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Absence of Phase Transitions in 2D $O(N)$ Spin Models and Renormalization Group Analysis

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Abstract

The classical $O(N)$ spin models in two dimensions have been believed free from any phase transitions if N is larger than or equal to 3. We show that if N is large, then the block-spin-type transformations can be applied through Fourier (duality) transformation. This enables us to prove the result claimed in the title of this paper.

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1 Introduction

Though quark confinement in 4 dimensional (4D) non-Abelian lattice gauge theories and spontaneous mass generations in 2D non-Abelian sigma models are widely believed [1], we still do not have a rigorous proof. These models exhibit no phase transitions in the hierarchical model approximation of Wilson-Dyson type or Migdal-Kadanov type [12].

In ref. [14], we considered a transformation of random walk (RW) which appears in the $O(N)$ spin models [3, 4]. This was extended by the cluster expansion [5, 11, 19, 20], and we showed in the 2D $O(N)$ sigma model that :

$$\frac{\beta_c}{N} \geq \text{const} \log N \tag{1.1}$$

In this paper, we apply a block-spin transformation to the functional integral of the system, and establish the following theorem:

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Main Theorem. *There exists no phase transition in two-dimensional O(N) invariant Heisenberg model for all β if N is large enough.*

To appeal to the $1/N$ expansion [17], we scale the inverse temperature β by N. ($N\beta$ is denoted simply β or β_c in [14] and in our bound (1.1).) The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$\langle f\rangle \equiv \frac{1}{Z_\Lambda(\beta)}\int f(\phi)\exp[-H_\Lambda(\phi)]\prod_i \delta(\phi_i^2 - N\beta)d\phi_i \tag{1.2}$$

Here

$$\Lambda = \Lambda_0 = [-(L/2)^M, (L/2)^M)^\nu \subset \mathbf{Z}^\nu$$

is the large square with center at the origin, where L is chosen odd (e.g. $L = 3$) and M is a large integer. Moreover $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $< 1 >= 1$. Moreover H_Λ is the Hamiltonian given by

$$H_\Lambda \equiv -\frac{1}{2}\sum_{|x-y|_1=1}\phi(x)\phi(y), \tag{1.3}$$

where $|x|_1 = \sum_{i=1}^{\nu}|x_i|$.

First substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]da/2\pi$ into eq.(1.2) with the condition [3, 4] that $\mathrm{Im}a_i < -\nu$. We set

$$\mathrm{Im}\, a_i = -(\nu + m^2/2), \quad \mathrm{Re}\, a_i = \frac{1}{\sqrt{N}}\psi_i \tag{1.4}$$

where $m^2 >$ will be determined soon. Thus we have

$$\begin{aligned} Z_\Lambda &= c^{|\Lambda|}\int\cdots\int \exp[-W_0(\phi,\psi)]\prod \frac{d\phi_j d\psi_j}{2\pi} \\ &= c^{|\Lambda|}\det(m^2-\Delta)^{-N/2}\int\cdots\int F(\psi)\prod\frac{d\psi_j}{2\pi} \end{aligned} \tag{1.5}$$

where

$$W_0(\phi,\psi) = \frac{1}{2}\langle\phi, (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)\phi\rangle - \sum_j i\sqrt{N}\beta\psi_j \tag{1.6a}$$

$$F(\psi) = \det{}^{-N/2}(1 + i\alpha G\psi)\exp[i\sqrt{N}\beta\sum_j\psi_j] \tag{1.6b}$$

$$\alpha = 2/\sqrt{N} \tag{1.6c}$$

Here c's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian, $G = (m^2 - \Delta)^{-1}$ is the covariant matrix. The two point functions are given by

$$\langle\phi_0\phi_x\rangle = \frac{1}{\tilde{Z}}\int\cdots\int (m^2 - \Delta + i\alpha\psi)^{-1}_{0x}F(\psi)\prod\frac{d\psi_j}{2\pi} \tag{1.7}$$

where $\tilde{Z}$ is the obvious normalization constant. Choose the mass parameter $m = m_0 > 0$ so that $G(0) = \beta$, where

$$G(x) \;=\; \int \frac{e^{ipx}}{m_0^2 + 2\sum(1-\cos p_i)} \prod_{i=1}^{\nu} \frac{dp_i}{2\pi} \tag{1.8}$$

This is possible for any β if and only $\nu \le 2$, and we find that $m^2 \sim 32e^{-4\pi\beta}$ as $\beta \to \infty$ for $\nu = 2$, which is consistent with the renormalizaiton group analysis, see e.g. [6]. Thus we can rewrite

$$F(\psi) \;=\; \det{}_3^{-N/2}(1 + i\alpha G\psi)\exp[-\langle\psi, G^{\circ 2}\psi\rangle] \tag{1.9}$$

for $\nu \le 2$, where $\det_3(1+A) = \det[(1+A)e^{-A+A^2/2}]$ and $G^{\circ 2}(x,y) = G(x,y)^2$ so that $\mathrm{Tr}(G\psi)^2 = \langle\psi, G^{\circ 2}\psi\rangle$. Moreover $F(\psi)$ is integrable if and only if $N > 2$, and thus $\nu \le 2$ and $N > 2$ are required.

If m is so chosen, the determinant $\det_3(1+i\alpha G\psi)^{-N/2}$ may be regarded as a small perturbation to the Gaussian measure $\sim \exp[-\langle\psi, G^{\circ 2}\psi\rangle]\prod d\psi$. This is the case if N is very large or if β is very small (e.g. $N\log N > \beta$), in which case $|||\alpha G|| \ll 1$ and we can disregard $\det_3^{-N/2}(1+i\alpha G\psi)$ and the model is exactly solvable in this limit. Thus we have

$$\begin{aligned} \langle\phi_0\phi_x\rangle \;&=\; \frac{1}{Z}\int (m_0^2 - \Delta + i\alpha\psi)^{-1}_{0x}\exp[-\mathrm{Tr}(G\psi)^2]\prod d\psi \\ &\le\; (m_0^2-\Delta)^{-1}_{0x} \le c\,\exp(-m_0|x|) \end{aligned} \tag{1.10}$$

But this argument fails for large β since G is of long-range and the expansion of the determinant is not justified at all.

On the other hand, this argument can be justified if the main part of the ψ integral consists of $|\psi| < N^{\epsilon}\beta^{-1/2}$ such that $\sum_x \psi_x \sim 0$. In this case, the expansion of the determinant is justified. Our main argument in this paper is to justify this argument.

The renormalization group (RG) method is the method to integrate the functional integration recursively introducing block spin operators C and C' defined by

$$\begin{aligned} \phi_1(x) \;&=\; (C\phi)(x) \\ &\equiv\; \frac{1}{L^2}\sum_{\zeta\in\Delta_0} f(Lx+\zeta) \qquad (1.11a) \\ \psi_1(x) \;&=\; (C'f)(x) \\ &\equiv\; L^2(Cf)(x) \qquad (1.11b) \end{aligned}$$

where $x \in \Lambda \cap L\Lambda$ and Δ_0 is the square of size $L \times L$ ($L \ge 2$)center at the origin. C and C' consist of averaging over the spins in the blocks and the scaling of the coordinates, i.e., $\Lambda = \Lambda_0 \to \Lambda_1$. We integrate out the remaining degrees of freedom which we call fluctuation fields (ξ and $\tilde{\psi}$) and continue these steps, $\phi_n \to \phi_{n+1} \to \cdots$, $\psi_n \to \psi_{n+1} \to \cdots$ and $\Lambda_n \to \Lambda_{n+1} \to \cdots$ $(n = 0, 1, 2, \cdots)$. We repeat this process by finding matrices A_n and $\tilde{A}_n$ such that

$$\phi_n = A_{n+1}\phi_{n+1} + Q\xi_n \quad (1.12a)$$
$$\psi_n = \tilde{A}_{n+1}\phi_{n+1} + Q\tilde{\psi}_n \quad (1.12b)$$

and

$$\langle \phi_n, G_n^{-1}\phi_n \rangle = \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} \rangle + \langle \xi_n, \Gamma_n^{-1}\xi_n \rangle \quad (1.13a)$$
$$\langle \psi_n, H_n^{-1}\psi_n \rangle = \langle \psi_{n+1}, \hat{H}_{n+1}^{-1}\psi_{n+1} \rangle + \langle \tilde{\psi}_n, Q^+ H_n^{-1} Q\tilde{\psi}_n \rangle \quad (1.13b)$$

where G_n^{-1} and H_n^{-1} are the main Gaussian parts in W_n, and

$$G_n = CG_{n-1}C^+ = C^n G_0 (C^+)^n \quad (1.14a)$$
$$(Q\xi)(x) = \begin{cases} \xi(x) & \text{if } x \in \Lambda_n' \\ -\sum_{\zeta\in\Delta(x),\zeta\neq x} \xi(\zeta) & \text{if } x \notin \Lambda_n' \end{cases} \quad (1.14b)$$
$$\Lambda_n' = \Lambda_n \backslash L\Lambda_n \quad (1.14c)$$

where $\Delta(x)$ is the square of size $L \times L$ center at x $(\in \Lambda_n \cap L\Lambda_n)$. Namely $Q : R^{\Lambda_n'} \to R^{\Lambda_n}$ $(n = 0, 1, 2, \cdots)$ is the operator to make zero-average fluctuations $Q\xi_n$ from $\{\xi_n(x) : x \in \Lambda_n'\}$.

In our case, we start with

$$\begin{aligned} G_0 &= (-\Delta + m_0)^{-1}(x, y) \\ &\sim \beta - \frac{1}{2\pi}\log|x-y| \\ H_0 &= \frac{1}{G^{\circ 2}}(x, y) \\ &\sim \frac{1}{|x-y|^4} \end{aligned}$$

where H_0^{-1} is derived from the formal $N \to \infty$ limit of $F(\psi)$. Thus we see that

$$\begin{aligned} G_1(x, y) &= (CG_0C^+)(x, y) \sim \frac{1}{L^4}\sum_{\zeta,\xi\in\Delta_0} \log(Lx - Ly + \zeta - \xi) \\ &\sim G_0(x, y) \\ H_1(x, y) &= (C'H_0C'^+)(x, y) \sim \sum_{\zeta,\xi\in\Delta_0} (Lx - Ly + \zeta - \xi)^{-4} \\ &\sim H_0(x, y) \end{aligned}$$

as $|x - y| \gg 1$. This means that the main Gaussian terms are left invariant by C and C' (self-similarity).

Define

$$\begin{aligned}
\mathcal{A}_n &= A_1 A_2 \cdots A_n && (1.15a)\\
\tilde{\mathcal{A}}_n &= \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_n && (1.15b)\\
\varphi_n &= \mathcal{A}_n \phi_n && (1.15c)\\
z_n &= \mathcal{A}_n Q \xi_n && (1.15d)\\
\mathcal{G}_n &= \mathcal{A}_n G_n \mathcal{A}_n^+ && (1.15e)\\
\mathcal{T}_n &= \mathcal{A}_n Q \Gamma_n Q^+ \mathcal{A}_n^+ && (1.15f)
\end{aligned}$$

so that

$$\begin{aligned}
\varphi_n &= \varphi_{n+1} + z_n && (1.16a)\\
\mathcal{G}_n &= \mathcal{G}_{n+1} + \mathcal{T}_n && (1.16b)\\
G_0 &= \sum \mathcal{T}_n && (1.16c)\\
\mathcal{G}_0^{\circ 2} &= \sum_n (\mathcal{G}_n^{\circ 2} - \mathcal{G}_{n+1}^{\circ 2}) && (1.16d)\\
&= \sum_n (\mathcal{T}_n^{\circ 2} + 2\mathcal{G}_{n+1} \circ \mathcal{T}_n) && (1.16e)
\end{aligned}$$

Since $\mathrm{Tr}(G\psi)^2 = \langle \psi, G^{\circ 2}\psi \rangle$ in (1.9), we will see that

$$H_n^{-1} \sim \mathcal{T}_n^{\circ 2} + 2\mathcal{G}_{n+1} \circ \mathcal{T}_n \sim 2\beta_{n+1}\mathcal{T}_n \quad (1.17)$$

Here we use the following notation (Hadamard product)

$$(A \circ B)(x, y) = A(x, y)B(x, y), \quad T^{\circ 2} = T \circ T$$

2 Hierarchical Model Revisited

Before beginning our BST, we study some remarkable features in this model by the hierarchical approximation of Dyson-Wilson type [13] in which the Gaussian part

$$\exp[-(1/2)\langle \phi_n, (-\Delta)\phi_n \rangle]$$

is replaced by the hierarchical one:

$$\exp[-(1/2)\langle \phi_{n+1}, (-\Delta)_{hcl}\phi_{n+1} \rangle - (1/2)\langle \xi_n, \xi_n \rangle],\ n = 0, 1, \cdots$$

Put $g_0(\phi) = \delta(\phi^2 - N\beta)$. Choosing a box of size $\sqrt{2} \times \sqrt{2}$ at the nth step including two spins ϕ_+ and ϕ_- (two ϕ_n's in the box), we put $\phi_\pm \equiv \phi \pm \xi$, where $\phi = \phi_{n+1}$ and $\xi = \xi_n$.

Then $2\xi^2 = \phi_+^2 + \phi_-^2 - 2\phi^2$ and put $\phi = (\varphi, 0) \in R_+ \times R^{N-1}$, $\xi = (s, u) \in R \times R^{N-1}$ and $f(x) = g_n(x)e^{-x/4}$. Then putting $x = \phi^2$, we have

$$\begin{aligned} g_{n+1}(x) &= e^{x/2}\int f((\phi+\xi)^2)f((\phi-\xi)^2)dsd^{N-1}u \\ &= e^{x/2}\int f((\varphi+s)^2+u^2)f((\varphi-s)^2+u^2)dsd^{N-1}u \\ &= \frac{e^{x/2}}{\sqrt{x}}\int_{\mathcal{D}} f(p)f(q)\mu(p,q,x)^{(N-3)/2}dpdq \\ \mu(p,q,x) &= \frac{p+q}{2} - x - \frac{(p-q)^2}{16x} \end{aligned}$$

where $\mathcal{D} \subset [0, N\beta]^{\times 2}$ is defined so that $\mu(p,q,x) \geq 0$ and

$$\frac{(p-q)^2}{16x} = \frac{(\phi_+^2 - \phi_-^2)^2}{16\phi^2} = \frac{\langle\phi,\xi\rangle^2}{\phi^2} \tag{2.1}$$

This is a part of the probability that two spins $\phi_\pm \equiv \phi \pm \xi$ form the block spin ϕ such that $\phi^2 = x$. If $f(p)$ has a peak at $p = N\beta$, $\exp[x/2 + (1/2)(N-3)\log(p-x)]$ has a peak at $x = N(\beta - 1 + O(N^{-1}))$.

What we learn from this model is the following which will appear in the real system:

1. The curvature of $V_n = -\log g_n$ at its bottom $x = N\beta_n$ is N^{-1}, and then the deviation of $x = \phi_n^2$ from $N\beta_n$ is $N^{1/2}$.
2. $\beta_n \sim \beta - O(n)$
3. The deviation $|\phi_n(x)\phi_n(y) - N\beta_n|$ is given by the Gaussian variables $u \in R^{N-1}$ of short correlation. In fact $|\phi_{n,+}\phi_{n,-} - N\beta_n| = |\phi_{n+1}^2 - N\beta_{n+1} + :u^2:_1| \sim N^{1/2}$
4. One block spin transformation yields the factor $x^{-1/2} \sim \beta_n^{-1/2}$. The factor $x^{-1/2}$ is relevant but logarithmic in the action. Thus its effects are negligible.
5. $g_{n+1}(x)$ in analytic in $0 < x < N\beta$ $(N \geq 3)$ if so is $g_n(x)$. $(g_1 = (e^{x/2}/\sqrt{x})(N\beta - x)^{(N-3/2)})$
6. The probability such that $x = \phi^2 > N\beta_{n_0}$ tends to zero rapidly as $(n_0 <)\, n \to \infty$, and $g_n(x) \to \delta(x)$. This is the mass generation in the hierarchical model.

Though this model is very much simplified, it is very surprising that this model contain almost all properties and problems which the real system has. The property (3) is important and related to the N^{-1} expansion since this means that $\varphi_n(x)\varphi_n(y)/N$ can replaced by $\mathcal{G}_n(x,y)$.

One serious problem is that the factor $(x)^{-1/2} = \exp[-\log(\phi^2)]$ and $\log(\phi^2)$ is relevant in the terminology of renormalization group analysis, i.e., the coefficient may grow exponentially fast as $n \to \infty$. To controll this, we introduce an artificial relevant potential $\delta_n(\phi_n^2 - N\beta_n)^2$ which absorb the effects of $\log(\varphi^2)$. We note that $(\phi_0^2 - N\beta)^2 = 0$ by the initial condition $\delta(\phi_0^2 - N\beta)$. Thus one of the main tasks in this paper is to show that δ_n are uniformly bounded in n.

Remark 1 *It is helpful to see the the asymptotic behavior of the partition function* Z_Λ

$$Z_\Lambda(\beta) = \int \exp\left[-\frac{1}{2}\langle\phi_1, G_1^{-1}\phi_1\rangle - \frac{1}{N}\sum(\phi_1^2(x) - N\beta_1)^2\right]\prod_{x\in\Lambda} d^N\phi_1(x) \quad (2.2a)$$

$$\sim \exp\left[-\frac{1}{2}|\Lambda|\log\beta + O(|\Lambda|N)\right] \quad (2.2b)$$

which holds for very large β. *This is obtained by putting* $\phi_i = r_i\omega_i$, $\omega_i \in S^{N-1}$ *and used the fact that the size of the* $(N-1)$ *unit sphere* $\int d\omega = |S^{N-1}|$ *is* $2(2\pi)^{(N-1)/2}/\Gamma((N-1)/2) = \exp[-(N/2)\log N + O(N)]$.

3 RG Flow of the Real System

We combine two types of block transformations to $W_0(\phi,\psi)$ which is the ν dimensional boson model of $\phi^2\psi$ type interaction with pure imaginary coupling. In this approach, we can expect all coefficients are bounded and small through the block spin transformations. Thus perturbative calculations are useful. We have two types of block spin transformations. One is the block spin transformation of the N component boson model of mass m_0^2, and the other is the block spin transformation of the auxiliary field ψ. The two dimensional boson field ϕ is dimensionless and the auxiliary field ψ has the dimension length^{-2}, and they have different scalings. The ψ field keeps $\phi_0 = \phi$ on the surface of the N dimensional ball of radius $(N\beta)^{1/2}$. We will see that by one step of the BSTs of ϕ and ψ, the radius is shrinked to $(N\beta_1)^{1/2}$, where $\beta_1 = \beta - O(1)$.

We turn to our model and sketch our main ideas and procedures. Our method of analysis depends on n. For $n < \log\beta$ we can forget the term $\log\phi^2$, but for $n > \log\beta$ this term is rather large and we cannot disregard $V_n^{(1)}$. Assume $n > \log\beta$ and assume that the Gibbs factor at the step n is given by

$$\exp[-W_n(\varphi_n,\psi_n) - \sum_X \delta W_n(X;\varphi_n,\psi_n)] \quad (3.1)$$

where $W_n(\varphi_n,\psi_n)$ is the main term which controls the system and $\delta W_n(X;\varphi_n,\psi_n)$ are polymers whose supports spread over paved set $X \subset \Lambda$. $\delta W_n(X;\varphi_n,\psi_n)$ are very small but analytic domain of φ_n may be small for large X. Our basic induction assumption is that the main part $W_n(\phi_n, \psi_n)$ is given by

$$\begin{aligned} W_n(\phi_n, \psi_n) &= \frac{1}{2}\langle \phi_n, G_n^{-1}\phi_n\rangle + \frac{i}{\sqrt{N}}\langle\langle : \phi_n^2 :_{G_n}, \psi_n\rangle + \langle \psi_n, H_n^{-1}\psi_n\rangle \\ &\quad + V_n^{(1)} + V_n^{(2)} \qquad (3.2a) \\ V_n^{(1)} &= \frac{1}{2N}\langle : \phi_n^2 :_{G_n}, \delta_n : \phi_n^2 :_{G_n}\rangle \qquad (3.2b) \\ V_n^{(2)} &= \frac{\gamma_n}{2}\langle : \phi_n^2 :_{G_n}, \tilde{A}_{n-1}E^{\perp}G_{n-1}^{-1}E^{\perp}\tilde{A}_{n-1}^{+} : \phi_n^2 :_{G_n}\rangle \qquad (3.2c) \end{aligned}$$

where $\tilde{A}_n$ is a constant matrix discussed later, $E^{\perp}$ is the projection operator to the set of block-wise zero-average functions, i.e. $\mathcal{N}(C) = \{f \in R^{\Lambda} : (Cf)(x) = 0, \forall x \in \Lambda_1\}$, and $: \phi_n^2 :_{G_n}$ is the Wick product of ϕ_n^2 with respect to G_n.

The point is that $E^{\perp}$ acts as a differential operator and $G_n^{-1} \sim -\Delta$. Thus $E^{\perp}(-\Delta)E^{\perp}$ contains $\prod_{i=1}^{4} \nabla_{\mu_i}$. The term $V_n^{(2)}$ corresponds to $(p-q)^2/16x$ and is irrelevant.

The relevant terms $V_n^{(1)}$ is a dummy and is not necessary in principle since $\langle : \varphi_0^2 :_{G_0}, : \varphi_0^2 :_{G_0}\rangle = 0$ at the beginning. The term $V_n^{(1)}$ is artificially inserted to control $\log \phi^2$. This is relevant, but we can show that the coefficient stays bounded. In the case of hierarchical model, we do not need any information of W_n or g_n for $\phi_n^2 < N\beta_n$ since the hierarchical Laplacian is local and (then) we have some a priori bound for g_n which are locally defined. But in the present model, however, it seems to be convenient to have the term $V_n^{(1)}$ to control $\log \varphi_n^2$.

We show that the change of the action W_n is absorbed by the parameters β_n, δ_n and γ_n. Here

$$\begin{aligned} \beta_n &= \beta - \text{const.}.n + o(n) \qquad (3.3a) \\ \delta_n &= O(1) \qquad (3.3b) \\ \gamma_n &= O((\beta_n N)^{-1}) \qquad (3.3c) \end{aligned}$$

$H_0^{-1} = 0$, $\gamma_0 = 0$ and $\beta_0 = \beta$ and we discarded irrelevant terms.

4 Outline of the Proof

We here sketch our proof which consists of several steps:

[**step 1**]
Let $\Lambda_n = L^{-n}\Lambda \cap Z^2$ and let ϕ_n be the nth block spin ($\phi_{n+1} = C\phi_n$): Set $\phi_n = A_{n+1}\phi_{n+1} + Q\xi_n$, where $\xi_n(x)$ are the fluctuation field living on $\Lambda_n' = \Lambda_n \backslash LZ^2$ and $Q : R^{\Lambda'} \to R^{\Lambda}$ is the zero-average matrix so that the block averages of $Q\xi$ are 0.

$$\langle \phi_n, G_n^{-1}\phi_n\rangle = \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1}\rangle + \langle \xi_n, \Gamma_n^{-1}\xi_n\rangle$$

where $G_{n+1}^{-1} = A_{n+1}^{+} G_n^{-1} A_{n+1}$ and $Q^+ G_n^{-1} Q = \Gamma_n^{-1}$. Namely $A_{n+1} = G_n C^+ G_{n+1}^{-1}$.

[**step 2**]

We have a relevant term, and then it is convenient to consider the Gaussian integral by $q(z) \equiv 2\varphi_n z_n + : z_n^2 :$ (not by z) since $: \varphi_n^2 :_{G_n} =: \varphi_{n+1}^2 :_{G_{n+1}} + q(z)$. Define

$$\begin{aligned} P(p) &= \int \exp[i\langle \lambda, (p-q)\rangle] d\mu(\xi) \prod d\lambda \\ z_n &= \mathcal{A}_n Q \Gamma_n^{1/2} \xi \\ d\mu(\xi) &= \exp\left[-\frac{1}{2}\langle \xi, \xi\rangle\right] \prod \frac{d\xi}{\sqrt{2\pi}} \end{aligned}$$

Then we have

$$\begin{aligned} P(p) &= \int \exp[i\langle\lambda, p\rangle] \exp\left[-i\langle\lambda,\ (2\varphi_{n+1}(\mathcal{A}_n Q \Gamma_n^{1/2}\xi) + : (\mathcal{A}_n Q\Gamma_n^{1/2}\xi)^2 :\rangle\right] d\mu(\xi) \prod d\lambda \\ &= \int \exp\left[-2i\langle \xi, \Gamma_n^{1/2} Q^+ \mathcal{A}_n^+ (\lambda\varphi_{n+1})\rangle - \frac{1}{2}\langle \xi, \left[1 + 2i\Gamma_n^{1/2} Q^+ \mathcal{A}_n^+ \lambda \mathcal{A}_n Q \Gamma_n^{1/2}\right] \xi\rangle\right] \\ &\quad \times \exp[i\langle\lambda, p\rangle + iN\langle\lambda, \mathcal{T}_n\rangle] \prod \frac{d\xi_x d\lambda(x)}{\sqrt{2\pi}} \end{aligned}$$

namely

$$\begin{aligned} P(p) &= \int \exp[i\langle\lambda, p\rangle + iN\langle\lambda, \mathcal{T}_n\rangle] \ \det{}^{-N/2}(1 + 2i\mathcal{T}_n\lambda) \\ &\quad \times \exp\left[-2\langle\lambda, (\varphi_{n+1}\varphi_{n+1}) \circ \left(\mathcal{A}_n Q \frac{1}{\Gamma_n^{-1} + 2iQ^+\mathcal{A}_n^+\lambda\mathcal{A}_n Q} Q^+ \mathcal{A}_n^+\right) \lambda\rangle\right] \prod d\lambda(x) \end{aligned} \tag{4.1}$$

We assume that we are outside of the domain wall region $D_w(\varphi_n)$ and large field region defined $D(\varphi_n)$ by

(1) $D_w(\varphi_n)$ = paved set such that
$|\varphi_n(x)\varphi_n(y) - N\mathcal{G}_n(x,y)| \geq k_0 N^{1/2+\varepsilon} \exp[\frac{c}{10L^n}|x-y|]$, $\forall x \in D_w, \exists y \in D_w$

(2) $D(\varphi_n)$ = minimal paved set such that
$| : \varphi_n^2(x) :_{G_n} | \leq k_0 N^{1/2+\varepsilon} \exp[\frac{c}{10L^n}|x-y|]$, $\forall x \in D(\varphi), \forall y \in D(\varphi)^c$

where $0 < \varepsilon < 1/2$ and paved set is a collection of squares $\{\square\}$ each of which consists of squares $\Delta \subset \Lambda$ of size $L \times L$. The power $N^{1/2}$ is related to the central limit theorem applied to the sum of N independent Gaussian variables $\sum_{i=1}^{N} : \xi_i^2 :$. To imagine why, consider spins $\varphi_n(x)$ located on the bottom of $(\varphi_n^2 - N\beta_n)^2$ and put $\varphi_n = \varphi_{n+1} + z_n$. Thus the parallel component of the fluctuation z_n is suppressed and only the orthogonal fluctuations occur.

We thus replace $\varphi_{n+1}\varphi_{n+1}$ by $N\mathcal{G}_{n+1}$ and expand the determinant up to the second order:

$$\begin{aligned}(4.1) \;=\; & \int \exp[i\langle\lambda, p\rangle - N\langle\lambda, (\mathcal{T}_n^{\circ 2} + 2\mathcal{G}_{n+1}\circ\mathcal{T}_n)\lambda\rangle] \\ & \times \det{}_3^{-N/2}(1 + 2i\Gamma_n^{1/2}Q^+\mathcal{A}_n^+\lambda\mathcal{A}_nQ\Gamma_n^{1/2}) \\ & \times \exp[-2\langle\lambda, (:\varphi_{n+1}\varphi_{n+1}:)\circ\mathcal{T}_n)\lambda\rangle + (\text{higher order terms})]\prod d\lambda(x) \\ \sim\; & \exp\left[-\frac{1}{4N}\langle p, \frac{1}{2\mathcal{G}_{n+1}\circ\mathcal{T}_n + \mathcal{T}_n^{\circ 2}}p\rangle\right] \qquad (4.2)\end{aligned}$$

The terms $:\varphi_{n+1}\varphi_{n+1}:$ are treated by polymer expansion and yields relevant terms $\langle :\varphi_{n+1}^2:, \delta_n :\varphi_{n+1}^2:\rangle$, which are fractions of $\log(\varphi_n^2)$.

Putting $p = Ap_1 + \tilde{Q}\tilde{p}$ with $p_1 = C^np$ and $C^nA = 1$, we see that $P(p)$ is given by

$$\exp\left[-\frac{1}{4N}\langle p_1, \frac{1}{C^n[2\mathcal{G}_{n+1}\circ\mathcal{T}_n + \mathcal{T}_n^{\circ 2}](C^+)^n}p_1\rangle - \frac{1}{4N}\langle\tilde{Q}\tilde{p}, \frac{1}{2\mathcal{G}_{n+1}\circ\mathcal{T}_n + \mathcal{T}_n^{\circ 2}}\tilde{Q}\tilde{p}\rangle\right] \qquad (4.3)$$

Here it is important to remark that

$$\begin{aligned} C^n\mathcal{T}_n(C^+)^n &= 0 \\ C^n\mathcal{T}_n^{\circ 2}(C^+)^n &\sim 1 \\ \mathcal{G}_{n+1}\circ\mathcal{T}_n &\sim \beta_n\mathcal{T}_n \end{aligned}$$

since $\mathcal{T}_n = \mathcal{A}_nQ\Gamma_nQ^+\mathcal{A}_n^+$, $C^n\mathcal{A}_n = 1$, $CQ = 0$ and $\mathcal{T}_n$ decays much faster than $\mathcal{G}_n$. This means that the blockwise constant part p_1 of p remains and the zero-average fluctuation part $\tilde{Q}\tilde{p}$ of p is almost absent.

[step 3]

In the present case, however, δ_n can be large ($\sim L^2$) and then we choose p which minimizes

$$\begin{aligned} F(p) &= \frac{1}{4N}\langle p, \frac{1}{2\mathcal{G}_{n+1}\circ\mathcal{T}_n + \mathcal{T}_n^{\circ 2}}p\rangle + \frac{1}{4N}\langle(:\varphi_{n+1}^2:_{G_{n+1}} + p), \delta_n(:\varphi_{n+1}^2:_{G_{n+1}} + p)\rangle \qquad (4.4) \\ &= \langle p, \frac{1}{D}p\rangle + \frac{1}{N}\langle(:\varphi_{n+1}^2:_{G_{n+1}}, \delta_n p\rangle + \frac{1}{2N}\langle(:\varphi_{n+1}^2:_{G_{n+1}}, \delta_n :\varphi_{n+1}^2:_{G_{n+1}}\rangle \qquad (4.5)\end{aligned}$$

where

$$\frac{1}{D} = \frac{1}{4N}\frac{1}{2\mathcal{G}_{n+1}\circ\mathcal{T}_n + \mathcal{T}_n^{\circ 2}} + \frac{1}{2N}\delta_n \qquad (4.6)$$

To diagonalize this, we again set $p = \mathcal{A}p_1 + \tilde{Q}\tilde{p}$ where

$$\mathcal{A} = D(C^+)^n[C^nD(C^+)^n]^{-1}, \quad C^n\tilde{Q} = 0 \qquad (4.7)$$

and

$$\begin{aligned} F(p) &= F_1(p)+F_2(p) &\quad (4.8\text{a})\\ F_1 &= \langle p_1, \frac{1}{C^n D (C^+)^n} p_1\rangle + \frac{1}{N}\langle (:\varphi_{n+1}^2:_{G_{n+1}}, \delta_n p\rangle \\ &\quad + \frac{1}{2N}\langle (E:\varphi_{n+1}^2:_{G_{n+1}}, \delta_n E:\varphi_{n+1}^2:_{G_{n+1}}\rangle &\quad (4.8\text{b})\\ F_2 &= \langle \tilde{Q}\tilde{p}, \frac{1}{D}\tilde{Q}\tilde{p}\rangle + \frac{1}{N}\langle (E^\perp:\varphi_{n+1}^2:_{G_{n+1}}, \delta_n \tilde{Q}\tilde{p}\rangle \\ &\quad + \frac{1}{2N}\langle (E^\perp:\varphi_{n+1}^2:_{G_{n+1}}, \delta_n E^\perp:\varphi_{n+1}^2:_{G_{n+1}}\rangle &\quad (4.8\text{c}) \end{aligned}$$

where E is the projection to blockwise constant functions (block of size $L^n \times L^n$) and $E^\perp = 1-E$. We moreover assume that δ_n is a constant diagonal matrix. Then F_1 and F_2 take their minima at the following points:

$$\begin{aligned} p_1 &= -\frac{1}{N} C^n D \delta_n : \varphi_{n+1}^2 :_{G_{n+1}} \\ &= \left[-1+\frac{1}{L^{2n}\delta_n}\frac{1}{C^n[2\mathcal{G}_{n+1}\circ \mathcal{T}_n + \mathcal{T}_n^{\circ 2}](C^+)^n}\right] C^n : \varphi_{n+1}^2 :_{G_{n+1}} &\quad (4.9)\\ \tilde{Q}\tilde{p} &= -\frac{1}{2N} E^\perp D \delta_n : \varphi_{n+1}^2 :_{G_{n+1}} \\ &= \left[-1+\frac{1}{\delta_n}\frac{1}{2\mathcal{G}_{n+1}\circ \mathcal{T}_n + \mathcal{T}_n^{\circ 2}}\right] E^\perp : \varphi_{n+1}^2 :_{G_{n+1}} &\quad (4.10) \end{aligned}$$

Since $Q\xi$ have L^2-1 degrees of freedom in each blocks, $\tilde{Q}\tilde{p}$ have L^2-2 degrees of freedom in each block. Anyway, we obtain

$$\begin{aligned} \min F_1 &= \frac{k}{4N}\langle C^n : \varphi_{n+1}^2 :, \frac{1}{C^n[2\mathcal{G}_{n+1}\circ \mathcal{T}_n + \mathcal{T}_n^{\circ 2}](C^+)^n} C^n : \varphi_{n+1}^2 :\rangle \\ \min F_2 &= \frac{1}{4N}\langle E^\perp : \varphi_{n+1}^2 :, \frac{1}{2\mathcal{G}_{n+1}\circ \mathcal{T}_n + \mathcal{T}_n^{\circ 2}} E^\perp : \varphi_{n+1}^2 :\rangle \end{aligned}$$

We integrate over p_1 and $\tilde{p}$ around the points (4.9) and (4.10) (steepest descent method) and we get some small terms coming from the integrations over p_1 and $E^\perp\tilde{p}$.

The term $\min F_1$ means that the δ term disappears and the coefficient of the relevant term $(:\varphi_{n+1}^2:)^2$ can be regarded as a constant for $n > \log\beta$ since $C^{n+1} : \varphi_{n+1}^2 :\sim: \phi_{n+1}^2 :$ (field on Λ_n) and $C^n[2\mathcal{G}_{n+1}\circ \mathcal{T}_n + \mathcal{T}_n^{\circ 2}](C^+)^n \sim 1$ (on Λ_n). This also implies that

$$\langle : \varphi_{n+1}^2 :_{G_{n+1}} + p, \psi_n\rangle \;\to\; \frac{1}{L^{2n}}\langle : \varphi_{n+1}^2 :_{G_{n+1}}, E\psi_n\rangle \qquad (4.11)$$

which is consistent with our choice of the scaling of ψ and $\tilde{A}_n$. The term $\min F_2$ is essentially $\mathcal{F}_n$ which is irrelevant. We remark that the log term is expanded and $: \varphi_{n+1}\varphi_{n+1} :$

is absorbed by $V_n^{(1)}$ and the Hamiltonian part of ϕ_{n+1} through

$$2 : \varphi_{n+1}(x)\varphi_{n+1}(y) :=: \varphi_{n+1}^2(x) : + : \varphi_{n+1}^2(y) : - : (\varphi_{n+1}(x) - \varphi_{n+1}(y))^2 :$$

The shifts of the variables p_1 and $\tilde{Q}\tilde{p}$ are in the admissible deviations of φ_{n+1} and q_n.
[**step 4**]
Thus we can iterate these steps. The most important point is that $q =: \varphi_n^2 : - : \varphi_{n+1}^2 :$ obeys the Gasussian distribution uniformly in n (CLT) and the coefficient δ_n is kept as a constant on the shell : $\varphi_n^2 :_{G_n} = 0$ near which the functional integrals have supports. This ensures our scenario.

5 Remaining Problems

The following problems remain:

1. Prove this for small N.

2. Prove this for quantum spins.

3. Solve the Millennium problem of quark confinement.

The present author hopes that the reader is ambitious enough to attack these problems.

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