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SPDEs deduced from evolutional models of two-dimensional Young diagrams

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Abstract

It is a survey on the study of limit shapes and fluctuation around relative limit shapes for both the static and the dynamic two-dimensional Young diagrams. In particular, by studying the non-equilibrium fluctuation problem [6] around their hydrodynamic limits established in [5] for the dynamics of two-dimensional Young diagrams associated with the uniform and restricted uniform statistics respectively, two linear stochastic partial differential equations are deduced to characterize the fluctuation limits. We will also see that the results for limit shapes and fluctuation obtained in the static level can be reinterpreted by that of the evolutional models of two-dimensional Young diagrams.

1 Introduction

This note is survey for random Young diagram both in static and dynamic levels. In particular, by the research on the fluctuation of the dynamics of two dimensional Young diagrams around its hydrodynamic limit [6], we connect the theory of stochastic partial differential equations (SPDEs, for short) with the research of large scale interacting systems, which is an important application of SPDEs.

The stochastic partial differential equation is one of the most dynamically developing areas of mathematics which lies at the cross section of probability, partial differential equations, population biology, mathematical physics, fluid mechanics, geophysics and finance and others. The research on SPDEs is especially attractive because of its interdisciplinary nature and the enormous richness of current and potential future applications. Roughly speaking, SPDEs are partial differential equations added with proper noises, like space-time Gaussian noise and colored noises. SPDEs have been used to characterize many important models from various fields, such as stochastic Navier-Stokes equations, stochastic Burgers equations, the curve of interest rate, filtering. Especially, we point out that SPDEs often appear in the study of stochastic analysis on large scale interacting systems. For example, Funaki and Olla [4] investigate the fluctuations of the height of a $\nabla\phi$ interface model on a wall and prove that its limit is the solution of stochastic heat differential equation with reflection. Please see the lecture note [3] by Funaki for more information relative to $\nabla\phi$ interface model.

In this note, we will deduce two SPDEs by studying the fluctuation of the evolutional model of two-dimensional Young diagrams, which are initially introduced by Funaki and Sasada [5] and the main results in published in [6] in 2013. Young diagrams are convenient graphical representation of the partition of integers, named after A. Young from 1900 and are very useful in the theory of group representation. Before the work of [5], all of the study on random Young diagrams is in the static level from the initial work of Vershik [9] in 1996. However, in the real world, everything is developed, so it is important to consider the evolutional model for Young diagrams.

The following is organized as follows. In the next section, we will give an explanation of Young diagrams and the review some result on the static model of Young diagrams. In Section 3, we will introduce the dynamic model of 2-D Young diagrams and consider the hydrodynamic limits of them based on [5] and then in the last section, we will review the results of fluctuation problems around the the hydrodynamic limits of the dynamic model of 2-D Young diagrams. It is shown that the limits are characterized by two SPDEs.

2 Static Model of 2-D Young diagrams

A Young diagram is a diagram relative to partition of positive integers, which is a combinatorial object useful in group representation theory of the symmetric and general linear groups and named after by A. Young from 1900. In this part, we mainly present some results on the limit shape and fluctuation around it of the static Young diagrams associated with the uniform and restricted uniform statistics respectively.

To make it clear, let us first introduce the partition of positive integers. For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the set of all partitions $p = (p_i)_{i\in\mathbb{N}}$ satisfying $p_1 \geq p_2 \geq \cdots \geq p_i \geq \cdots$, $p_i \in \mathbb{Z}_+$ and $n(p) := \sum_{i\in\mathbb{N}} p_i = n$. For each partition $p \in \mathcal{P}_n$, the two dimensional Young diagram associated to it is the diagram by piling up j sticks of height 1 and side-length p_i, see Figure 2 below.

Vershik in 1996 [9] connected the Young diagrams with the theory of probability. He introduced some kinds of probability measures on Young diagrams known as statistical mechanics and then studied the limit shapes of the boundary of the Young diagrams, i.e., the law of large numbers in probability terminology. There are many examples, which particularly include the uniform and restricted uniform statistics. These probabilities are known as canonical in Physics and the results relative to them will be mainly summarized for our goal.

For each n, the uniform statistics is a uniform probability measure denoted by μ_U^n on the $\mathcal{P}_n$ and the restricted uniform statistics is a uniform probability measure denoted by μ_R^n considered on the set $\mathcal{Q}_n$, which is collection of all partitions with $q = (q_i)_{i\in\mathbb{N}} \in \mathcal{P}_n$ and $q_i > q_{i+1}$ if $q_i > 0$. It is clear that $\mathcal{Q}_n$ is a proper subset of $\mathcal{P}_n$. The uniform statistics is also called the Bose statistics, which applies only to those particles not limited to single occupancy of the same state. The restricted uniform statistics is also called the Fermi statistics, that is, the particles in the systems satisfies the Pauli exclusion principle: two particles can not occupy the same state at the same time.

To state his results for limit shapes, let us here introduce the height function of the Young diagrams,

$$\psi_p(u) := \sum_{i\in\mathbb{N}} 1_{\{u<p_i\}} = \sharp\{i \in \mathbb{N} : \ p_i > u\}, \ u \geq 0.$$

and the scaled height function in the static level

$$\tilde{\psi}_p^N(u) = \frac{1}{N}\psi_p(Nu), \ u \geq 0.$$

Clearly, under such scaling, the area equals to 1, that is, $\int_0^\infty \tilde{\psi}_p^N(u)du = 1$. Furthermore, it is known that $\tilde{\psi}_p^N(u)$ is non-negative, non-increasing and cadlag(right continuous with left limits).

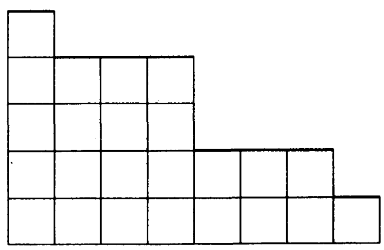

Figure 1: Young Diagram and the height for $n = 24 = (8, 7, 4, 4, 1)$.

Let us also denote by $\mathcal{D}(A)$ the space of all functions ψ defined on A which are $\tilde{\psi}_p^N(u)$ is non-negative, non-increasing and cadlag, where $A = \mathbb{R}_+$ or $A = \mathbb{R}_+^\circ := (0, \infty)$. We will endow it with the topology of uniform convergence on any compact subsets,

In [9], Vershik proved the law of large numbers holds, see also [7]. Such kind of results are usually called the equivalence of ensembles in the field of statistical physics.

Proposition 2.1. *For the uniform statistics, under the probability measure $\mu_U^{N^2}$, it is shown that $\tilde{\psi}_p^N(u)$ converges to the curve*

$$\psi_U(u) = -\frac{1}{\alpha}\log(1 - e^{-\alpha u}),$$

in probability in $\mathcal{D}(\mathbb{R}_+)$, where $\alpha = \pi/\sqrt{6}$.

Similarly, for the restricted uniform statistics, under the probability measure $\mu_R^{N^2}$, $\tilde{\psi}_q^N(u)$ converges to the curve

$$\psi_R(u) = \frac{1}{\beta}\log(1 + e^{-\beta u}).$$

in probability in $\mathcal{D}(\mathbb{R}_+^\circ)$ where $\beta = \pi/\sqrt{3}$.

From the view point of physics, the limit shapes obtained in the above give the explanation of the limit distribution of the particles' energy. In addition, the limit shapes ψ_U and ψ_R obtain in the above will be called Vershik's curves.

Now it is natural to study the fluctuations of the random interfaces around their limit shape under some proper scaling. Such problem is an equivalent of the central limit theorem in probability theory, which is also studied by many authors, see for example, Vershik et. al.[2] and Yakubovich [10]. Let

$$\Psi_N^U(u) := \sqrt{N}(\tilde{\psi}_p^N(u) - \psi_U(u)), \ u > 0$$
$$\Psi_N^R(u) := \sqrt{N}(\tilde{\psi}_q^N(u) - \psi_R(u)), \ u \geq 0,$$

where $\psi_U(u)$ and $\psi_R(u)$ are Vershik's curve, i.e., the limit shapes obtained in Proposition 2.1. Then the following fluctuations around the Vershik's curves are established in [2, 8, 10], see also [6] for the grandcanonical ensembles.

Proposition 2.2. *The fluctuation fields $\Psi_U^N(u)$ and $\Psi_R^N(u)$ weakly converge to $\Psi_U(u)$ and $\Psi_R(u)$ under $\mu_U^{N^2}$ and $\mu_R^{N^2}$, where Ψ_U, Ψ_R are mean 0 Gaussian processes with covariance structures*

$$C_U(u,v) = \frac{1}{\alpha}\rho_U(u \vee v), \quad u,v > 0 \quad \text{and} \quad C_R(u,v) = \frac{1}{\beta}\rho_R(u \vee v), \quad u,v \geq 0,$$

and $\rho_U = -\psi_U'$, $\rho_R = -\psi_R'$ are slopes of the Vershik's curves, respectively.

These results can be reinterpreted from the dynamic point of view by identifying the static fluctuation limits with the invariant measures of the limit SPDEs, which are obtained in Section 4 below.

3 Hydrodynamic limits of for the evolutional model of Young diagrams

In the above section, we reviewed the convergence of height function of static random Young diagrams and the fluctuation around its limit shape. From this section, we will introduce the dynamics of two-dimensional Young diagrams based on the paper [5] by Funaki and Sasada.

Let us first introduce the grandcanonical ensembles μ_U^ε on $\mathcal{P} := \cup_n \mathcal{P}_n$ and μ_U^ε on $\mathcal{Q} := \cup_n \mathcal{Q}_n$ respectively, where $\varepsilon \in (0,1)$. Let define

$$\mu_U^\varepsilon(p) = \varepsilon^{n(p)}/Z_U(\varepsilon), p \in \mathcal{P}$$

on $\mathcal{P}$ and

$$\mu_R^\varepsilon(q) = \varepsilon^{n(q)}/Z_R(\varepsilon), q \in \mathcal{Q}$$

with the normalizing constants

$$Z_U(\varepsilon) = \prod_{k=1}^{\infty}(1-\varepsilon^k)^{-1} \text{ and } Z_R(\varepsilon) = \prod_{k=1}^{\infty}(1+\varepsilon^k).$$

It is known that $\mu_U^\varepsilon(p)$ and $\mu_R^\varepsilon(q)$ are two probability measures and the conditional probabilities on $\mathcal{P}_n$ and $\mathcal{Q}_n$ respectively identify with canonical ensembles introduced in Section 2.

Let us now introduce the dynamics of two-dimensional Young diagrams constructed in [5].

For $0 < \varepsilon < 1$, the dynamics $p_t := p_t^\varepsilon = (p_i(t))_{i\in\mathbb{N}}$ on $\mathcal{P}$ and $q_t := q_t^\varepsilon = (q_i(t))_{i\in\mathbb{N}}$ on $\mathcal{Q}$ are introduced as Markov processes on these spaces having generators $L_{\varepsilon,U}$ and $L_{\varepsilon,R}$, respectively, defined as follows. The operator $L_{\varepsilon,U}$ acts on functions $f : \mathcal{P} \to \mathbb{R}$ as

$$L_{\varepsilon,U}f(p) = \sum_{i\in\mathbb{N}}\left[\varepsilon 1_{\{p_{i-1}>p_i\}}\{f(p^{i,+}) - f(p)\} + 1_{\{p_i>p_{i+1}\}}\{f(p^{i,-}) - f(p)\}\right],$$

while the operator $L_{\varepsilon,R}$ acts on functions $f : \mathcal{Q} \to \mathbb{R}$ as

$$L_{\varepsilon,R}f(q) = \sum_{i\in\mathbb{N}}\left[\varepsilon 1_{\{q_{i-1}>q_i+1\}}\{f(q^{i,+}) - f(q)\} + 1_{\{q_i>q_{i+1}+1 \text{ or } q_i=1\}}\{f(q^{i,-}) - f(q)\}\right],$$

where $p^{i,\pm} = (p_j^{i,\pm})_{j\in\mathbb{N}} \in \mathcal{P}$ are defined for $i \in \mathbb{N}$ and $p \in \mathcal{P}$ by $p_j^{i,\pm} = p_j$ for $j \neq i$, and $p_i \pm 1$ for $j = i$ and $q^{i,\pm} \in \mathcal{Q}$ similarly for $q \in \mathcal{Q}$. By convention, we set $p_0 = q_0 = \infty$.

For each initial probability measure ν on $\mathcal{P}$ or $\mathcal{Q}$, we will denote by $\mathbb{P}_\nu^N$ or $\mathbb{Q}_\nu^N$ the law of the trajectory of the Markov process p_t^ε or q_t^ε on $\mathcal{D}(\mathbb{R}_+^\circ, \mathcal{P})$ or $\mathcal{D}(\mathbb{R}_+, \mathcal{Q})$ respectively. Then, for every $\varepsilon \in (0,1)$, these Markov processes constructed in the above have the grandcanonical ensembles $\mu_U^\varepsilon(p)$ and $\mu_R^\varepsilon(q)$ as their invariant measures, respectively.

In the following, we will choose $\varepsilon = \varepsilon(N)(= \varepsilon_U(N), \varepsilon_R(N))$ in such a way that in each case the averaged size of the Young diagrams under μ_U^ε or μ_R^ε is equal to N^2, i. e.

$$E_{\mu_U^{\varepsilon(N)}}[n(p)] = N^2 \quad \text{and} \quad E_{\mu_R^{\varepsilon(N)}}[n(q)] = N^2.$$

Then under the above conditions, the behaviors of $\varepsilon(N)$ as $N \to \infty$ are formulated as below.

Lemma 3.1.

$$\varepsilon_U(N) = 1 - \frac{\alpha}{N} + O\Big(\frac{\log N}{N^2}\Big), \quad \alpha = \pi/\sqrt{6}$$

and

$$\varepsilon_R(N) = 1 - \frac{\beta}{N} + O\Big(\frac{\log N}{N^2}\Big), \quad \beta = \pi/\sqrt{12}.$$

Such precise behaviors are very important to study the hydrodynamical limit and especially, to the fluctuations of the evolutional model of Young diagrams.

Define the corresponding height functions scaled in space and time of the dynamics of the two dimensional Young diagrams by

$$\tilde{\psi}_U^N(t,u) := \tilde{\psi}_{p^\varepsilon_{N^2t}}^N(u) = \frac{1}{N}\psi_{p^\varepsilon_{N^2t}}(Nu) \quad \text{and} \quad \tilde{\psi}_R^N(t,u) := \tilde{\psi}_{q^\varepsilon_{N^2t}}^N(u) = \frac{1}{N}\psi_{q^\varepsilon_{N^2t}}(Nu),$$

with $\varepsilon = \varepsilon(N)$ and recalling that $\tilde{\psi}_p^N(u) = \frac{1}{N}\sum_{i\in\mathbb{N}} 1_{\{Nu<p_i\}}$, $u \in \mathbb{R}_+$.

Let us now roughly summarize the results of hydrodynamic limit for dynamics of Young diagrams obtained in [5] for the uniform statistics and the restricted statistics respectively, please see Theorems 2.1 and Theorems 2.2 [5].

Theorem 3.2. *Let $(\nu^N)_{N\in\mathbb{N}}$ be a sequence of probability measures on $\mathcal{P}$ with the property: there exists a function $\psi_0 \in X_U$ such that for many $\delta > 0$ and every function f with compact support in $\mathbb{R}_+^\circ$*

$$\lim_{N\to\infty} \nu^N \left[\left| \int_0^\infty f(u)\tilde{\psi}_p^N(u)du - \int_0^\infty f(u)\psi_0(u)du \right| > \delta \right] = 0.$$

Then for each $t > 0$, $\tilde{\psi}_U^N(t,u)$ converges to $\psi_U(t,u)$ in probability in the following sense: for any $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\nu^N}^N \left[\left| \int_0^\infty f(u)\tilde{\psi}_U^N(t,u)du - \int_0^\infty f(u)\psi(t,u)du \right| > \delta \right] = 0.$$

Here $\psi(t,u) := \psi_U(t,u)$ is a unique solution of the following nonlinear partial differential equation (PDE) in X_U:

$$\partial_t\psi = \left(\frac{\psi'}{1-\psi'}\right)' + \alpha\frac{\psi'}{1-\psi'}, \quad u \in \mathbb{R}_+^\circ, \tag{3.1}$$

with the initial condition $\psi(0,\cdot) = \psi_{U,0}(\cdot)$, *where*

$$X_U = \left\{\psi : \mathbb{R}_+^\circ \to \mathbb{R}_+^\circ; \psi \in C^1, \psi' < 0, \lim_{u\downarrow 0}\psi(u) = \infty, \lim_{u\uparrow\infty}\psi(u) = 0\right\}.$$

Theorem 3.3. *Let* $(\nu^N)_{N\in\mathbb{N}}$ *be a sequence of probability measures on* $\mathcal{Q}$ *with the property: there exists a function* $\psi_0 \in X_R$ *such that for many* $\delta > 0$ *and every function* f *with compact support in* $\mathbb{R}_+$

$$\lim_{N\to\infty} \nu^N\left[\left|\int_0^\infty f(u)\tilde{\psi}_q^N(u)du - \int_0^\infty f(u)\psi_0(u)du\right| > \delta\right] = 0.$$

Then for each $t > 0$, $\tilde{\psi}_U^N(t,u)$ *converges to* $\psi_U(t,u)$ *in probability in the following sense: for any* $\delta > 0$

$$\lim_{N\to\infty} \mathbb{Q}_{\nu^N}^N\left[\left|\int_0^\infty f(u)\tilde{\psi}_R^N(t,u)du - \int_0^\infty f(u)\psi(t,u)du\right| > \delta\right] = 0.$$

Here $\psi(t,u) := \psi_U(t,u)$ *is a unique solution of the following nonlinear partial differential equation (PDE) in* X_R*:*

$$\partial_t\psi = \psi'' + \beta\,\psi'(1+\psi'), \quad u \in \mathbb{R}_+, \tag{3.2}$$

with the initial condition $\psi(0,\cdot) = \psi_{R,0}(\cdot)$, *where*

$$X_R := \left\{\psi : \mathbb{R}_+ \to \mathbb{R}_+; \psi \in C^1, -1 \le \psi' \le 0,\ \psi'(0) = -1/2, \lim_{u\uparrow\infty}\psi(u) = 0\right\}.$$

It is easy to see that the Vershik's curves ψ_U and ψ_R obtained in the law of large numbers in Section 2 are the unique stationary solution in PDEs (3.1) and (3.2). In this sense, the derivation of the Vershik's curves is understandable from the dynamical point of view.

In the end of this section, we give some ideas to the proof of Theorem 3.2 and Theorem 3.3. For the grandcanonical uniform and restricted uniform statistics, it is known that if we consider the gradients of their height functions of the dynamics of p_t^ε and q_t^ε, then they can be transformed to a weakly asymmetric zero-range process, respectively a weakly asymmetric simple exclusion process on a set of positive integers with a stochastic reservoir at the boundary site 0 in both processes as natural time evolutions of the Young diagrams. The weakly asymmetric zero-range process can be further transformed into a weakly asymmetric simple exclusion process $\bar{\eta}_t \in \{0,1\}^{\mathbb{Z}}$ on the whole integer lattice $\mathbb{Z}$ without any boundary conditions by rotating the xy-plane around the origin by 45 degrees counterclockwise and projecting the system to the x-axis rescaled by $\sqrt{2}$. This involves quite a nonlinearity as observed in Section 4 of [5]. Anyway, to prove the hydrodynamic limits above, we can turn to study that of a weakly asymmetric simple exclusion process with a stochastic reservoir at the boundary site 0 and the discrete type of Hopf-Cole transformation is adapted, which avoid the one-block and two-block estimates. We point out for the restricted case, the study of the boundary behavior of the transformed process is vital.

4 Fluctuation in evolutional Young diagrams and invariant measures

Similar to the static level, when the hydrodynamic limit for dynamics of Young diagrams, it is natural to consider the fluctuation of the interface around hydrodynamic limit. In this part, we will review some results on the non-equilibrium dynamic fluctuation for two-dimensional evolutional Young diagrams associated with both uniform and restricted uniform statistics based on the work [6].

The theory of the equilibrium dynamic fluctuation around the hydrodynamic limit is well established based on the so-called Boltzmann-Gibbs principle. However, the results on the non-equilibrium dynamic fluctuations are rather limited and is due to a special feature of the models.

We will also see that the fluctuations studied in the static level can be reinterpreted from the dynamic point of view by identifying the static fluctuation limits with the invariant measures of the limit SPDEs.

More precisely, we consider the fluctuations processes $\tilde{\psi}_U^N(t,u)$ and $\tilde{\psi}_R^N(t,u)$ around their limits:

$$\Psi_U^N(t,u) := \sqrt{N}\big(\tilde{\psi}_U^N(t,u) - \psi_U(t,u)\big) \quad \text{and} \quad \Psi_R^N(t,u) := \sqrt{N}\big(\tilde{\psi}_R^N(t,u) - \psi_R(t,u)\big),$$

which are elements of $D([0,T], D(\mathbb{R}_+^\circ))$ and $D([0,T], D(\mathbb{R}_+))$, respectively. Here let us first state the result for the case of the restricted uniform statistics [6]. In the following, we will omit all of the assumptions for our main results, please see [6] for details.

Theorem 4.1. *(RU-case) Under some proper assumption, the fluctuation process $\Psi_R^N(t,u)$ converges weakly to $\Psi_R(t,u)$ as $N \to \infty$ on the space $D([0,T], D(\mathbb{R}_+))$ for every $T > 0$. The limit $\Psi_R(t,u)$ is in $C([0,T], C(\mathbb{R}_+))$ (a. s.) and characterized as a solution of the following SPDE:*

$$\begin{cases} \partial_t \Psi_R(t,u) = \Psi_R''(t,u) + \beta(1 - 2\rho_R(t,u))\Psi_R'(t,u) \\ \qquad\qquad\qquad + \sqrt{2\rho_R(t,u)(1-\rho_R(t,u))}\dot{W}(t,u), \\ \Psi_R'(t,0) = 0, \\ \Psi_R(0,u) = \Psi_{R,0}(u), \end{cases} \tag{4.1}$$

where $\rho_R(t,u) = -\psi_R'(t,u)$ and $\dot{W}(t,u)$ is the space-time white noise on $[0,T] \times \mathbb{R}_+$.

Here space-time white noise $\dot{W}(t,u)$ means that it is a Gaussian type space-time noise with covariance

$$\mathbb{E}[\dot{W}(t,u)\dot{W}(s,v)] = \delta(t-s)\delta(u-v).$$

Let us give some explanation for the concept of the solutions to the SPDE (4.1). We first introduce the concept of weak solutions. Similar to that in PDEs, we say

$\Psi_R(t,u)$ is a solution of the SPDE (4.1) if it is adapted, satisfies $\Psi_R \in C([0,T], C(\mathbb{R}_+)) \cap C([0,T], L^2_e(\mathbb{R}_+))$ (a. s.) and for every $f \in C_0^{1,2}([0,T] \times \mathbb{R}_+)$ satisfying $f'(t,0) = 0$ the following holds:

$$\begin{aligned}\langle \Psi_R(t), f(t)\rangle =&\langle \Psi_{R,0}, f(0)\rangle + \int_0^t \langle \Psi_R(s), f''(s) - \beta((1-2\rho_R(s))f(s))' + \partial_s f\rangle ds \\ &+ \int_0^t \int_{\mathbb{R}_+} f(s,u)\sqrt{2\rho_R(s,u)(1-\rho_R(s,u))}W(dsdu) \text{ a. s.}\end{aligned} \tag{4.2}$$

As in the theory of PDEs, we can show that the solution of (4.1) is equivalent to its mild form, that is, $\Psi_R(t,u)$ is an $L^2_e(\mathbb{R}_+)$-valued adapted process and the following holds:

$$\begin{aligned}\Psi_R(t,u) = &\int_{\mathbb{R}_+} g(t,u,v)\Psi_{R,0}(v)dv + \int_0^t \int_{\mathbb{R}_+} 2\beta g(t-s,u,v)\rho_R'(s,v)\Psi_R(s,v)dvds \\ &- \int_0^t \int_{\mathbb{R}_+} \frac{\partial}{\partial v} g(t-s,u,v)\beta(1-2\rho_R(s,v))\Psi_R(s,v)dvds \\ &+ \int_0^t \int_{\mathbb{R}_+} g(t-s,u,v)\sqrt{2\rho_R(s,v)(1-\rho_R(s,v))}W(dsdv) \text{ a. s.,}\end{aligned}$$

where $g(t,u,v)$ is the fundamental solution to $\partial_t \Psi(t,u) = \Psi''(t,u)$ with the homogeneous Neumann boundary condition at 0, that is

$$g(t,u,v) = \frac{1}{\sqrt{4\pi t}}\{e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\}, u, v \in \mathbb{R}_+.$$

It can be proved that the uniqueness of it also holds in the restricted uniform case.

The following, the two SPDEs will obtained by considering the problems of fluctuations. The definition for solutions can be introduced similarly as above. So we will omit the explanation at that time.

Before we state the result for the fluctuation of the uniform case, let us consider the fluctuation for its rotation process of it. In fact, a natural idea in the U-case is to investigate the fluctuation of the curve $\check{\psi}_U^N(t)$ around $\check{\psi}_U(t)$, which are obtained by rotating the original curves $\tilde{\psi}_U^N(t)$ and $\psi_U(t) = \{(u,y); y = \psi_U(t,u), u \in \mathbb{R}_+^\circ\}$ located in the first quadrant of the uy-plane by 45 degrees counterclockwise around the origin O, respectively, where $\tilde{\psi}_U^N(t)$ is a continuous indented curve obtained from the graph $\{(u,y); y = \tilde{\psi}_U^N(t,u), u \in \mathbb{R}_+\}$ of the original function $\tilde{\psi}_U^N(t,u)$ by filling all jumps by vertical segments. In particular, this contains a part of y-axis: $\{(0,y); y \geq \tilde{\psi}_U^N(t,0)\}$.

Then it is also interesting to consider the diffusive fluctuation

$$\check{\Psi}_U^N(t,v) := \sqrt{N}(\check{\psi}_U^N(t,v) - \check{\psi}_U(t,v)), \quad v \in \mathbb{R},$$

which is an element of $D([0,T], C(\mathbb{R}))$. The fluctuation $\check{\Psi}_U^N(t)$ defined as above is a natural object to study, since the Young diagrams corresponding to the class $\mathcal{P}$ belong to the same class under the reflection with respect to the line $\{y = u\}$, while those corresponding to $\mathcal{Q}$ do not have such property in general.

Now we can state the result for $\check{\Psi}_U^N(t)$ and then apply it to $\Psi_U^N(t)$.

Theorem 4.2. *(U-case under rotation) Under some proper assumption, the fluctuation process $\check{\Psi}_U^N(t,v)$ converges weakly to $\check{\Psi}_U(t,v)$ as $N \to \infty$ on the space $D([0,T], C(\mathbb{R}))$ for every $T > 0$. The limit $\check{\Psi}_U(t,v)$ is in $C([0,T], C(\mathbb{R}))$ (a. s.) and characterized as a solution of the following SPDE:*

$$\begin{cases} \partial_t \check{\Psi}_U(t,v) = \frac{1}{2}\check{\Psi}_U''(t,v) + \frac{\alpha}{\sqrt{2}}(1 - 2\rho(t,\sqrt{2}v))\check{\Psi}_U'(t,v) \\ \qquad\qquad + 2^{\frac{3}{4}}\sqrt{\rho(t,\sqrt{2}v)(1-\rho(t,\sqrt{2}v))}\dot{W}(t,v), \\ \check{\Psi}_U(0,v) = \check{\Psi}_{U,0}(v), \end{cases} \tag{4.3}$$

where $\rho(t,\cdot)$ is the solution of the PDE

$$\begin{cases} \partial_t \rho(t,v) = \rho''(t,v) + \alpha(\rho(t,v)(1-\rho(t,v)))', \quad t > 0, v \in \mathbb{R}, \\ \rho(0,v) = \rho_0(v). \end{cases}$$

and $\dot{W}(t,v)$ is the space-time white noise on $[0,T] \times \mathbb{R}$.

Although the directions of the fluctuations are different in $\check{\Psi}_U^N$ and Ψ_U^N, we still are able to deduce the next theorem from Theorem 4.2. As pointed out before, the transformation is nonlinear, so it is important that the convergence in Theorem 4.2 is shown in a function space $D([0,T], C(\mathbb{R}))$.

Theorem 4.3. *(U-case) Under some proper assumption, the fluctuation process $\Psi_U^N(t,u)$ converges weakly to $\Psi_U(t,u)$ as $N \to \infty$ on the space $D([0,T], D(\mathbb{R}_+^\circ))$ for every $T > 0$. The limit $\Psi_U(t,u)$ is in $C([0,T], C(\mathbb{R}_+^\circ))$ (a. s.) and a solution of the following SPDE:*

$$\begin{cases} \partial_t \Psi_U(t,u) = \left(\frac{\Psi_U'(t,u)}{(1+\rho_U(t,u))^2}\right)' + \alpha\frac{\Psi_U'(t,u)}{(1+\rho_U(t,u))^2} + \sqrt{\frac{2\rho_U(t,u)}{1+\rho_U(t,u)}}\dot{W}(t,u), \\ \Psi_U(0,u) = \Psi_{U,0}(u), \end{cases} \tag{4.4}$$

where $\rho_U(t,u) = -\psi_U'(t,u)$ and $\dot{W}(t,u)$ is the space-time white noise on $[0,T] \times \mathbb{R}_+^\circ$.

The existence of SPDE (4.4) is shown by our proof which depends on a nonlinear transformation. However, because of the nonlinearity, the uniqueness of it is just proved in some small function space, see Section 3[6].

Finally, let us study the invariant measures for the SPDEs (4.4) and (4.1). Roughly speaking, an invariant measure to a Markov process is a probability measure μ such that for any time, the process has the same law μ. Let

$$Q_U = -\frac{\partial}{\partial u}\Big\{\frac{1}{\rho_U(u)(1+\rho_U(u))}\frac{\partial}{\partial u}\Big\}$$

and

$$Q_R = -\frac{\partial}{\partial u}\Big\{\frac{1}{\rho_R(u)(1-\rho_R(u))}\frac{\partial}{\partial u}\Big\}.$$

Then we have the following results, see Theorem 5.2 and Theorem 5.4 in [6].

Theorem 4.4. *The Gaussian measure* $N(0, Q_U^{-1})$ *is the unique invariant measure of the SPDE* (4.4)*, which appeared in Theorem 4.3.*

Similarly, the Gaussian measure $N(0, Q_R^{-1})$ *is the unique invariant measure of the SPDE* (4.1)*, which appeared in Theorem 4.1.*

Since $C_U(u,v)$ and $C_R(u,v)$ are the Green kernels of Q_U^{-1} and Q_R^{-1} respectively, the above theorem gives another proof of fluctuation in the static level for the uniform and the restricted uniform case.

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