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A computer-assisted proof of the Kolmogorov problem of incompressible viscous fluid

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Abstract

A computer-assisted proof which proves the existence of non-trivial steady-state solutions for the Kolmogorov flows is presented. The method is based on the infinite-dimensional fixed-point theorem using Newton-like operator. This paper also proposes a numerical verification algorithm which generates automatically on a computer a set including the exact non-trivial solution with mathematical rigorous error bounds. All discussed numerical results are taken into account of the effects of rounding errors in the floating point computations.

1 Introduction

Consider the Navier-Stokes equations:

$$u_t + uu_x + vu_y = \nu\Delta u - \frac{1}{\rho}p_x + \gamma \sin\left(\frac{\pi y}{b}\right), \tag{1}$$

$$v_t + uv_x + vv_y = \nu\Delta v - \frac{1}{\rho}p_y, \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where (u, v), ρ, p and ν are velocity vector, mass density, pressure and kinematic viscosity, respectively and γ is a constant representing the strength of the sinusoidal outer force. Also $*_\xi := \partial/\partial\xi(\xi = t, x, y)$ and $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$. The flow region is a rectangle $[-a, a] \times [-b, b]$ and the periodic boundary conditions are imposed in both directions. The aspect ratio is denoted by $\alpha := b/a$.

The above equations (1–3) describe the Navier-Stokes flows in a two-dimensional flat torus under a special driving force proposed by Kolmogorov [1, 5], [6, Chapter 5] and have a basic solution which is written as

$$(u, v, p) = (k \sin(\pi y/b), 0, d),$$

where $k := b^2\gamma/(\pi^2\nu)$ and d is any constant. It is known that non-trivial solutions bifurcate from the basic solution at a certain Reynolds number, which is defined below, if and only if $0 < \alpha < 1$ [1]. Okamoto-Shoji [5] computed numerically bifurcation diagrams with the Reynolds number as a bifurcation parameter varying the aspect ratio as a splitting parameter. They also strongly suggested stability of the bifurcating solutions for all $0 < \alpha < 1$. Nagatou [2] took an another approach to this stability problem by employing the theory of verified computation and showed that the stability of the bifurcating solutions is mathematical rigorously assured for the cases of $\alpha = 0.4$, 0.7 and 0.8.

In the previous paper [10], we proposed a method to prove the existence and the local uniqueness of the steady-state solutions of the Navier-Stokes equations (1–3) for a given Reynolds number and aspect ratio by a computer-assisted proof with some verified results. It was also the first theoretical results to the non-trivial solutions of the equations (1–3).

The aim of this paper is to apply our other verification method: FN-Int[3, 4, 9] to prove the existence of the steady-state solutions of problem (1–3) and to ascertain its effectiveness in the actual numerical computation.

In FN-Int, the equation is decomposed into the finite-dimensional part and the infinite-dimensional error part, and if the both part lead to the retraction maps under suitable assumptions, an infinite-dimensional fixed-point theorem implies the existence of the solution in a certain function set. In the self-validating process in computer, Newton-like iteration is executed for the finite-dimensional part, and the computation comes down to solving interval linear systems. Note that we have also proposed some verification algorithms which assure the local uniqueness of the solution in the enclosed set [11, 12] . We will discuss about them in future works. We also note that our verification methods described above can be formulated as a more general form and one may apply it to many kind of differential equations and integral equations which can be transformed into fixed-point equations.

We admit that our study in this paper has some restrictions (a driving force, two-dimensional rectangle region, boundary condition, etc.), however, we believe that our idea, not our results themselves, will pave the way to a tool to study the global bifurcation structure for partial differential equations arising in more practical, or even industrial problems.

2 Nondimensionalization and function spaces

The letter $\mathbf{T}_\alpha$ denotes the rectangular region $(-\pi/\alpha, \pi/\alpha) \times (-\pi, \pi)$ for a given aspect ratio $0 < \alpha < 1$ (see Fig. 1). Introducing the stream function ϕ satisfying $u = \phi_y$ and $v = -\phi_x$ so

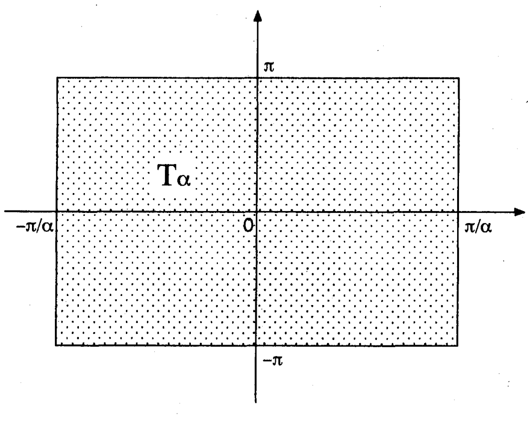


Figure 1: Shape of $\mathbf{T}_\alpha$

that $u_x + v_y = 0$, the equations (1–3) can be rewritten as

$$(\Delta\phi)_t - \nu\Delta^2\phi - J(\phi, \Delta\phi) = \frac{\gamma\pi}{b}\cos\left(\frac{\pi y}{b}\right) \tag{4}$$

by cross-differentiating equations (1) and (2) and eliminating the pressure p. Here J is a bilinear form defined by

$$J(u, v) := u_x v_y - u_y v_x. \tag{5}$$

The equation (4) is nondimensionalized by using change of variables

$$(x', y') = \left(\frac{\pi x}{b}, \frac{\pi y}{b}\right), \quad t' = \frac{\gamma b}{\nu \pi} t, \quad \phi'(t', x', y') = \frac{\nu \pi^3}{\gamma b^3} \phi(t, x, y)$$

and the Reynolds number $R := \dfrac{\gamma b^3}{\nu^2 \pi^3}$. After dropping the primes, an equation

$$(\Delta\phi)_t - \frac{1}{R}\Delta^2\phi - J(\phi, \Delta\phi) = \frac{1}{R}\cos(y) \tag{6}$$

is obtained.

We shall find *steady-state solutions*, where $(\Delta\phi)_t$ is equated to 0 in equation (6) in the region $\mathbf{T}_\alpha$, namely consider the following nonlinear problem:

$$\Delta^2\phi = -R\,J(\phi, \Delta\phi) - \cos(y) \quad \text{in} \quad \mathbf{T}_\alpha. \tag{7}$$

Assume that ϕ is subject to periodicity conditions in x and y, and the symmetry condition

$$\phi(x, y) = \phi(-x, -y) \tag{8}$$

as well as the normalization $\int_\Omega \phi\, dxdy = 0$ [2], then the equation (7) has a trivial solution $\phi = -\cos(y)$ for any $R > 0$ (Fig. 2). The aim of this paper is to enclose a non-trivial solution

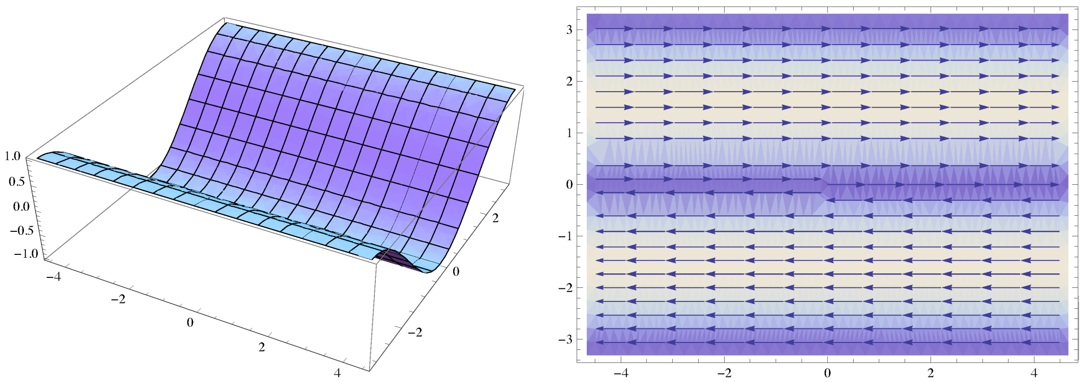


Figure 2: Shape of the trivial solution $\phi = -\cos(y)$ and stream line of $[\phi_y, -\phi_x]^T$.

of (7) by computer-assisted proof.

3 Function spaces

From the periodicity, the stream function ϕ can be expanded to double Fourier series by

$$\phi(x, y) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}} a_{m,n} e^{im\alpha x + iny}, \qquad a_{m,n} \in \mathbb{C}.$$

Note that if ϕ is a solution of (7), $\phi + c$ ($\forall c \in \mathbb{C}$) is also the solution. Then we exclude the case $(m, n) = (0, 0)$. By using Euler's formula and symmetry condition $\phi(x, y) = \phi(-x, -y)$, it holds that

$$\phi(x, y) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}} a_{m,n}\big(\cos(m\alpha x + ny) + i\sin(m\alpha x + ny)\big),$$

$$\phi(-x,-y) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}} a_{m,n}(\cos(m\alpha x+ny) - i\sin(m\alpha x+ny)).$$

Then adding equations and translating coefficient $a_{m,n}$, we have

$$\phi(x,y) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}} a_{m,n}\cos(m\alpha x+ny).$$

Now decomposing indices and using the property of cosine together with replacing $a_{m,n}$, we obtain

$$\phi(x,y) = \sum_{1\leq n\leq\infty} a_n\cos(ny) + \sum_{1\leq m\leq\infty}\sum_{-\infty\leq n\leq\infty} a_{m,n}\cos(m\alpha x+ny).$$

Consequently, we can define function space X^k $(k\geq 0)$ by the closure in $H^k(\mathbf{T}_\alpha)$ of the linear hull of all functions

$$\cos(m\alpha x+ny), \qquad m\in\mathbb{N}_0,\ n\in\mathbb{Z},\ (m,n)\neq(0,0).$$

Especially we define

$$X := X^3.$$

For each $\psi\in X^k$ can be represented by

$$\psi = \sum_{(m,n)\in Q} A_{mn}\cos(m\alpha x+ny), \qquad A_{mn}\in\mathbb{R},$$

where

$$Q := \left\{(m,n)\in\mathbb{Z}\times\mathbb{Z} \;\middle|\; \begin{array}{l} \text{“}m=0 \text{ and } 1\leq n\leq\infty\text{'' or} \\ \text{“}1\leq m\leq\infty \text{ and } -\infty\leq n\leq\infty\text{''} \end{array}\right\}, \tag{9}$$

and it is noted that the base function of X^k satisfies

$$(\cos(m\alpha x+ny),\cos(k\alpha x+ly))_{L^2} = \begin{cases} \dfrac{2\pi^2}{\alpha} & \text{if } k=m \text{ and } l=n \\ 0 & \text{else} \end{cases} \tag{10}$$

for any $(m,n),(k,l)\in Q$, where $(\cdot,\cdot)_{L^2}$ means the usual L^2-inner product in $\mathbf{T}_\alpha$.

4 Projection and an a priori error estimate

Let X_N be the finite-dimensional subspace of X, which depends on a non-negative integer parameter N, defined by

$$X_N := \left\{\sum_{(m,n)\in Q_N} A_{mn}\cos(m\alpha x+ny) \;\middle|\; A_{mn}\in\mathbb{R}\right\}, \tag{11}$$

where

$$Q_N := \left\{(m,n)\in\mathbb{Z}\times\mathbb{Z} \;\middle|\; \begin{array}{l} \text{“}m=0 \text{ and } 1\leq n\leq N\text{'' or} \\ \text{“}1\leq m\leq N \text{ and } -N\leq n\leq N\text{''} \end{array}\right\}. \tag{12}$$

Then

$$K := \dim X_N = 2N(N+1).$$

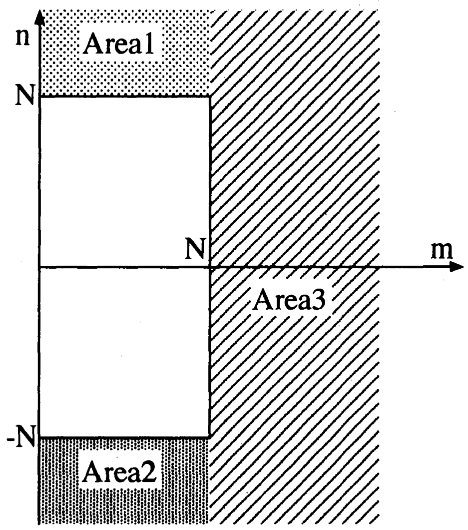
Let X_* denote the orthogonal complement of X_N in X such that $X = X_N \oplus X_*$, then for any $\psi_* \in X_*$ can be represented by

$$\psi_* = \sum_{(m,n)\in Q_*} A_{mn} \cos(m\alpha x + ny), \qquad A_{mn} \in \mathbb{R}, \tag{13}$$

where

$$Q_* := \left\{ (m,n) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; \begin{array}{l} \text{“}0 \le m \le N \text{ and } N+1 \le n \le \infty\text{''} \text{ or} \\ \text{“}1 \le m \le N \text{ and } -\infty \le n \le -N-1\text{''} \text{ or} \\ \text{“}N+1 \le m \le \infty \text{ and } -\infty \le n \le \infty\text{''} \end{array} \right\}. \tag{14}$$

Figure 3 indicates the area of Q_*.



Area1 $= \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \le m \le N$ and $N+1 \le n \le \infty\}$,
Area2 $= \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \le m \le N$ and $-\infty \le n \le -N-1\}$,
Area3 $= \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid N+1 \le m \le \infty$ and $-\infty \le n \le \infty\}$.

Figure 3: Area of Q_*

Now we define the projection $X \to X_N$ by the N-th truncation of Fourier expansion. Note that by the orthogonality of of the basis P_N coinsides with usual H_0^2-projection:

$$(\,\Delta(\psi - P_N\psi), \Delta\psi_N\,)_{L^2} = 0, \qquad \forall \psi_N \in X_N, \tag{15}$$

and we obtain the following a priori error estimate for the liner problem of $\Delta^2\xi = g$.

Lemma 4.1 For each $g \in X^0$ let $\xi \in X^4$ the solution of $\Delta^2\xi = g$, then

$$\|\xi - P_N\xi\|_X \le C_5\|g\|_{L^2}$$

holds, where

$$C_5 = \frac{1}{\alpha(N+1)}. \tag{16}$$

5 Some estimations for X

Since

$$\begin{aligned}|\psi|^2_{H^3(\Omega)} &= \|u_{xxx}\|^2_{L^2} + 3\|u_{xxy}\|^2_{L^2} + 3\|u_{xyy}\|^2_{L^2} + \|u_{yyy}\|^2_{L^2} \\ &= \frac{2\pi^2}{\alpha} \sum_{(m,n)\in Q} (\alpha^6 m^6 + 3\alpha^4 m^4 n^2 + 3\alpha^2 m^2 n^4 + n^6) A^2_{mn} \\ &= \frac{2\pi^2}{\alpha} \sum_{(m,n)\in Q} (\alpha^2 m^2 + n^2)^3 A^2_{mn},\end{aligned}$$

we have

$$|\psi|_{H^3(\Omega)} = \pi\sqrt{\frac{2}{\alpha}} \times \sqrt{\sum_{(m,n)\in Q} (\alpha^2 m^2 + n^2)^3 A^2_{mn}}. \tag{17}$$

Therefore since semi-norm $|\psi|_{H^3(\Omega)}$ becomes norm of X, we define the norm and the inner-product of X by

$$\begin{aligned}\|\phi\|_X &:= |\psi|_{H^3(\Omega)}, \\ (u, v)_X &:= (u_{xxx}, v_{xxx})_{L^2} + 3(u_{xxy}, v_{xxy})_{L^2} + 3(u_{xyy}, v_{xyy})_{L^2} + (u_{yyy}, v_{yyy})_{L^2},\end{aligned}$$

respectively.

For the norm $\|\cdot\|_X$, the following estimations hold.

Lemma 5.1 For any $\psi \in X$, and $\forall\psi_* \in X_*$, it can be checked that

$$\begin{array}{rcl@{\qquad}rcl}
\|\psi\|_{L^2} &\le& \alpha^{-3}\|\psi\|_X, & \|\psi_*\|_{L^2} &\le& C_1\|\psi_*\|_X,\\
\|\psi_x\|_{L^2} &\le& \alpha^{-2}\|\psi\|_X, & \|(\psi_*)_x\|_{L^2} &\le& C_2\|\psi_*\|_X,\\
\|\psi_y\|_{L^2} &\le& C_3\|\psi\|_X, & \|(\psi_*)_y\|_{L^2} &\le& C_4\|\psi_*\|_X,\\
\|\nabla\psi\|_{L^2} &\le& \alpha^{-2}\|\psi\|_X, & \|\nabla\psi_*\|_{L^2} &\le& C_2\|\psi_*\|_X,\\
\|\nabla\psi_x\|_{L^2} &\le& \alpha^{-1}\|\psi\|_X, & \|\nabla(\psi_*)_x\|_{L^2} &\le& C_5\|\psi_*\|_X,\\
\|\nabla\psi_y\|_{L^2} &\le& C_6\|\psi\|_X, & \|\nabla(\psi_*)_y\|_{L^2} &\le& C_7\|\psi_*\|_X,\\
\|\Delta\psi\|_{L^2} &\le& \alpha^{-1}\|\psi\|_X, & \|\Delta\psi_*\|_{L^2} &\le& C_5\|\psi_*\|_X,\\
\|\Delta\psi_x\|_{L^2} &\le& \|\psi\|_X, & \|\Delta(\psi_*)_x\|_{L^2} &\le& \|\psi_*\|_X,\\
\|\Delta\psi_y\|_{L^2} &\le& \|\psi\|_X, & \|\Delta(\psi_*)_y\|_{L^2} &\le& \|\psi_*\|_X,
\end{array}$$

where

$$\begin{array}{rcl@{\qquad}rcl}
C_1 &=& \dfrac{1}{\alpha^3(N+1)^3}, & C_2 &=& \dfrac{1}{\alpha^2(N+1)^2},\\
C_3 &=& \max\left\{1,\ \dfrac{2\sqrt{3}}{9\alpha^2}\right\}, & C_4 &=& \max\left\{\dfrac{1}{(N+1)^2}, \dfrac{2\sqrt{3}}{9\alpha^2(N+1)^2}\right\},\\
C_5 &=& \dfrac{1}{\alpha(N+1)}, & C_6 &=& \max\left\{1,\ \dfrac{1}{2\alpha}\right\},\\
C_7 &=& \max\left\{\dfrac{1}{N+1}, \dfrac{1}{2\alpha(N+1)}\right\}. & & &
\end{array}$$

In actual calculations, L^∞-estimates proposed by Plum [7] are also needed.

Lemma 5.2 (Plum,1992 [7]) For any $\psi \in X$, the following assertion holds true:

$$\|\psi\|_{L^\infty} \leq C_8\|\psi\|_{L^2} + C_9\|\nabla\psi\|_{L^2} + C_{10}\|\Delta\psi\|_{L^2}, \tag{18}$$

where $\|\cdot\|_{L^\infty}$ is the sup-norm and

$$C_8 = \frac{\sqrt{\alpha}}{2\pi}, \quad C_9 = \frac{1.1548}{\sqrt{3}}\sqrt{\frac{\alpha^2+1}{\alpha}}, \quad C_{10} = \pi\frac{0.44722}{3}\sqrt{\frac{9\alpha^4+10\alpha^2+9}{5\alpha^3}}.$$

Lemma 5.1 and Lemma 5.2 imply L^∞-estimates immediately.

Lemma 5.3 For $\forall\psi \in X$ and $\forall\psi_* \in X_*$, it is ture that

$$\begin{array}{ll} \|\psi\|_{L^\infty} \leq C_{11}\|\psi\|_X, & \|\psi_*\|_{L^\infty} \leq C_{12}\|\psi\|_X, \\ \|\psi_x\|_{L^\infty} \leq C_{13}\|\psi\|_X, & \|(\psi_*)_x\|_{L^\infty} \leq C_{14}\|\psi_*\|_X, \\ \|\psi_y\|_{L^\infty} \leq C_{15}\|\psi\|_X, & \|(\psi_*)_y\|_{L^\infty} \leq C_{16}\|\psi_*\|_X, \end{array}$$

where

$$\begin{array}{lllllll} C_{11} & = & \alpha^{-3}C_8 + \alpha^{-2}C_9 + \alpha^{-1}C_{10}, & & C_{12} & = & C_1C_8 + C_2C_9 + C_5C_{10}, \\ C_{13} & = & \alpha^{-2}C_8 + \alpha^{-1}C_9 + C_{10}, & & C_{14} & = & C_2C_8 + C_5C_9 + C_{10}, \\ C_{15} & = & C_3C_8 + C_6C_9 + C_{10}, & & C_{16} & = & C_4C_8 + C_7C_9 + C_{10}. \end{array}$$

We mention about partial integrations at finite-dimensional part. Let

$$Y^1 := \left\{ v = \sum_{(m,n)\in Q} A_{mn}\sin(\alpha mx + ny) \,|A_{mn} \in \mathbb{R},\ \|\nabla v\|_{L^2} < \infty \right\},$$

then it holds true.

Lemma 5.4

$$\begin{array}{ll} (\psi_x, \phi)_{L^2} = -(\psi, \phi_x)_{L^2}, & \psi \in X^1,\ \phi \in Y^1. \\ (\psi_y, \phi)_{L^2} = -(\psi, \phi_y)_{L^2}, & \psi \in X^1,\ \phi \in Y^1. \\ (\Delta\psi, \phi)_{L^2} = (\psi, \Delta\phi)_{L^2}, & \psi, \phi \in X^2. \\ (\Delta\psi, \Delta\phi)_{L^2} = (\Delta^2\psi, \phi)_{L^2}, & \psi \in X^4,\ \phi \in X^2. \\ (J(u,v), w)_{L^2} = (J(w,u), v)_{L^2} = -(J(u,w), v)_{L^2}, & u, v, w \in X^2. \end{array}$$

The following is an important property of Jacobian for (7).

Lemma 5.5 $\forall\psi_1, \psi_2 \in X^1$,

$$J(\psi_1, \psi_2) \in X^0,$$

namely $J(\psi_1, \psi_2)$ can be re-expanded by $\cos(m\alpha x + ny)$ $((m, n) \in Q)$.

6 Verification procedure

This section is devoted to detailed verification procedure of the steady-state Kolmogorov problem (7).

6.1 Matrices

For fixed $\phi_N \in X_N$, define $H, D, L, G \in \mathbb{R}^{K\times K}$ $(1 \le i, j \le K)$ by

$$\begin{aligned} H_{ij} &:= ((\phi_j)_{xxx}, (\phi_i)_{xxx})_{L^2} + 3((\phi_j)_{xxy}, (\phi_i)_{xxy})_{L^2} + \\ &\quad 3((\phi_j)_{xyy}, (\phi_i)_{xyy})_{L^2} + ((\phi_j)_{yyy}, (\phi_i)_{yyy})_{L^2}, \end{aligned} \tag{19}$$

$$D_{ij} := (\Delta\phi_j, \Delta\phi_i)_{L^2}, \tag{20}$$

$$L_{ij} := (\phi_j, \phi_i)_{L^2}, \tag{21}$$

$$G_{ij} := (\Delta\phi_j, \Delta\phi_i)_{L^2} + R(J(\phi_N, \Delta\phi_j) + J(\phi_j, \Delta\phi_N), \phi_i)_{L^2}. \tag{22}$$

6.2 Residual and fixed-point formulation

By setting

$$r_{2N} := -\Delta^2\phi_N - RJ(\phi_N, \Delta\phi_N) - \cos y, \tag{23}$$

r_{2N} is able to be re-expanded as an element in X_{2N} and we can compute its inner-product with $\{\phi_i\}_{i=1}^K$ and L^2-norm by interval arithmetic.

For fixed approximate solution $\phi_N \in X_N$ of (7), setting

$$\phi = \phi_N + \psi, \tag{24}$$

we try to find residual term ψ. Substituting (24) to (7), we obtain a residual equation

$$\Delta^2\psi = -R\ J(\phi_N + \psi, \Delta\phi_N + \Delta\psi) - \cos(y) - \Delta^2\phi_N \ \text{ in } \ \mathbf{T}_\alpha. \tag{25}$$

By denoting the right hand side of (25) by

$$f(\psi) := -R\ J(\phi_N + \psi, \Delta\phi_N + \Delta\psi) - \cos(y) - \Delta^2\phi_N, \tag{26}$$

from Lemma 5.5, $f : X \to X^0$ is continuous and maps any bounded set of X to a bounded set of X^0.

Denote $F := \Delta^{-2}f : X \to X$, then F becomes compact operator and problem (25) is equivalent to a fixed-point equation $\psi = F(\psi)$ in X.

6.2.1 Newton-like operator

By using the projection P_N, the fixed-point residual equation (25) can be decomposed into finite-dimensional part X_N and infinite-dimensional part X_* as

$$\left\{ \begin{array}{rcl} P_N\psi &=& P_N F(\psi), \\ (I - P_N)\psi &=& (I - P_N)F(\psi). \end{array} \right.$$

Now we define Newton-like operator $\mathcal{N}_N : X \to X_N$ by

$$\mathcal{N}_N(\psi) := P_N\psi - [I - F'[0]]_N^{-1} P_N(\psi - F(\psi)),$$

and re-formulate the finite-dimensional part equivalently to

$$P_N\psi = \mathcal{N}_N(\psi).$$

Here $[I - F'[0]]_N^{-1} : X_N \to X_N$ is the inverse operator of $P_N(I - F'[0]) : X \to X_N$ whose definition is restricted to X_N. Next we definie Newton-like operator T on X by

$$T(\psi) := \mathcal{N}_N(\psi) + (I - P_N)F(\psi)$$

which is the compact map.

6.2.2 Candidate set

Let $\mathbb{IR}$ be the set of K-dimensional interval vector. A finite-dimensional set $U_N \subset X_N$ is taken to be a set of linear combinations of base functions in X_N with interval coefficient $\{B_i\}_{1\le i\le K}$ such as

$$U_N := \sum_{i=1}^{K} B_i\phi_i, \tag{27}$$

where B_i has upper and lower bounds such that $B_i = [\underline{B}_i, \overline{B}_i]$. Here $\sum_{i=1}^{K} B_i\phi_i$ is interpreted as the set of functions in which each element is linear combination of $\{\phi_i\}_{1\le i\le K}$ whose coefficient of ϕ_i belongs to the corresponding interval $[\underline{B}_i, \overline{B}_i]$ for each $1 \le i \le K$, namely,

$$U_N = \left\{ \sum_{i=1}^{K} v_i\phi_i \in X_N \;\middle|\; v_i \in \mathbb{R},\; v_i \in B_i, \quad 1 \le i \le K \right\}. \tag{28}$$

For $\alpha > 0$, a infinite-dimensional set $U_* \subset X_*$ and a candidate set $U \subset X$ is taken to be

$$U_* := \{\psi_* \in X_* \mid \|\psi_*\|_X \le \beta\}, \tag{29}$$
$$U := U_N + U_*. \tag{30}$$

6.2.3 Verification condition

Theorem 6.1 Assume that the candidate set $U \subset X$ is defined by (29) and (27) with (30), and that any element $\psi \in U$ is represented by

$$\psi = \psi_N + \psi_*, \qquad \psi_N \in U_N, \quad \psi_* \in U_*.$$

Let $\boldsymbol{d} = [d_i] \in \mathbb{IR}^K$ denote an interval enclosure of the set whose i-th component consists of

$$\{ (f(\psi) - f'[0]\psi_N, \phi_i)_{L^2} \in \mathbb{R} \mid \psi \in U \}, \qquad 1 \le i \le K. \tag{31}$$

If, for an interval vector $\boldsymbol{v} = [v_i] \in \mathbb{IR}^K$ enclosing the solution $\boldsymbol{x} \subset \mathbb{IR}^K$ for the linear equation

$$G\boldsymbol{x} = \boldsymbol{d}, \tag{32}$$

the conditions

$$v_i \subset B_i, \qquad 1 \le i \le K, \tag{33}$$

and

$$\sup_{\psi\in U} \|(I - P_N)F(\psi)\|_X \le \beta \tag{34}$$

hold, then there exists a fixed-point of F in U.

7 Some verification results

We use Sun Fortran 95 Ver.8.6 Linux_i386 (supporting interval arithmetic) and the interval arithmetic toolbox INTLAB [8] Version 6 with MATLAB 7.14.0.739 (R2012a) on Fujitsu PRIMERGY TX300 S5 (CPU: Intel Xeon E5520 2.27GHz, OS: Red Hat Enterprise Linux Server release 5.6).

In the case of $\alpha = 0.7$, the basic flow (trivial solution) $-\cos(y)$ loses stability at a critical Reynolds number R_c and another steady state bifurcates. Okamoto-Shoji strongly suggested that there is no secondary bifurcation from this branch. Nagatou [2] also enclosed the R_c in the interval $[3.011528364444, 3.011528364446]$.

Figure 4 shows the bifurcation diagram, where $|A_{0,1}|$ means the absolute value of the coefficient to the base $\cos(ny)$ for obtained approximate solution.

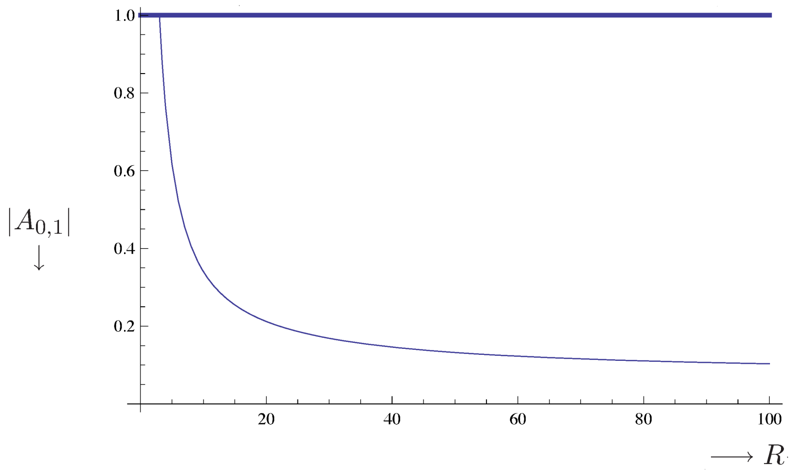


Figure 4: Bifurcation diagram for $\alpha = 0.7$

Figure 5 shows verification results for $R = 3.015, 3.02, 3.05, 3.1, 3.2,$ and 3.5 by IN-Linz. We will report on various verification results for various α in another article.

Acknowledgments

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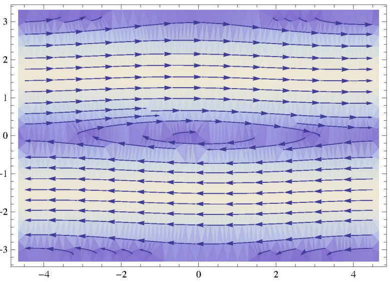
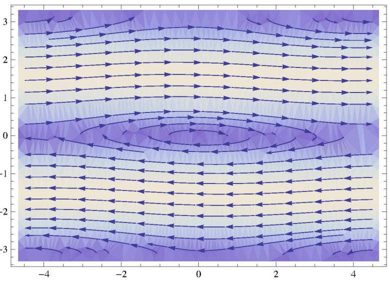
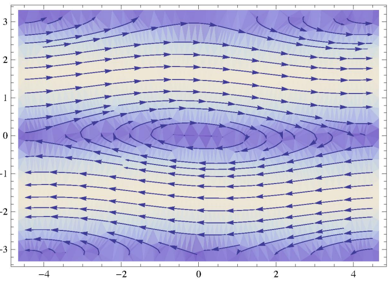


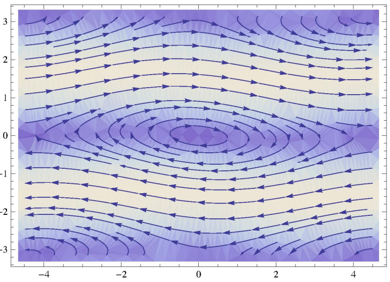
$R = 3.015$
$\|\phi - \phi_N\|_{L^\infty} \leq 2.0677 \times 10^{-11}$



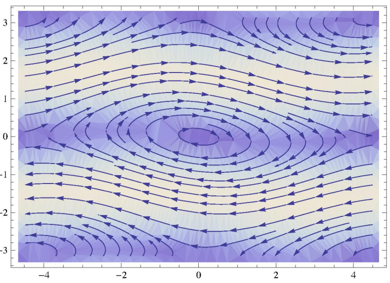
$R = 3.02$
$\|\phi - \phi_N\|_{L^\infty} \leq 1.2599 \times 10^{-11}$



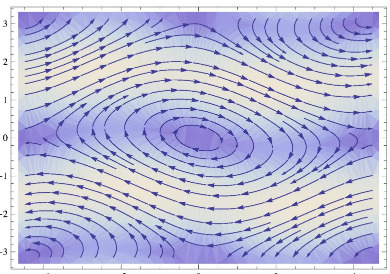
$R = 3.05$
$\|\phi - \phi_N\|_{L^\infty} \leq 9.8987 \times 10^{-12}$



$R = 3.1$
$\|\phi - \phi_N\|_{L^\infty} \leq 1.085610^{-11}$



$R = 3.2$
$\|\phi - \phi_N\|_{L^\infty} \leq 1.1318 \times 10^{-11}$



$R = 3.5$
$\|\phi - \phi_N\|_{L^\infty} \leq 1.4785 \times 10^{-11}$

Figure 5: Shape of the stream line of $[(\phi_N)_y, -(\phi_N)_x]^T$ when $\alpha = 0.7$, $N = 20$.

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