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L_p-L_q maximal regularity and its application

Yoshihiro SHIBATA *
Department of Mathematics and Research Institute of Science and Engineering,
Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan.
e-mail address: yshibata@waseda.jp

Miho MURATA
Department of Pure and Applied Mathematics, Graduate School of
Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan.
e-mail address: m-murata@ruri.waseda.jp

1 Introduction

We consider a local in time unique existence theorem for the motion of a compressible barotropic viscous fluid occupying a domain Ω of the N dimensional Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ with the boundary slip condition. Let $\rho = \rho(x,t)$ be the density of the fluid, $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), \ldots, v_N(x,t))$ the velocity vector field, and $P(\rho)$ the pressure function with $x = (x_1, \ldots, x_N) \in \Omega$ and t being the time variable. The motion is described by the following equations:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 & \text{in } \Omega \times (0,T), \\ \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{Div}\mathbf{S}(\mathbf{u}) + \nabla P(\rho) = 0 & \text{in } \Omega \times (0,T), \\ \mathbf{D}(\mathbf{u})\mathbf{n} - <\mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n}> \mathbf{n} = 0, \ \ \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma \times (0,T), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_* + \theta_0, \mathbf{u}_0) & \text{in } \Omega \end{cases} \tag{1.1}$$

(cf.[4, 6]), where $\partial_t = \partial/\partial t$, ρ_* is a positive constant describing the mass density of the reference body Ω, Γ the boundary of Ω and $\mathbf{n}$ the unit outward normal to Γ. Moreover, $P(\rho)$ is a C^∞ function defined on $\rho > 0$ satisfying the condition: $P'(\rho) > 0$ for $\rho > 0$ and $\mathbf{S}(\mathbf{u})$ the stress tensor of the form:

$$\mathbf{S}(\mathbf{u}) = \alpha\mathbf{D}(\mathbf{u}) + (\beta - \alpha)\operatorname{div}\mathbf{u}\mathbf{I},$$

where α and β are positive constants describing the first and second viscosity coefficients, respectively, $\mathbf{D}(\mathbf{v})$ denotes the deformation tensor whose (j,k) components are $D_{jk}(\mathbf{v}) =$

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$\partial_j v_k + \partial_k v_j$ with $\partial_j = \partial/\partial x_j$, and $\mathbf{I}$ the $N \times N$ identity matrix. Finally, for any matrix field $\mathbf{K}$ with components K_{jk}, $j, k = 1, \ldots, N$, the quantity $\operatorname{Div} K$ is an N-vector with j-th component $\sum_{k=1}^N \partial_k K_{jk}$, and also for any vector of functions $\mathbf{u} = (u_1, \ldots, u_N)$ we set $\operatorname{div} \mathbf{u} = \sum_{j=1}^N$ and $\mathbf{u} \cdot \nabla = \sum_{j=1}^N u_j \partial_j$ with $\nabla = (\partial_1, \ldots, \partial_N)$.

A local in time unique existence theorem was proved by Burnat and Zajączkowski [2] in a bounded domain of 3-dimensional Euclidean space $\mathbb{R}^3$, where their velocity field $\mathbf{u}$ and density of the fluid ρ belong to Sobolev-Slobodetskii spaces $W_2^{2+\alpha,1+\alpha/2}$ and $W_2^{1+\alpha,1/2+\alpha/2}$ with $\alpha \in (1/2, 1)$, respectively. Moreover, a global in time unique existence theorem was proved by Kobayashi and Zajączkowski [5] in the same class as in the local in time unique existence theorem by the energy method. The purpose of this paper is to prove a local in time unique existence theorem in a uniform $W_q^{3-1/q}$ * and our velocity field $\mathbf{u}$ and density of the fluid ρ belong to $W^{2,1}_{q,p}(\Omega \times (0,T))$ and $W^{1,1}_{q,p}(\Omega \times (0,T))$ with $2 < p < \infty$ and $N < q < \infty$, where we have set

$$W^{\ell,m}_{q,p}(\Omega \times (0,T)) = L_p((0,T), W^\ell_q(\Omega)) \cap W^m_p((0,T), L_q(\Omega)). \tag{1.2}$$

One of the merits of our approach is less compatibility condition than [2].

To obtain such local in time unique existence theorem, it is key to prove the L_p-L_q maximal regularity for the linearized problem of the following form:

$$\begin{cases} \partial_t \rho + \gamma_2 \operatorname{div} \mathbf{u} = f & \text{in } \Omega \times (0,\infty), \\ \gamma_0 \partial_t \mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}) + \nabla(\gamma_1 \rho) = \mathbf{g} & \text{in } \Omega \times (0,\infty), \\ \alpha[\mathbf{D}(u)\mathbf{u} - < \mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n} > \mathbf{n}] = \mathbf{h} - < \mathbf{h}, \mathbf{n} > \mathbf{n}, \quad \mathbf{u} \cdot \mathbf{n} = \tilde{h} & \text{on } \Gamma \times (0,\infty), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{1.3}$$

$\gamma_i = \gamma_i(x)$ $(i = 0, 1, 2)$ are uniformly continuous functions defined on $\bar{\Omega}$ satisfying the assumptions:

$$\rho_*/2 \le \gamma_0(x) \le 2\rho_*, \quad 0 \le \gamma_k(x) \le \rho_1 \quad (x \in \Omega, k = 1, 2), \quad \|\nabla \gamma_\ell\|_{L_r(\Omega)} \le \rho_1 \quad (\ell = 0, 1, 2) \tag{1.4}$$

with some positive constant ρ_1.

In order to show L_p-L_q maximal regularity, we prove the existence of $\mathcal{R}$-bounded solution operator to the following generalized resolvent problem corresponding to time dependent problem (1.3):

$$\begin{cases} \lambda\theta + \gamma_2 \operatorname{div} \mathbf{v} = f & \text{in } \Omega, \\ \gamma_0 \lambda \mathbf{v} - \operatorname{Div} \mathbf{S}(\mathbf{v}) + \nabla(\gamma_1 \theta) = \mathbf{g} & \text{in } \Omega, \\ \alpha[\mathbf{D}(\mathbf{v})\mathbf{n} - < \mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n} > \mathbf{n}] = \mathbf{h} - < \mathbf{h}, \mathbf{n} > \mathbf{n}, \quad \mathbf{v} \cdot \mathbf{n} = \tilde{h} & \text{on } \Gamma. \end{cases} \tag{1.5}$$

In fact, we prove that for any $\epsilon \in (0, \pi/2)$, there exist a constant $\lambda_0 \ge 1$ and an operator family $R(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W^2_q(\Omega)^N))$ such that for any $f \in W^1_q(\Omega$, $\mathbf{g} \in L_q(\Omega)^N$, $\mathbf{h} \in W^1_q(\Omega)^N$ and $\tilde{h} \in W^2_q(\Omega)$, problem (1.5) admits a unique solution $(\rho, \mathbf{v}) = R(\lambda)(\mathbf{f}, \mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h}, \lambda\tilde{h}, \lambda^{1/2}\nabla\tilde{h}, \nabla^2\tilde{h})$ and $(\lambda, \lambda^{1/2}\nabla P_v, \nabla^2 P_v)R(\lambda)$ is $\mathcal{R}$-bounded for $\lambda \in \Sigma_{\epsilon,\lambda_0} \cap K_\epsilon$ with value in $\mathcal{L}(\mathcal{X}_q(\Omega), W^1_q(\Omega) \times L_q(\Omega)^{\tilde{N}})$. Here, P_v is the

*The definition of $W_q^{3-1/q}$ domain is given in Definition 1.1, below.

projection such that $P_v(\rho, \mathbf{u}) = \mathbf{u}$, $\tilde{N} = N + N^2 + N^3$,

$$\begin{aligned}
\Sigma_{\epsilon,\lambda_0} &= \{\lambda \in \mathbb{C} \mid |\lambda| \geq \lambda_0, |\arg\lambda| \leq \pi - \epsilon\}, \\
K_\epsilon &= \{\lambda \in \mathbb{C} \mid (\lambda + \frac{\gamma}{\alpha+\beta} + \epsilon)^2 + (\mathrm{Im}\,\lambda)^2 \geq (\frac{\gamma}{\alpha+\beta} + \epsilon)^2\}, \qquad (1.6) \\
\mathcal{X}_q(\Omega) &= \{F = (F_1, \ldots, F_7) \mid F_1 \in W^1_q(\Omega), F_5 \in L_q(\Omega), F_2, F_3, F_6 \in L_q(\Omega)^N, \\
&\qquad\qquad F_4, F_7 \in L_q(\Omega)^{N^2}\},
\end{aligned}$$

with $\gamma = \sup_{x\in\overline{\Omega}} \gamma_1(x)\gamma_2(x)$, and F_1, F_2, F_3, F_4, F_5, F_6 and F_7 are independent variables corresponding to f, $\mathbf{g}$, $\lambda^{1/2}\mathbf{h}$ $\nabla\mathbf{h}$, $\lambda\tilde{h}$, $\lambda^{1/2}\nabla\tilde{h}$ and $\nabla^2\tilde{h}$, respectively. Moreover, $\mathrm{Hol}\,(U, \mathcal{L}(X, Y))$ denotes the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on a complex domain U and $\mathcal{L}(X, Y)$ the set of all bounded linear operators from a Banach space X into another Banach space Y. Since the solution $(\rho, \mathbf{u})$ to problem (1.3) is represented by the inverse Laplace transform of the solution $(\theta, \mathbf{v})$ to problem (1.5), so that the maximal L_p-L_q result for problem (1.3) is obtained with help of Weis' operator valued Fourier multiplier theorem [11].

Before ending the introduction, we summarize several symbols and functional spaces used throughout the paper. For the differentiations of scalar f and N-vector $\mathbf{g}$, we use the following symbols:

$$\begin{aligned}
\nabla f &= (\partial_1 f, \ldots, \partial_N f), & \nabla^2 f &= (\partial_i\partial_j f \mid i, j = 1, \ldots, N), \\
\nabla \mathbf{g} &= (\partial_i g_j \mid i, j = 1, \ldots, N), & \nabla^2 \mathbf{g} &= (\partial_i\partial_j g_k \mid i, j, k = 1, \ldots, N)
\end{aligned}$$

with $\partial_j \doteq \partial/\partial x_j$. For any Banach space X with norm $\|\cdot\|_X$, X^d denotes the d-product space of X, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. For any domain D, $L_q(D)$ and $W^m_q(D)$ denote the usual Lebesgue space and Sobolev space, while $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W^m_q(D)}$ denote their norms, respectively. We set

$$W^{m,\ell}_q(D) = \{(f, \mathbf{g}) \mid f \in W^m_q(D), \ \ \mathbf{g} \in W^\ell_q(D)^N\}$$

with $W^0_q(D) = L_q(D)$. For $1 \leq p \leq \infty$, $L_p((a, b), X)$ and $W^m_p((a, b), X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on the interval (a, b), while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W^m_p((a,b),X)}$ denote their norms, respectively. Set

$$\begin{aligned}
L_{p,\gamma_1}(\mathbb{R}, X) &= \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R}, X) \mid e^{-\gamma_1 t} f(t) \in L_p(\mathbb{R}, X)\}, \\
L_{p,\gamma_1,0}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0 \ (t < 0)\}, \\
W^m_{p,\gamma_1}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid e^{-\gamma_1 t}\partial^j_t f(t) \in L_p(\mathbb{R}, X) \ (j = 1, \ldots, m)\}, \\
W^m_{p,\gamma_1,0}(\mathbb{R}, X) &= W^m_{p,\gamma_1}(\mathbb{R}, X) \cap L_{p,\gamma_1,0}(\mathbb{R}, X), \\
\|e^{-\gamma t} f\|_{W^m_p(I,X)} &= \sum_{j=0}^m \Big(\int_I (e^{-\gamma t}\|\partial^j_t f(t)\|_X)^p \, dt\Big)^{1/p}
\end{aligned}$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}^{-1}_\xi = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by

$$\mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \quad \mathcal{F}^{-1}_\xi[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\, d\xi.$$

We also write $\hat{f}(\xi) = \mathcal{F}_x[f](\xi)$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform, respectively, which are defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\, dt, \quad \mathcal{L}^{-1}[g](\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\tau)\, d\tau$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. Given $s \in \mathbb{R}$ and X-valued function $f(t)$, we set

$$\Lambda_\gamma^s f(t) = \mathcal{L}_\lambda^{-1}[\lambda^s \mathcal{L}[f](\lambda)](t).$$

Moreover, the Bessel potential space of X-valued functions of order s is defined by the following:

$$\begin{aligned} H^s_{p,\gamma_1}(\mathbb{R}, X) &= \{f \in L_p(\mathbb{R}, X) \mid e^{-\gamma t}\Lambda^s_\gamma[f](t) \in L_p(\mathbb{R}, X) \text{ for any } \gamma \geq \gamma_1\},\\ H^s_{p,\gamma_1,0}(\mathbb{R}, X) &= \{f \in H^s_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0 \ (t < 0)\}. \end{aligned}$$

The letter C denotes generic constants and the constant $C_{a,b,\cdots}$ depends on a, b, $\cdots$. The values of constants C and $C_{a,b,\cdots}$ may change from line to line. $\mathbb{N}$ and $\mathbb{C}$ denote the set of all natural numbers and complex numbers, respectively, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Finally, we introduce the definition of a uniform $W_r^{3-1/r}$ domain, $\mathcal{R}$ boundedness of operator families and the Weis operator valued Fourier multiplier theorem.

Definition 1.1. Let $1 < r < \infty$ and let Ω be a domain in $\mathbb{R}^N$ with boundary Γ. We say that Ω is a uniform $W_r^{3-1/r}$ domain if there exists positive constants α, β and K such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma$ there exist a coordinate number j and a $W_r^{3-1/r}$ function $h(x')$ $(x' = (x_1, \ldots, \hat{x}_j, \ldots, x_N))$ defined on $B'_\alpha(x'_0)$ with $x'_0 = (x_{01}, \ldots, \hat{x}_{0j}, \ldots x_{0N})$ and $\|h\|_{W_r^{3-1/r}(B'_\alpha(x'_0))} \leq K$ such that

$$\begin{aligned} \Omega \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j > h(x') \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0),\\ \Gamma \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j = h(x') \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0). \end{aligned}$$

Here, $B'_\alpha(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\}$, $B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$ and $W_r^{3-1/r}(B'_\alpha(x'_0))$ denotes the set of all function $h \in W_r^2(B'_\alpha(x'_0))$ such that

$$\left\{\iint_{B'_\alpha(x_0)\times B'_\alpha(x_0)} \frac{|\partial_k\partial_l h(x') - \partial_k\partial_l h(y')|^r}{|x' - y'|^{N-2+r}} dx'dy'\right\}^{1/r} < \infty$$

for $k, l \neq j$ with $\partial_k\partial_l h = \partial^2 h/\partial x_k \partial x_l$.

Definition 1.2. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\left\{\int_0^1 \|\sum_{j=1}^n r_j(u) T_j x_j\|_Y^p\, du\right\}^{\frac{1}{p}} \leq C\left\{\|\sum_{j=1}^n r_j(u) x_j\|_X^p\, du\right\}^{\frac{1}{p}}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact supports and the Schwartz space of rapidly decreasing X valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)). \tag{1.7}$$

The following theorem is obtained by Weis [11].

Theorem 1.3. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\tau\frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}) \leq \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (1.7) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leq C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 1.4. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

2 Main Results

The following local in time unique existence theorem for (1.1) is our main result.

Theorem 2.1. *Let $2 < p < \infty$, $N < q < \infty$, $R > 0$ and assume that Ω be a uniform $W_q^{3-1/q}$ domain in $\mathbb{R}^N$. Let ρ_* be a positive constant and let $P(\rho)$ be a C^∞ function defined on $\rho > 0$ such that $\rho_1 < P'(\rho) < \rho_2$ with some positive constants ρ_1 and ρ_2 for any $\rho \in (\rho_*/4, 4\rho_*)$. Let $B_{q,p}^{2(1-1/p)}(\Omega)$ be the space defined in (2.2). Then, there exists a time T depending on R such that for any initial data $(\theta_0, \mathbf{u}_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)$ with $\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq R$ satisfying the range condition:*

$$\rho_*/2 < \rho_* + \theta_0(x) < 2\rho_* \quad (x \in \Omega) \tag{2.1}$$

problem (1.1) admits a unique solution $(\rho, \mathbf{u})$ with

$$\rho \in W_{q,p}^{1,1}(\Omega \times (0, T)), \quad \mathbf{u} \in W_{q,p}^{2,1}(\Omega \times (0, T)).$$

To obtain Theorem 2.1, first, we show the following result for the equation (1.3) with $f = 0$, $\mathbf{g} = 0$, $\mathbf{h} = 0$ and $\tilde{h} = 0$.

Theorem 2.2. *Let $1 < p, q < \infty$, $N < r < \infty$, $\max(q, q') \leq r$ $(q' = q/(q-1))$, $0 < \epsilon < \pi/2$, $\delta_0 > 0$ and $\lambda_0 > 0$. Assume that Ω is a uniform $W_r^{3-1/r}$ domain and that $|\delta| \leq \delta_0$. Set*

$$B_{q,p}^{2(1-1/p)}(\Omega) = (W_q^{1,0}(\Omega), \mathcal{D}_q(\Omega))_{1-1/p,p} \tag{2.2}$$

with real interpolation functor $(\cdot,\cdot)_{\theta,p}$. *Then, for any* $(\rho_0, \mathbf{u}_0) \in W^1_q(\Omega) \times B^{2(1-1/p)}_{q,p}(\Omega)$, *problem* (1.3) *with* $f = 0$, $\mathbf{g} = 0$, $\mathbf{h} = 0$ *and* $\tilde{h} = 0$ *admits a unique solution* $(\rho, \mathbf{u})$ *with*

$$\rho \in W^1_{p,\gamma_1}((0,\infty), W^1_q(\Omega)), \quad \mathbf{u} \in L_{p,\gamma_1}((0,\infty), W^2_q(\Omega)) \cap W^1_{p,\gamma_1}((0,\infty), L_q(\Omega))$$

possessing the estimate:

$$\begin{aligned}&\|e^{-\gamma t}\rho\|_{W^1_p(0,\infty),W^1_q(\Omega))} + \|e^{-\gamma t}\partial_t \mathbf{u}\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{-\gamma t}\mathbf{u}\|_{L_p((0,\infty),W^2_q(\Omega))}\\ &\quad\le C(\|\rho_0\|_{W^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)})\end{aligned}$$

for any $\gamma \ge \gamma_1$ *with some constant* C *depending on* p, q, γ_1 *and* N.

Secondly, we show the maximal L_p-L_q regularity result for the equation (1.3) with $\rho_0 = 0$ and $\mathbf{u}_0 = 0$.

Theorem 2.3. *Let* $1 < p, q < \infty$, $N < r < \infty$, $\max(q, q') \le r$ $(q' = q/(q-1))$, $0 < \epsilon < \pi/2$, $\delta_0 > 0$ *and* $\lambda_0 > 0$. *Assume that* Ω *is a uniform* $W^{3-1/r}_r$ *domain and that* $|\delta| \le \delta_0$. *Then, there exists a positive constant* γ_2 *such that for any* $(f, \mathbf{g}) \in L_{p,\gamma_2,0}(\mathbb{R}, W^{1,0}_q(\Omega))$, $\mathbf{h} \in L_{p,\gamma_2,0}(\mathbb{R}, W^1_q(\Omega)^N) \cap H^{1/2}_{p,\gamma_2,0}(\mathbb{R}, L_q(\Omega)^N)$, *and* $\tilde{h} \in L_{p,\gamma_2,0}(\mathbb{R}, W^2_q(\Omega)) \cap W^1_{p,\gamma_2,0}(\mathbb{R}, L_q(\Omega))$, *problem* (1.3) *with* $\rho = 0$ *and* $\mathbf{u} = 0$ *admits a unique solution* $(\rho, \mathbf{u})$ *with*

$$\rho \in W^1_{p,\gamma_2,0}(\mathbb{R}, W^1_q(\Omega)), \quad \mathbf{u} \in L_{p,\gamma_2,0}(\mathbb{R}, W^2_q(\Omega)^N) \cap W^1_{p,\gamma_2,0}(\mathbb{R}, L_q(\Omega)^N),$$

possessing the estimate

$$\begin{aligned}&\|e^{-\gamma t}(\gamma\rho, \partial_t\rho)\|_{L_p(\mathbb{R},W^1_q(\Omega))} + \|e^{-\gamma t}(\gamma\mathbf{u}, \partial_t\mathbf{u})\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\mathbf{u}\|_{L_p(\mathbb{R},W^2_q(\Omega))}\\ &\quad\le C(\|e^{-\gamma t}(f,\mathbf{g})\|_{L_p(\mathbb{R},W^{1,0}_q(\Omega))} + \|e^{-\gamma t}(\nabla\mathbf{h}, \Lambda^{1/2}_\gamma\mathbf{h})\|_{L_p(\mathbb{R},L_q(\Omega))}\\ &\qquad + \|e^{-\gamma t}\tilde{h}\|_{W^1_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\tilde{h}\|_{L_p(\mathbb{R},W^2_q(\Omega))}) \qquad (2.3)\end{aligned}$$

for any $\gamma \ge \gamma_2$ *with some constant* C *depending on* N, p *and* q.

Remark 2.4. As was seen in Shibata and Shimizu [8], we know that

$$\begin{aligned}&H^{1/2}_{p,\gamma_2,0}(\mathbb{R}, W^1_q(\Omega)) \subset L_{p,\gamma_2,0}(\mathbb{R}, W^2_q(\Omega)^N) \cap W^1_{p,\gamma_2,0}(\mathbb{R}, L_q(\Omega)^N),\\ &\|e^{-\gamma t}\Lambda^{1/2}_\gamma \nabla f\|_{L_p(\mathbb{R},L_q(\Omega))} \le C\{\|e^{-\gamma t}f\|_{W^1_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}f\|_{L_p(\mathbb{R},W^2_q(\Omega))}\}.\end{aligned}$$

3 $\mathcal{R}$-boundedness

Employing the similar argumentation to that in Enomoto and Shibata [3], Shibata and Shimizu [8], we can show Theorem 2.2 and Theorem 2.3 by $\mathcal{R}$-boundedness of solution operators to the generalized resolvent problem. Thus, the main part of this section is devoted to the proof of the existence of $\mathcal{R}$-bounded solution operators to problem (1.5). First, we are concerned with generalized resolvent problem (1.5).

Theorem 3.1. *Let $1 < q < \infty$, $N < r < \infty$, $\max(q, q') \le r$ ($q' = q/(q-1)$), and $0 < \epsilon < \pi/2$. Assume that Ω is a uniform $W_r^{3-1/r}$ domain. Let $\Sigma_{\epsilon,\lambda_0}$, K_ϵ and $\mathcal{X}_q(\Omega)$ be the sets defined in (1.6) and set $\Lambda_{\epsilon,\lambda_0} = \Sigma_{\epsilon,\lambda_0} \cap K_\epsilon$. Moreover, we define the space $X_q(\Omega)$ by*

$$X_q(\Omega) = \{(f, \mathbf{g}, \mathbf{h}, \tilde{h}) \mid (f, \mathbf{g}) \in W_q^{1,0}(\Omega),\ \ \mathbf{h} \in W_q^1(\Omega)^N,\ \ \tilde{h} \in W_q^2(\Omega)\}.$$

Then, there exist a positive constant λ_0 and an operator family

$$R(\lambda) \in \mathrm{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), \mathrm{W}_q^{1,2}(\Omega)))$$

such that for any $(f, \mathbf{g}, \mathbf{h}, \tilde{h}) \in X_q(\Omega)$ and $\lambda \in \Lambda_{\epsilon,\lambda_0}$, $(\theta, \mathbf{v}) = R(\lambda)(f, \mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}, \lambda\tilde{h}, \lambda^{1/2}\nabla\tilde{h}, \nabla^2\tilde{h})$ is a unique solution to problem (1.5).

Moreover, there exists a constant C depending on ϵ, λ_0, q and N such that

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda R(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma R(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^2})}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla P_v R(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) &\le C,\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^3})}(\{(\tau\partial_\tau)^\ell(\nabla^2 P_v R(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) &\le C
\end{aligned} \tag{3.1}$$

with $\lambda = \gamma + i\tau$ and $\ell = 0, 1$.

To prove Theorem 3.1 in the case $\lambda \ne 0$, inserting the formula: $\theta = \lambda^{-1}(f - \gamma_1(x)\mathrm{div}\,\mathbf{v})$ into the second equation in (1.5), we have

$$\lambda\mathbf{v} - \gamma_0^{-1}\mathrm{Div}\,\mathbf{S}(\mathbf{v}) - \lambda^{-1}\gamma_0^{-1}\nabla(\gamma_1\gamma_2\mathrm{div}\,\mathbf{v}) = \mathbf{g} - \lambda^{-1}\gamma_0^{-1}\nabla(\gamma_1 f).$$

Thus, instead of (1.5), we mainly consider the equations:

$$\begin{cases} \gamma_0\lambda\mathbf{v} - \mathrm{Div}\,\mathbf{S}(\mathbf{v}) - \delta\nabla(\gamma_3\mathrm{div}\,\mathbf{v}) = \mathbf{f} & \text{in } \Omega \\ \alpha[\mathbf{D}(\mathbf{v})\mathbf{n} - <\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}>\mathbf{n}] = \mathbf{h} - <\mathbf{h}, \mathbf{n}>\mathbf{n}, \quad \mathbf{v}\cdot\mathbf{n} = \tilde{h} & \text{on } \Gamma, \end{cases} \tag{3.2}$$

with $\gamma_3 = \gamma_1\gamma_2$. Let $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. As δ, we consider the following three cases:

(C1) $\delta = \lambda^{-1}$ and $\lambda \in \Lambda_{\epsilon,\lambda_0}$;

(C2) $\delta \in \Sigma_\epsilon$ with $\mathrm{Re}\,\delta < 0$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \ge \left|\frac{\mathrm{Re}\,\delta}{\mathrm{Im}\,\delta}\right||\mathrm{Im}\,\lambda|$ and $|\lambda| \ge \lambda_0$;

(C3) $\mathrm{Re}\,\delta \ge 0$, and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \ge \lambda_0|\mathrm{Im}\,\lambda|$ and $|\lambda| \ge \lambda_0$.

In the following, for $\delta_0 > 0$ we assume that $|\delta| \le \delta_0$ in any cases of (C1), (C2) and (C3). In particular, in (C1), $\delta_0 = \lambda_0^{-1}$. In (C2), we note that $\mathrm{Im}\,\delta \ne 0$, because $\delta \in \Sigma_\epsilon$ and $\mathrm{Re}\,\delta < 0$. We may include the case where $\delta = 0$ in (C1), which is corresponding to the Lamé system. The case (C1) is used to prove the existence of an $\mathcal{R}$-bounded solution operator pertaining to (1.5) and the cases (C2) and (C3) enable us the application of a homotopic argument in proving the exponential stability of the analytic semigroup associated with (1.3) in a bounded domain. Such homotopic argument already appeared in [9] and [3] in the non-slip condition case. For the sake of simplicity, we introduce the set $\Gamma_{\epsilon,\lambda_0,\delta_0}$ defined by

$$\Gamma_{\epsilon,\lambda_0,\delta_0} = \begin{cases} \Lambda_{\epsilon,\lambda_0} & \text{for (C1)} \\ \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \left|\frac{\mathrm{Re}\,\delta}{\mathrm{Im}\,\delta}\right| |\mathrm{Im}\,\lambda|, \ |\lambda| \geq \lambda_0\} & \text{for (C2)} \\ \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \lambda_0 |\mathrm{Im}\,\lambda|, \ |\lambda| \geq \lambda_0\} & \text{for (C3)} \end{cases} \tag{3.3}$$

with $|\delta| \leq \delta_0$.

We have Theorem 3.1 immediately with help of the following Theorem 3.2 in the case (C1).

Theorem 3.2. *Let* $1 < q < \infty$, $N < r < \infty$, $\max(q, q') \leq r$ $(q' = q/(q-1))$, $0 < \epsilon < \pi/2$, $\delta_0 > 0$ *and* $\lambda_0 > 0$. *Assume that* Ω *is a uniform* $W_r^{3-1/r}$ *domain and that* $|\delta| \leq \delta_0$. *Let* $\Gamma_{\epsilon,\lambda_0,\delta_0}$ *be the set defined in* (3.3). *Set*

$$\begin{aligned} &Y_q(\Omega) = \{(\mathbf{f}, \mathbf{h}, \tilde{h}) \mid \mathbf{f} \in L_q(\Omega)^N, \ \mathbf{h} \in W_q^1(\Omega)^N, \ \tilde{h} \in W_q^2(\Omega)\}, \\ &\mathcal{Y}_q(\Omega) = \{F = (F_2, \ldots, F_7) \mid F_2, F_3, F_6 \in L_q(\Omega)^N, \ F_4, F_7 \in L_q(\Omega)^{N^2}, \ F_5 \in L_q(\Omega)\}. \end{aligned} \tag{3.4}$$

Then, there exist a positive constant λ_0 *and an operator family* $\tilde{\mathcal{A}}(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0,\delta_0}, \mathcal{L}(\mathcal{Y}_q(\Omega), W_q^2(\Omega)^N))$ *such that for any* $(\mathbf{f}, \mathbf{h}, \tilde{h}) \in Y_q(\Omega)$ *and* $\lambda \in \Lambda_{\epsilon,\lambda_0}$, $\mathbf{v} = \mathcal{A}(\lambda) F_\lambda(\mathbf{f}, \mathbf{h}, \tilde{h})$ *is a unique solution to problem* (3.2), *and* $\mathcal{A}(\lambda)$ *satisfies the following estimates:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\tau \frac{d}{d\tau})^\ell (G_\lambda \mathcal{A}(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1)$$

with some constant C *depending on* ϵ, λ_0, δ_0, a, b, q *and* N. *Here and in the following, we set* $\tilde{N} = N^3 + N^2 + 2N$, $G_\lambda \mathbf{v} = (\lambda \mathbf{v}, \gamma \mathbf{v}, \lambda^{1/2} \nabla \mathbf{v}, \nabla^2 \mathbf{v})$, *and* $F_\lambda(\mathbf{f}, \mathbf{h}, \tilde{h}) = (\mathbf{f}, \lambda^{1/2} \mathbf{h}, \nabla \mathbf{h}, \lambda \tilde{h}, \lambda^{1/2} \nabla \tilde{h}, \nabla^2 \tilde{h})$.

Secondly, we are concerned with time dependent problem (1.3). Let $\mathcal{B}$ be a linear operator defined by

$$\mathcal{B}(\theta, \mathbf{v}) = (-\gamma_2 \mathrm{div}\, \mathbf{v}, \gamma_0^{-1} \mathrm{Div}\, \mathbf{S}(\mathbf{v}) - \gamma_0^{-1} \nabla(\gamma_1 \theta)) \quad \text{for } (\theta, \mathbf{v}) \in W_q^1(\Omega) \times \mathcal{D}_q(\Omega)$$

with $\mathcal{D}_q(\Omega) = \{\mathbf{v} \in W_q^2(\Omega) \mid [\mathbf{D}(\mathbf{v})\mathbf{n} - < \mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n} > \mathbf{n}]|_\Gamma = 0, \ \mathbf{v} \cdot \mathbf{n} = 0\}$. Since Definition 1.2 with $n = 1$ implies the boundedness of the operator family $\mathcal{T}$, it follows from Theorem 3.1 that $\Lambda_{\epsilon,\lambda_0}$ is contained in the resolvent set of $\mathcal{B}$ and for any $\lambda \in \Lambda_{\epsilon,\lambda_0}$ and $(f, \mathbf{g}, \mathbf{h}, \tilde{h}) \in X_q(\Omega)$,

$$\begin{aligned} &|\lambda| \|\theta\|_{W_q^1(\Omega)} + \|(\lambda \mathbf{v}, \lambda^{1/2} \nabla \mathbf{v}, \nabla^2 \mathbf{v})\|_{L_q(\Omega)} \\ &\leq C(\|(f, \mathbf{g})\|_{W_q^{1,0}(\Omega)} + \|(\lambda^{1/2} \mathbf{h}, \nabla \mathbf{h})\|_{L_q(\Omega)} + \|(\lambda \tilde{h}, \lambda^{1/2} \nabla \tilde{h}, \nabla^2 \tilde{h})\|_{L_q(\Omega)}). \end{aligned} \tag{3.5}$$

By (3.5), we have the following theorem.

Theorem 3.3. *Let* $1 < q < \infty$, $N < r < \infty$, $\max(q, q') \le r$ $(q' = q/(q-1))$, $0 < \epsilon < \pi/2$, $\delta_0 > 0$ *and* $\lambda_0 > 0$. *Assume that* Ω *is a uniform* $W^{3-1/r}_r$ *domain and that* $|\delta| \le \delta_0$. *Then, the operator* $\mathcal{B}$ *generates an analytic semigroup* $\{T(t)\}_{t\ge0}$ *on* $W^{1,0}_q(\Omega)$. *Moreover, there exists constants* $\gamma_0 > 0$ *and* $M > 0$ *such that for any* $(f, \mathbf{g}) \in W^{1,0}_q(\Omega)$, $(\rho(t), \mathbf{u}(t)) = T(t)(f, \mathbf{g})$ *solves* (1.3) *with* $f = 0$, $\mathbf{g} = 0$, $\mathbf{h} = 0$ *and* $\tilde{h} = 0$ *and satisfies the following estimate:*

$$\|T(t)(f,\mathbf{g})\|_{W^{1,0}_q(\Omega)} + t^{1/2}\|\nabla P_v T(t)(f,\mathbf{g})\|_{L_q(\Omega)} + t\|\nabla^2 P_v T(t)(f,\mathbf{g})\|_{L_q(\Omega)} \le Me^{\gamma_0 t}\|(f,\mathbf{g})\|_{W^{1,0}_q(\Omega)}. \qquad (3.6)$$

Combining Theorem 3.3 with a real interpolation method (cf. Shibata and Shimizu [8, Proof of Theorem 3.9]), we have Theorem 2.2.

Employing the similar argumentation to that in Shibata and Shimizu [8], we can show the existence part of Theorem 2.3 by using Theorem 3.1 and Theorem 1.3. Moreover, the uniqueness of solutions to (1.3) can be proved by using the existence of solutions to the dual problem as was seen also in Shibata and Shimizu [8]. Thus, we may omit the proof of Theorem 2.2 and Theorem 2.3.

4 Local in time unique existence theorem for nonlinear problem (1.1)

As was done in Burnat and Zajaczkowski [2] and Kobayashi and Zajaczkowski [5], in order to prove Theorem 2.1, we formulate (1.1) in Lagrangian coordinates. Let velocity fields $\mathbf{v}(\xi, t)$ and $\mathbf{u}(x, t)$ be known as vectors of functions of the Lagrange coordinates ξ and the Euler coordinates x of the same fluid particle, respectively. In this case, the connection between the Lagrange coordinate and the Euler coordinate is written in the form:

$$x = \xi + \int_0^t \mathbf{v}(\xi, s)ds \equiv \mathbf{X}_{\mathbf{v}}(\xi, t), \qquad (4.1)$$

and $\mathbf{v}(\xi, t) = \mathbf{u}(x, t)$. Let $A_{\mathbf{v}}$ be the Jacobi matrix of the transformation $x = \mathbf{X}_{\mathbf{v}}(\xi, t)$ with element $a_{ij} = \delta_{ij} + \int_0^t (\partial v_i/\partial\xi_j)(\xi, s)\,ds$. There exists a small number σ such that $A_{\mathbf{v}}$ is invertible, that is $\det A_{\mathbf{v}} \ne 0$, provided that

$$\max_{i,j=1\dots,N} \|\int_0^t (\partial v_i/\partial\xi_j)(\cdot, s)\,ds\|_{L_\infty(\Omega)} < \sigma \quad (0 < t < T). \qquad (4.2)$$

In this case, we have $\nabla_x = A_{\mathbf{v}}^{-1}\nabla_\xi = (\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla\mathbf{v}(\xi, s)\,ds))\nabla\xi$, where $\mathbf{V}_0(\mathbf{K})$ is an $N \times N$ matrix of C^∞ functions with respect to $\mathbf{K} = (k_{ij})$ defined on $|\mathbf{K}| < 2\sigma$ and $\mathbf{V}_0(0) = 0$, where k_{ij} are corresponding variables to $\int_0^t (\partial v_i/\partial\xi_j)(\cdot, s)\,ds$. Assume that $\rho(x, t)$ and $\mathbf{u}(x, t)$ are solutions of (1.1) in the Euler coordinate. Setting $\rho(\mathbf{X}_{\mathbf{v}}(\xi, t), t) = \rho_* + \theta_0(\xi) + \theta(\xi, t)$, we write (1.1) in the Lagrangian coordinate introduced by (4.1) as follows:

$$
\begin{cases}
\partial_t\theta + (\rho_* + \theta_0)\mathrm{div}\,\mathbf{v} = F(\theta, \mathbf{v}) & \text{in } \Omega \times (0, T), \\
(\rho_* + \theta_0)\partial_t\mathbf{v} - \mathrm{Div}\,\mathbf{S}(\mathbf{v}) + \nabla(P'(\rho_* + \theta_0)\theta) = \mathbf{g} + G(\theta, \mathbf{v}) & \text{in } \Omega \times (0, T), \\
\alpha[\mathbf{D}(\mathbf{v})\mathbf{n} - <\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}> \mathbf{n}] = H(\mathbf{v}) & \text{on } \Gamma \times (0, T), \\
\mathbf{v}\cdot\mathbf{n} = -\mathbf{v}\cdot(\hat{\mathbf{n}}_{\mathbf{v}} - \mathbf{n}) & \text{on } \Gamma \times (0, T), \\
(\theta, \mathbf{v})|_{t=0} = (0, \mathbf{u}_0) & \text{in } \Omega,
\end{cases} \tag{4.3}
$$

where $\hat{\mathbf{n}}_{\mathbf{v}} = \mathbf{n}(\mathbf{X}_{\mathbf{v}}(\xi, t))$, $\mathbf{g} = -P'(\rho_*+\theta_0)\nabla\theta_0$ and $F(\theta, \mathbf{v})$, $G(\theta, \mathbf{v})$ and $H(\mathbf{v})$ are nonlinear functions of the following forms:

$$
\begin{aligned}
&F(\theta, \mathbf{v}) = -\theta\mathrm{div}\,\mathbf{v} - (\rho_* + \theta_0 + \theta)\mathbf{V}_1(\int_0^t \nabla\mathbf{v}ds)\nabla\mathbf{v}, \\
&G(\theta, \mathbf{v}) = -\theta\partial_t\mathbf{v} + \mathrm{Div}\,[\alpha\mathbf{V}_2(\int_0^t \nabla\mathbf{v}ds)\nabla\mathbf{v} + (\beta - \alpha)\mathbf{V}_1(\int_0^t \nabla\mathbf{v}ds)\nabla\mathbf{v}\mathbf{I}] \\
&\quad + \mathbf{V}_3(\int_0^t \nabla\mathbf{v}ds)\nabla[\alpha\mathbf{D}(\mathbf{v}) + \alpha\mathbf{V}_2(\int_0^t \nabla\mathbf{v}ds)\nabla\mathbf{v} \\
&\qquad\qquad + (\beta - \alpha)\mathrm{div}\,\mathbf{v}\mathbf{I} + (\beta - \alpha)\mathbf{V}_1(\int_0^t \nabla\mathbf{v}ds)\nabla\mathbf{v}] \\
&\quad - \nabla(\int_0^1 P''(\rho_* + \theta_0 + \tau\theta)(1 - \tau)\,d\tau\theta^2) - P'(\rho_* + \theta_0 + \theta)\mathbf{V}_0(\int_0^t \nabla\mathbf{v}ds)\nabla(\theta_0 + \theta), \\
&H(\mathbf{v}) = -\alpha\mathbf{D}(\mathbf{v})(\hat{\mathbf{n}}_{\mathbf{v}} - \mathbf{n}) - \alpha\{<\mathbf{D}(\mathbf{v})\hat{\mathbf{n}}_{\mathbf{v}}, \hat{\mathbf{n}}_{\mathbf{v}}> \hat{\mathbf{n}}_{\mathbf{v}} - <\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}> \mathbf{n}\} \\
&\quad - \alpha\{(\mathbf{V}_2(\int_0^t \nabla\mathbf{v}\,ds)\nabla\mathbf{v})\hat{\mathbf{n}}_{\mathbf{v}} - <(\mathbf{V}_2(\int_0^t \nabla\mathbf{v}\,ds)\nabla\mathbf{v})\hat{\mathbf{n}}_{\mathbf{v}}, \hat{\mathbf{n}}_{\mathbf{v}}> \hat{\mathbf{n}}_{\mathbf{v}}\}.
\end{aligned} \tag{4.4}
$$

Here, $\mathbf{V}_0(\mathbf{K})$, $\mathbf{V}_1(\mathbf{K})$, $\mathbf{V}_2(\mathbf{K})$ and $\mathbf{V}_3(\mathbf{K})$ are some matrices of C^∞ functions with respect to $\mathbf{K}$ defined on $|\mathbf{K}| \le \sigma$, which satisfy conditions:

$$
\mathbf{V}_0(0) = 0, \quad \mathbf{V}_1(0) = 0, \quad \mathbf{V}_2(0) = 0, \quad \mathbf{V}_3(0) = 0 \tag{4.5}
$$

and relations: $A_{\mathbf{v}}^{-1} = I + \mathbf{V}_0(\int_0^t \nabla\mathbf{v}\,ds)$, $\mathrm{div}\,_x\mathbf{w} = \mathrm{div}\,_\xi\hat{\mathbf{w}} + \mathbf{V}_1(\int_0^t \nabla\hat{\mathbf{w}}ds)\nabla\mathbf{w}$, $\mathbf{D}_x(\mathbf{w}) = \mathbf{D}_\xi(\hat{\mathbf{w}}) + \mathbf{V}_2(\int_0^t \nabla\hat{\mathbf{w}}\,ds)\nabla\hat{\mathbf{w}}$ and $\mathrm{Div}\,_x\mathbf{K}(\mathbf{w}) = \mathrm{Div}\,_\xi\hat{\mathbf{K}} + \mathbf{V}_3(\int_0^t \nabla\hat{\mathbf{w}}ds)\nabla\hat{\mathbf{K}}$ with $\hat{\mathbf{w}} = \mathbf{w}(\mathbf{X}_v(\xi, t), t)$ and $\hat{\mathbf{K}} = \mathbf{K}(\mathbf{X}_{\mathbf{v}}(\xi, t), t)$.

Since we can show eventually that the correspondence $x = \mathbf{X}_{\mathbf{v}}(\xi, t)$ is invertible by using the argument due to Ströhmer [10], our main task is to prove the following theorem.

Theorem 4.1. *Let $N < q < \infty$, $2 < p < \infty$ and $R > 0$. If the initial data $(\theta_0, \mathbf{u}_0)$ for (1.1) satisfy the condition (2.1) and $\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le R$, then there exists a time $T > 0$ depending on R such that the problem (4.3) admits a unique solution $(\theta, \mathbf{v})$ with*

$$
\theta \in W_p^1((0, T), W_q^1(\Omega)), \quad \mathbf{v} \in W_p^1(0, T), L_q(\Omega)) \cap L_p((0, T), W_q^2(\Omega)).
$$

To prove Theorem 4.1, first we show the maximal L_p-L_q regularity for the following time local linear problem:

$$\begin{cases} \partial_t\rho + \gamma_2 \mathrm{div}\,\mathbf{u} = f, \quad \gamma_0\partial_t\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) + \nabla(\gamma_1\rho) = \mathbf{g} & \text{in } \Omega\times(0,T), \\ \alpha[\mathbf{D}(\mathbf{u})\mathbf{n} - <\mathbf{D}(\mathbf{u})\mathbf{n},\mathbf{n}>\mathbf{n}] = \mathbf{h} - <\mathbf{h},\mathbf{n}>\mathbf{n}, \quad \mathbf{u}\cdot\mathbf{n} = \tilde h & \text{on } \Gamma\times(0,T), \\ (\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{4.6}$$

For this purpose, we have to replace the nonlocal operator $\Lambda_\gamma^{1/2}$ with value in $L_q(\Omega)$ by the local operator ∂_t with value in $W_q^{-1}(\Omega)$. For this purpose, according to Shibata [7], we introduce the extension map $\iota : L_{1,\mathrm{loc}}(\Omega) \to L_{1,\mathrm{loc}}(\mathbb{R}^N)$ having the following properties:

(e-1) For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\iota f \in W_q^1(\mathbb{R}^N)$, $\iota f = f$ in Ω and $\|\iota f\|_{W_q^\ell(\mathbb{R}^N)} \le C_q\|f\|_{W_q^\ell(\Omega)}$ for $\ell = 0, 1$ with some constant C_q depending on q, r and Ω.

(e-2) For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\|(1-\Delta)^{-1/2}\iota(\nabla f)\|_{L_q(\mathbb{R}^N)} \le C_q\|f\|_{L_q(\Omega)}$ with some constant C_q depending on q, r and Ω.

Here, $(1-\Delta)^{-1/2}$ is the operator defined by $(1-\Delta)^{-1/2}f = \mathcal{F}_\xi^{-1}[(1+|\xi|^2)^{1/4}\mathcal{F}[f](\xi)]$. In the following, such extension map ι is fixed. We define $W_q^{-1}(\Omega)$ by

$$W_q^{-1}(\Omega) = \{f \in L_{1,\mathrm{loc}}(\Omega) \mid (1-\Delta)^{-1/2}\iota f \in L_q(\Omega)\}.$$

Employing the similar argumentation to the proof of Proposition 2.8 in Shibata and Shimizu [8] (cf. also the appendix in Shibata [7]), we have

$$W^1_{p,\gamma_0,0}(\mathbb{R}, W_q^{-1}(\Omega)) \cap L_{p,\gamma_0,0}(\mathbb{R}, W_q^1(\Omega)) \subset H^{1/2}_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega)), \tag{4.7}$$

$$\|e^{-\gamma t}\Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R},L_q(\Omega))} \le C\{\|e^{-\gamma t}\partial_t[(1-\Delta)^{-1/2}(\iota f)]\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t} f\|_{L_p((\mathbb{R},W_q^1(\Omega))}\}. \tag{4.8}$$

Combining Theorem 2.2, Theorem 2.3 and (4.7), we have the following theorem.

Theorem 4.2. *Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \le r$ ($q' = q/(q-1)$). Let T and R be any positive numbers and $B^{2(1-1/p)}_{q,p}(\Omega)$ be the same set as in (2.2). Assume that $\theta_0 \in W_q^1(\Omega)$ satisfies the range condition (2.1) and $\|\nabla\theta_0\|_{L_q(\Omega)} \le R$. Then, there exists a positive number $\gamma_0 = \gamma_0(R)$ depending on R and ρ_* such that for any initial data $(\rho_0, \mathbf{u}_0) \in W_q^1(\Omega) \times B^{2(1-1/p)}_{q,p}(\Omega)$ and right members f, $\mathbf{g}$, $\mathbf{h}$ and $\tilde h$ with*

$$\begin{aligned} &\mathbf{f} \in W_p^1((0,T), W_q^1(\Omega)^N)), \quad \mathbf{g} \in L_p((0,T), L_q(\Omega)), \\ &\mathbf{h} \in L_p((0,T), W_q^1(\Omega)) \cap W_p^1((0,T), W_q^{-1}(\Omega)), \\ &\tilde h \in L_p((0,T), W_q^2(\Omega) \cap W_p^1((0,T), L_q(\Omega)) \end{aligned}$$

satisfying the conditions: $\mathbf{h}|_{t=0} = 0$ and $\tilde h|_{t=0} = 0$, problem (4.6) admit unique solutions ρ and $\mathbf{u}$ with

$$\rho \in W_p^1((0,T), W_q^1(\Omega)), \quad \mathbf{u} \in L_p((0,T), W_q^2(\Omega)) \cap W_p^1((0,T), L_q(\Omega))$$

possessing the estimate:

$$\|\rho\|_{W^1_p((0,t),W^1_q(\Omega))} + \|\mathbf{u}\|_{L_p((0,t),W^2_q(\Omega))} + \|\mathbf{u}\|_{W^1_p((0,t),L_q(\Omega))}$$
$$\leq C(R)e^{\gamma t}\{\|(\rho_0, \mathbf{u}_0)\|_{E_{q,p}(\Omega)} + \|(f, \mathbf{g})\|_{L_p((0,t),W^{1,0}_q(\Omega))} + \|\mathbf{h}\|_{L_p((0,t),W^1_q(\Omega))}$$
$$+ \|\partial_s[(1-\Delta)^{-1/2}\iota\mathbf{h}]\|_{L_p((0,t),L_q(\Omega))} + \|\tilde{h}\|_{L_p((0,t),W^2_q(\Omega))} + \|\tilde{h}\|_{W^1_p((0,t),L_q(\Omega))}\}$$

for any $t \in (0,T]$ and $\gamma \geq \gamma_0$ with some constant $C(R)$ depending on R but independent of $\gamma \geq \gamma_0$ and $t \in (0,T]$.

Proof of Theorem 4.1. In the following, we assume that $2 < p < \infty$ and $N < q < \infty$, that Ω is a uniform $W^{3-1/q}_q$ domain in $\mathbb{R}^N$ $(N \geq 2)$. By Sobolev's embedding theorem we have

$$W^1_q(\Omega) \subset L_\infty(\Omega), \quad \|\prod_{j=1}^m f_j\|_{W^1_q(\Omega)} \leq C\prod_{j=1}^m \|f_j\|_{W^1_q(\Omega)}. \tag{4.9}$$

Let T and L be any positive numbers and we define a space $\mathcal{I}_{L,T}$ by

$$\mathcal{I}_{L,T} = \{(\theta, \mathbf{v}) \mid \theta \in W^1_p((0,T), W^1_q(\Omega)), \ \mathbf{v} \in W^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), W^2_q(\Omega))$$
$$(\theta, \mathbf{v})|_{t=0} = (0, \mathbf{u}_0) \quad \text{in } \Omega, \quad \|\theta\|_{W^1_p((0,T),W^1_q(\Omega))} + \mathbb{I}_{\mathbf{v}}(0,T) \leq L\}, \tag{4.10}$$

where we have set $\mathbb{I}_{\mathbf{v}}(0,T) = \|\mathbf{v}\|_{L_p((0,T),W^2_q(\Omega))} + \|\partial_t\mathbf{v}\|_{L_p((0,T),L_q(\Omega))}$. Since we choose $T > 0$ small enough eventually, we may assume that $0 < T \leq 1$ in the following. Given $(\kappa, \mathbf{w}) \in \mathcal{I}_{L,T}$, let θ and $\mathbf{v}$ be solutions to problem:

$$\begin{cases} \partial_t\theta + (\rho_* + \theta_0)\mathrm{div}\,\mathbf{v} = F(\kappa, \mathbf{w}) & \text{in } \Omega \times (0,T), \\ (\rho_* + \theta_0)\partial_t\mathbf{v} - \mathrm{Div}\,\mathbf{S}(\mathbf{v}) + \nabla(P'(\rho_* + \theta_0)\theta) = \mathbf{g} + G(\kappa, \mathbf{w}) & \text{in } \Omega \times (0,T), \\ \alpha[\mathbf{D}(\mathbf{v})\mathbf{n} - <\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}> \mathbf{n}] = H(\mathbf{w}) & \text{on } \Gamma \times (0,T), \\ \mathbf{v}\cdot\mathbf{n} = -\mathbf{w}\cdot(\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n}) & \text{on } \Gamma \times (0,T), \\ (\theta, \mathbf{v})|_{t=0} = (0, \mathbf{u}_0) & \text{in } \Omega, \end{cases} \tag{4.11}$$

First, we estimate the right-hand sides of (4.11). By (4.9), Hölder's inequality and the identity: $\kappa(\cdot, t) = \int_0^t \partial_s\kappa(\cdot, s)\,ds$, we have

$$\sup_{t\in(0,T)} \|\int_0^t \nabla\mathbf{w}(\cdot, s)\,ds\|_{L_\infty(\Omega)} \leq M_1T^{1/p'}L, \quad \sup_{t\in(0,T)} \|\int_0^t \nabla\mathbf{w}(\cdot, s)\,ds\|_{W^1_q(\Omega)} \leq CT^{1/p'}L,$$
$$\sup_{t\in(0,T)} \|\kappa(\cdot, t)\|_{L_\infty(\Omega)} \leq M_1T^{1/p'}L, \qquad \sup_{t\in(0,T)} \|\kappa(\cdot, t)\|_{W^1_q(\Omega)} \leq CT^{1/p'}L, \tag{4.12}$$

with $p' = p/(p-1)$. Here and in the following, C denotes a generic constant independent of T and R and we use the letters M_1 to denote a special constants independent of T and L. The value of C may change from line to line. To treat polynomials of functions with respect to κ and $\int_0^t \nabla \mathbf{w}(\cdot, s)\, ds$, in view of (2.1) and (4.2) we choose T so small that $M_1 T^{1/p'} L \le \rho_*/4$ and $M_1 T^{1/p'} L \le \sigma$, so that

$$\rho_*/4 \le \rho_* + \theta_0 + \tau\kappa \le 4\rho_* \ \ (\tau \in [0,1]), \qquad \sup_{t\in(0,T)} \| \int_0^t \nabla \mathbf{w}(\cdot, s)\, ds\|_{L_\infty(\Omega)} < \sigma. \tag{4.13}$$

Recall that $\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le R$. By (4.12), (4.13), and (4.5), we have

$$\sup_{t\in(0,T)} \|\mathbf{V}_i(\int_0^t \nabla \mathbf{w}(\cdot, t)\, ds)\|_{W_q^1(\Omega)} \le CT^{1/p'}L, \ \sup_{t\in(0,T)} \|\nabla \mathbf{W}(\int_0^t \nabla \mathbf{w}(\cdot, t)\, ds)\|_{L_q(\Omega)} \le CT^{1/p'}L$$
$$\sup_{t\in(0,T)} \|\nabla \int_0^1 P''(\rho_* + \theta_0 + \tau\kappa)(1-\tau)\, d\tau\|_{L_q(\Omega)} \le C(R + T^{1/p'}L), \tag{4.14}$$

where $i = 1, 2, 4, 5$ and 6, and $\mathbf{W} = \mathbf{W}(\mathbf{K})$ is any matrix of polynomials with respect to $\mathbf{K}$. By (4.9), (4.4), (4.12), (4.13) and (4.14), we have

$$\begin{aligned}
&\|\mathbf{F}(\kappa, \mathbf{w})\|_{W_p^1((0,T),W_q^1(\Omega))} \le C(L^2T^{1/p'} + L^3(T^{1/p'})^2), \quad \|\mathbf{g}\|_{L_p((0,T),L_q(\Omega))} \le CRT^{1/p},\\
&\|G(\kappa, \mathbf{w})\|_{L_p((0,T),L_q(\Omega))} \le C\{L^2T^{1/p'} + L^3(T^{1/p'})^2\\
&\qquad + (1 + R + LT^{1/p'})(LT^{1/p'})^2T^{1/p} + (R + LT^{1/p'})(LT^{1/p'})T^{1/p}\}.
\end{aligned} \tag{4.15}$$

To obtain

$$\sup_{t\in(0,T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le C(\mathbb{I}_{\mathbf{w}}(0,T) + e^{\gamma T}\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}), \tag{4.16}$$

we use the embedding relation:

$$L_p((0,\infty), X_1) \cap W_p^1((0,\infty), X_0) \subset BUC(J, [X_0, X_1]_{1-1/p,p}) \tag{4.17}$$

for any two Banach spaces X_0 and X_1 such that X_1 is dense in X_0 and $1 < p < \infty$ (cf. [1]). Since $B_{q,p}^{2(1-1/p)}(\Omega) \subset W_q^1(\Omega)$ as follows from the assumption: $2 < p < \infty$, by (4.16) and (4.13)

$$\begin{aligned}
&\sup_{t\in(0,T)} \|\mathbf{w}(\cdot, t)\|_{W_q^1(\Omega)} \le C(L + e^{\gamma T}\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}),\\
&\sup_{t\in(0,T)} \|\partial_t \mathbf{W}(\int_0^t \nabla \mathbf{w}(\cdot, s)\, ds)\|_{L_q(\Omega)} \le C(L + e^{\gamma T}\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}).
\end{aligned} \tag{4.18}$$

In the following, we assume that $LT^{1/p'} \le 1$ and $0 < \sigma \le 1$ to obtain

$$\sup_{t\in(0,T)} \|\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n}\|_{W_q^1(\Omega)} \le C(R + L)T. \tag{4.19}$$

we write $\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n} = \int_0^1 (\nabla \mathbf{n})(\xi + \tau \int_0^t \mathbf{w}(\xi, s)\, ds)\, d\tau (\int_0^t \mathbf{w}(\xi, s)\, ds)$. Using the following estimates

$$\|\mathbf{n}\|_{W_\infty^1(\mathbb{R}^N)} \le C, \quad \|\hat{\mathbf{n}}_{\mathbf{w}}\|_{W_\infty^1(\Omega)} \le C, \quad \|(\nabla \mathbf{n})(\cdot + \tau \int_0^t \mathbf{w}(\cdot, s)\, ds)\|_{L_\infty(\Omega)} \le C \ \ (\tau \in [0,1]), \tag{4.20}$$

(4.9) and (4.18) we have $\|\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n}\|_{L_q(\Omega)} \le C(R+L)T$. To estimate $\nabla(\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n})$, we write $\nabla(\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n}) = A_1 + A_2$ with

$$\begin{aligned} A_1 &= \int_0^1 (\nabla \mathbf{n})(\xi + \tau \int_0^t \mathbf{w}(\xi, s)\, ds)\, d\tau (\int_0^t \nabla \mathbf{w}(\xi, s)\, ds), \\ A_2 &= \int_0^1 (\nabla^2 \mathbf{n})(\xi + \tau \int_0^t \mathbf{w}(\xi, s)\, ds)(\mathbf{I} + \tau \int_0^t \nabla \mathbf{w}(\xi, s)\, ds)\, d\tau (\int_0^t \mathbf{w}(\xi, s)\, ds). \end{aligned}$$

By (4.18) and (4.20), we have $\|A_1\|_{L_q(\Omega)} \le C(R+L)T$. On the other hand, by properties of uniform $W_r^{3-1/r}$ domain and (4.12) we have $\|A_2\|_{L_q(\Omega)} \le \sup_{t \in (0,T)} CT \|\mathbf{w}(\cdot, t)\|_{W_q^1(\Omega)}$, which combined with (4.18) furnishes (4.19). We also have

$$\|f(\nabla^2 \mathbf{n})\|_{L_q(\Omega)} \le C\|f\|_{L_q(\Omega)}, \quad \|f(\nabla^2 \hat{\mathbf{n}}_{\mathbf{w}})\|_{L_q(\Omega)} \le C\|f\|_{W_q^1(\Omega)}. \tag{4.21}$$

By (4.9), (4.14), (4.18), (4.19), (4.20), and (4.21) we have

$$\begin{aligned} &\|H(\mathbf{w})\|_{L_p((0,T), W_q^1(\Omega))} \le C((R+L)LT + L^2 T^{1/p'}), \\ &\|\mathbf{w} \cdot (\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n})\|_{L_p((0,T), W_q^2(\Omega))} \le C((R+L)LT + (R+L)T^{1/p}). \end{aligned} \tag{4.22}$$

Since $\partial_t \hat{\mathbf{n}}_{\mathbf{w}} = (\nabla \mathbf{n})(\mathbf{X}_{\mathbf{w}})\mathbf{w}$, by (4.20) we have

$$\|\partial_t \hat{\mathbf{n}}_{\mathbf{w}}\|_{L_q(\Omega)} \le C\|\mathbf{w}(\cdot, t)\|_{L_q(\Omega)}, \tag{4.23}$$

so that by (4.9), (4.18), (4.19) and (4.23), we have

$$\|\partial_t(\mathbf{w} \cdot (\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n}))\|_{L_p((0,T), L_q(\Omega))} \le C((R+L)LT + (R+L)^2 T^{1/p}). \tag{4.24}$$

To estimate $\|\partial_t[(1-\Delta)^{-1/2} \iota H(\mathbf{w})]\|_{L_p((0,T), L_q(\mathbb{R}^N))}$, we prepare the following lemma.

Lemma 4.3. *Let $1 < p < \infty$, $N < q, r < \infty$ and let Ω be a uniform $W_r^{3-1/r}$ domain. Let ι be the extension map satisfying the properties* (e-1) *and* (e-2)*. Then,*

$$\begin{aligned} &\|\partial_t[(1-\Delta)^{-1/2} \iota((\nabla f) g)]\|_{L_p((0,T), L_q(\mathbb{R}^N))} \\ &\le C\Big\{\Big(\int_0^T (\|\partial_t f(\cdot, t)\|_{L_q(\Omega)} \|g(\cdot, t)\|_{W_q^1(\Omega)})^p\, dt\Big)^{1/p} \\ &\qquad + \Big(\int_0^T (\|\nabla f(\cdot, t)\|_{L_q(\Omega)} \|\partial_t g(\cdot, t)\|_{L_q(\Omega)})^p\, dt\Big)^{1/p}\Big\}. \end{aligned}$$

Applying Lemma 4.3 and using (4.20) and (4.23), we have

$$\begin{aligned} &\|\partial_t[(1-\Delta)^{-1/2} H(\mathbf{w})]\|^p_{L_p((0,T), L_q(\mathbb{R}^N))} \le C \int_0^t \|\partial_s \mathbf{w}(\cdot, s)\|^p_{L_q(\Omega)} \|\hat{\mathbf{n}}_{\mathbf{w}} - \mathbf{n}\|^p_{W_q^1(\Omega)}\, ds \\ &+ \int_0^t \|\nabla \mathbf{w}(\cdot, s)\|^p_{L_q(\Omega)} \|\mathbf{w}(\cdot, s)\|^p_{L_q(\Omega)}\, ds + \int_0^t \|\partial_s \mathbf{w}(\cdot, s)\|^p_{L_q(\Omega)} \|\mathbf{V}_2(\int_0^s \nabla \mathbf{w}(\cdot, r)\, dr)\|^p_{W_q^1(\Omega)}\, ds \\ &+ \int_0^t (\|\nabla \mathbf{w}(\cdot, s)\|^2_{L_q(\Omega)} + \|\mathbf{V}_2(\int_0^s \nabla \mathbf{w}(\cdot, r)\, dr)\|_{L_\infty(\Omega)} \|\nabla \mathbf{w}(\cdot, s)\|_{L_q(\Omega)} \|\mathbf{w}(\cdot, s)\|_{L_q(\Omega)})^p\, ds. \end{aligned} \tag{4.25}$$

Applying (4.13), (4.14), (4.18) and (4.19) to the right-hand side of (4.25), we have

$$\|\partial_t[(1-\Delta)^{-1/2}H(\mathbf{w})]\|_{L_p((0,T),L_q(\mathbb{R}^N))} \leq C((R+L)LT+(R+L)^2T+L^2T^{1/p'}). \quad (4.26)$$

Noting that θ_0 satisfies the condition (2.1) and $\|\nabla\theta_0\|_{L_q(\Omega)} \leq R$, by Theorem 4.2, (4.15), (4.22), (4.24) and (4.26), we have

$$\|\theta\|_{W_p^1((0,T),W_q^1(\Omega))} + \mathbb{I}_{\mathbf{v}}(0,T) \leq C(R)\{\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \mathcal{C}(R,L,T)\}, \quad (4.27)$$

where we have set

$$\begin{aligned}\mathcal{C}(R,L,T) = M_2\{&L^2T^{1/p'} + L^3(T^{1/p'})^2 + (1+R+LT^{1/p'})(LT^{1/p'})^2T^{1/p} \\ &+ (R+LT^{1/p'})(LT^{1/p'})T^{1/p} + (R+L)LT + (R+L)^2T + (R+L)T^{1/p}\}\end{aligned}$$

with some special constant M_2 independent of R and T. Recalling that $\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq R$, setting $L = 2C(R)R$ and choosing $T \in (0,1)$ so small that $\mathcal{C}(R,L,T) \leq R$, by (4.27) we have

$$\|\theta\|_{W_p^1((0,T),W_q^1(\Omega))} + \mathbb{I}_{\mathbf{v}}(0,T) \leq L. \quad (4.28)$$

If we define a map Φ by $\Phi(\kappa,\mathbf{w}) = (\theta,\mathbf{v})$, then by (4.28) Φ is the map from $\mathcal{I}_{L,T}$ into itself. Considering the difference $\Phi(\kappa_1,\mathbf{w}_1) - \Phi(\kappa_2,\mathbf{w}_2)$ for $(\kappa_i,\mathbf{w}_i) \in \mathcal{I}_{L,T}$ $(i=1,2)$, employing the same argument and choosing $T \in (0,1)$ smaller if necessary, we see that Φ is a contraction map on $\mathcal{I}_{L,T}$, so that by the Banach fixed point theorem there exists a unique fixed point $(\theta,\mathbf{v}) \in \mathcal{I}_{L,T}$ such that $(\theta,\mathbf{v}) = \Phi(\theta,\mathbf{v})$ which solves the nonlinear equation (4.3). Since the existence of solutions to (4.3) is proved by the contraction mapping principle, the uniqueness of solutions automatically follows, which completes the proof of Theorem 4.1.

Proof of Theorem 2.1. Following Ströhmer [10], we see that the correspondence $x = \mathbf{X}_{\mathbf{v}}(\xi,t)$ is injective for any $t \in [0,T]$ provided that (4.2) holds with some small constant $\sigma > 0$. Let Γ be defined by $F(x) = 0$ locally. Since $\hat{\mathbf{n}}_{\mathbf{v}}$ is parallel to $(\nabla F)(\mathbf{X}_{\mathbf{v}}(\xi,t))$, so that the fact that $\mathbf{v}\cdot\hat{\mathbf{n}}_{\mathbf{v}} = 0$ on Γ implies that $\partial F(\mathbf{X}_{\mathbf{v}}(\xi,t))/\partial t = 0$, which furnishes that $F(\mathbf{X}_{\mathbf{v}}(\xi,t)) = F(\mathbf{X}_{\mathbf{v}}(\xi,0)) = F(\xi) = 0$ for $\xi \in \Gamma$. Therefore, we have $\{\mathbf{X}_{\mathbf{v}}(\xi,t) \mid \xi \in \Gamma\} \subset \Gamma$ for each $t \in (0,T)$. Note that $\mathbf{X}_{\mathbf{v}} \in C^0([0,T], W_q^2(\Omega) \cap W_p^1((0,T),W_q^2(\Omega))$. Let us fix $t \in (0,T)$. Employing the same argument as in proving the inverse mapping theorem for a non-degenerate C^1 map, we see that $\mathbf{X}_{\mathbf{v}}$ gives us a local diffeomorphism. We see that $\{\mathbf{X}_{\mathbf{v}}(\xi,t) \mid \xi \in \Gamma\}$ is non-empty, open and closed subset of Γ, so that the connectedness of Γ implies that $\Gamma = \{\mathbf{X}_{\mathbf{v}}(\xi,t) \mid \xi \in \Gamma\}$. Thus, we also see that $\{\mathbf{X}_{\mathbf{v}}(\xi,t) \mid \xi \in \Omega\} \subset \Omega$, because of the injectivity of $\mathbf{X}_{\mathbf{v}}$. Since Ω is a connected open set, we also see that $\Omega = \{\mathbf{X}_{\mathbf{v}}(\xi,t) \mid \xi \in \Omega\}$. Moreover, the inverse map: $\xi = \mathbf{X}_{\mathbf{v}}^{-1}(x,t) \in W_p^1((0,T),W_q^2(\Omega))$, and therefore setting $\rho = \theta(\mathbf{X}_{\mathbf{v}}^{-1}(x,t),t)$ and $\mathbf{u} = \mathbf{v}(\mathbf{X}_{\mathbf{v}}^{-1}(x,t),t)$, we see that $(\rho,\mathbf{u})$ satisfies the non-linear equations (1.1) and

$$\rho \in W_q^1((0,T),L_q(\Omega)) \cap L_p((0,T),W_q^1(\Omega)), \quad \mathbf{u} \in W_p^1((0,T),L_q(\Omega)) \cap L_p((0,T),W_q^2(\Omega)).$$

The uniqueness for (1.1) follows from the uniqueness of solutions to problem (4.3), which completes the proof of Theorem 2.1.

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