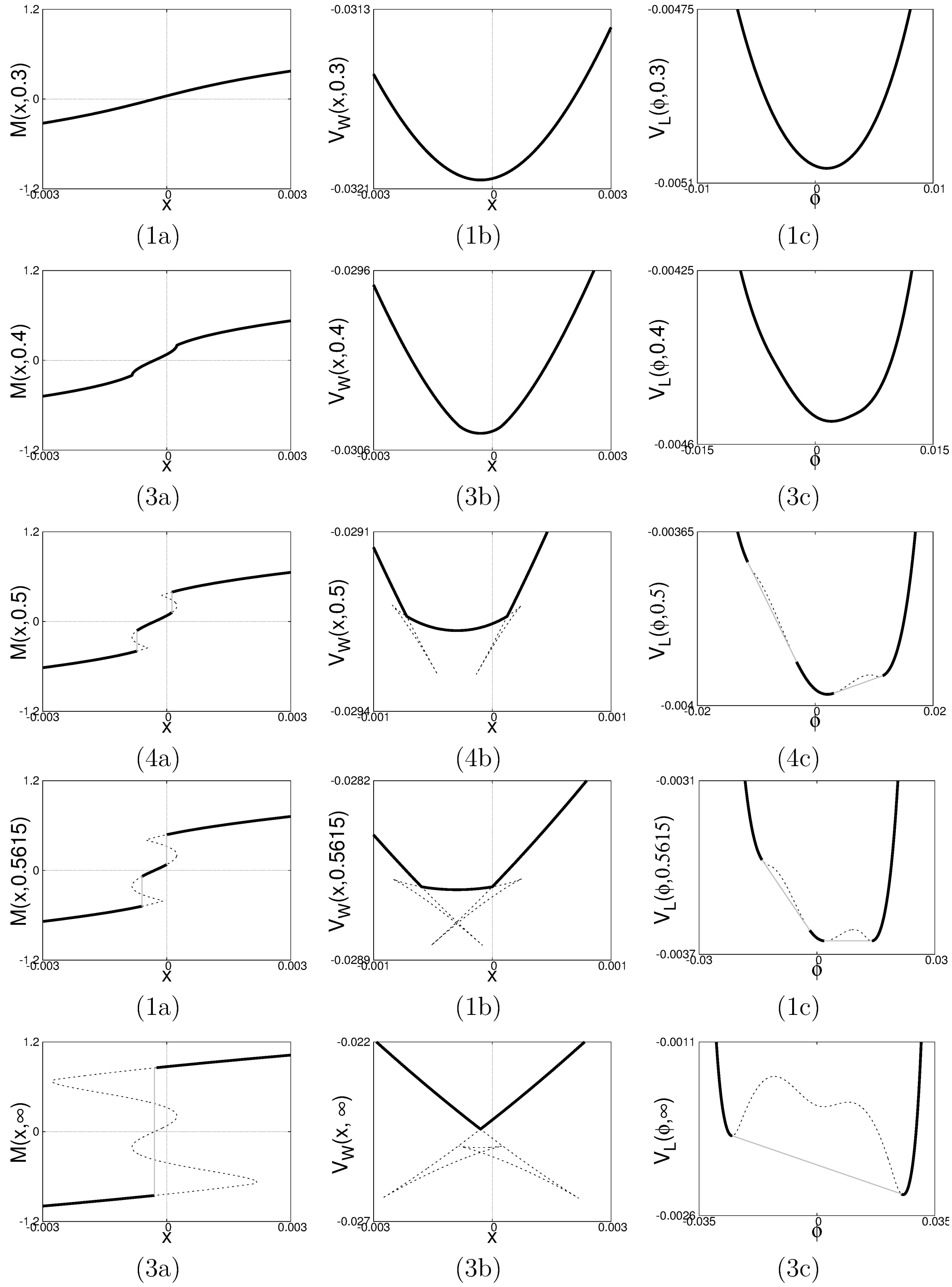

Figure 6: RG evolution of physical quantities by weak solution with non-zero bare mass ($G_0 = 1.7G_c$, $m_0 = 0.01\Lambda_0$, $\mu = 0.7$, $t = 0.3, 0.4, 0.5, 0.5615, \infty$). (a) Mass function. (b) Fermion potential. (c) Legendre effective potential. The thick solid lines denote the weak solution, and the dashed lines denote the local strong solutions dropped by the equal area rule. The thin solid line in (c) denotes the envelope. Origins of y-axis in (b) and (c) are arbitrary.

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