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Long time existence for the 3D incompressible Euler equations with high-speed rotation

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1 Introduction and Main Result

This note is a brief survey of our paper [14], which is a joint work with Professors Youngwoo Koh and Sanghyuk Lee.

We consider the initial value problem for the 3D rotating incompressible Euler equations:

$$\begin{cases} \partial_t u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p = 0 & t>0,\ x\in\mathbb{R}^3, \\ \operatorname{div} u = 0 & t>0,\ x\in\mathbb{R}^3, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^3, \end{cases} \tag{E_Ω}$$

where $u=(u_1(t,x),u_2(t,x),u_3(t,x))$ and $p=p(t,x)$ denote the unknown velocity field and the unknown scalar pressure, respectively, while $u_0=(u_{0,1}(x),u_{0,2}(x),u_{0,3}(x))$ denotes the given initial velocity field satisfying $\operatorname{div} u_0 = 0$. The constant $\Omega\in\mathbb{R}$ represents the speed of rotation of the fluids around the vertical unit vector $e_3=(0,0,1)$, which is called the Coriolis parameter.

In this note, we prove the long time existence of unique classical solutions to the Euler equations with the Coriolis force provided the speed of rotation is sufficiently high. More precisely, we shall show that for given initial velocity u_0 and for given time $0<T<\infty$ there exists a positive constant Ω_0 depending on s, T and the norm of u_0 such that if $|\Omega|\geqslant\Omega_0$ then (E_Ω) admits a unique classical solution on the given time interval $[0,T]$.

Let $\mathbb{P} := (\delta_{jk}+R_jR_k)_{1\leqslant j,k\leqslant 3}$ be the Helmholtz projection onto the divergence-free vector fields, where R_j denotes the Riesz transform in $\mathbb{R}^3$. Applying the projection $\mathbb{P}$ to both sides of the first equation of (E_Ω), we obtain the following system for the velocity fields:

$$\begin{cases} \partial_t u + \Omega\mathbb{P}(e_3 \times u) + \mathbb{P}(u\cdot\nabla)u = 0 & t>0,\ x\in\mathbb{R}^3, \\ \operatorname{div} u = 0 & t>0,\ x\in\mathbb{R}^3, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^3. \end{cases} \tag{E'_Ω}$$

We now summarize the local existence results on the original Euler equations for $\Omega = 0$. Kato [10] proved that for given integer $m \in \mathbb{Z}$ with $m > 5/2$ and for given initial velocity $u_0 \in H^m(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, there exists a positive time $T = T(m, \|u_0\|_{H^m})$ such that the equation (E_0') admits a unique classical solution u in the class $C([0,T]; H^m(\mathbb{R}^3)) \cap C^1([0,T]; H^{m-1}(\mathbb{R}^3))$. Kato and Ponce [11] extended this result to the fractional ordered Sobolev spaces $W^{s,p}(\mathbb{R}^3)$ for $s > 3/p + 1$, $1 < p < \infty$. Chae [2] and Chen, Miao and Zhang [7] obtained a local well-posedness for (E_0') in the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^3)$ for $s > 3/p + 1$, $1 < p, q < \infty$. Chae [3] also obtained the local well-posedness in the Besov spaces $B^s_{p,q}(\mathbb{R}^3)$ for $s > 3/p + 1$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and for $s = 3/p + 1$, $1 < p < \infty$, $q = 1$. Pak and Park [17] extended these results to the Besov space $B^1_{\infty,1}(\mathbb{R}^3)$.

For large Coriolis parameter $|\Omega|$, Dutrifoy [9] showed the asymptotics of solutions to vortex patches or Yudovich solutions as the Rossby number goes to zero for some particular initial data, and gave a lower bound on the lifespan T_Ω of the solution to (E_Ω') as $T_\Omega \gtrsim \log\log|\Omega|$. Similar results are obtained for the quasigeostrophic systems by Dutrifoy [8] and Charve [4].

Now, let us state our main result.

Theorem 1.1. *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then, for $0 < T < \infty$ and $u_0 \in H^{s+1}(\mathbb{R}^3)$ satisfying $\operatorname{div} u_0 = 0$, there exists a positive constant $\Omega_0 = \Omega_0(s, T, \|u_0\|_{H^{s+1}})$ such that if $|\Omega| \geqslant \Omega_0$ then (E_Ω') possesses a unique classical solution u satisfying*

$$u \in C([0,T]; H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T]; H^s(\mathbb{R}^3)).$$

In particular, for $2 < q < \infty$ there exist a positive absolute constant C_0 and a positive constant $C_1 = C_1(s,q)$ such that the parameter Ω_0 can be taken so that

$$\Omega_0 \geqslant C_0 \left[1 + \|u_0\|_{H^{s+1}} T \exp\left\{C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}}\right\}\right]^q. \tag{1.1}$$

Remark 1.2. From the characterization (1.1) it follows that for sufficiently high speed of rotation $|\Omega|$, the maximal existence time $T_\Omega \geqslant 1$ of the solution to (E_Ω') has a lower bound

$$T_\Omega \geqslant \frac{C_1'}{\|u_0\|_{H^{s+1}}} \log\left(\frac{|\Omega|}{C_0'}\right) \tag{1.2}$$

with some positive constants $C_0' = C_0'(q)$ and $C_1' = C_1'(s,q)$. On the other hand, in [8,9] it was shown that $T_\Omega \gtrsim \log\log|\Omega|$ as $|\Omega|$ tends to infinity. Although Theorem 1.1 does not cover the data such as vortex patches or Yudovich solutions which are treated in [8,9], the lower bound (1.2) can be regarded as an improvement for the lifespan of the solution to (E_Ω'). This is due to the use of a single exponential estimate for the blow-up criterion (see Section 3).

This note is organized as follows. In Section 2, we recall the definitions of function spaces, and the commutator estimates in these spaces. In Section 3, we state the local existence results and the blow-up criterion of Beale-Kato-Majda type for the local solutions. In Section 4, we shall give the Strichartz estimates for the propagator generated by the Coriolis force. In Section 5, we present the proof of Theorem 1.1.

Throughout this paper, we denote by C the constants which may differ from line to line. In particular, $C = C(\cdot, \dots, \cdot)$ will denote the constant which depends only on the quantities appearing in parentheses. For $A, B \geqslant 0$, $A \lesssim B$ means that there exists some positive constant C such that $A \leqslant CB$. Also, $A \gtrsim B$ is defined in the same way as $A \lesssim B$. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

2 Preliminaries

Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class, and let $\mathscr{S}'(\mathbb{R}^3)$ be the space of tempered distributions. We first recall the definition of the Littlewood–Paley decomposition. Let φ_0 be a function in $\mathscr{S}(\mathbb{R}^3)$ satisfying $0 \leqslant \widehat{\varphi_0}(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^3$, $\operatorname{supp}\widehat{\varphi_0} \subset \{\xi \in \mathbb{R}^3 \mid 1/2 \leqslant |\xi| \leqslant 2\}$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi_j}(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\varphi_j(x) := 2^{3j}\varphi_0(2^j x)$. We set $\widehat{\chi}(\xi) := 1 - \sum_{j\geqslant 1}\widehat{\varphi_j}(\xi)$. Let $\{\Delta_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley operator defined by $\Delta_j f := \varphi_j * f$ for $f \in \mathscr{S}'(\mathbb{R}^3)$. Then, we recall the definitions of the inhomogeneous and the homogeneous Besov spaces.

Definition 2.1. (i) For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{B^s_{p,q}} := \|\chi * f\|_{L^p} + \left\| \{2^{sj}\|\Delta_j f\|_{L^p}\}_{j=1}^{\infty} \right\|_{\ell^q} < \infty.$$

(ii) For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \left\| \{2^{sj}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \right\|_{\ell^q} < \infty.$$

Next, let $H^s(\mathbb{R}^3)$ denotes the Sobolev space of order $s \in \mathbb{R}$ with the inner product

$$\langle f, g\rangle_{H^s} := \int_{\mathbb{R}^3} (1-\Delta)^{\frac{s}{2}} f(x)\overline{(1-\Delta)^{\frac{s}{2}} g(x)}dx = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

and the norm $\|f\|_{H^s} := \sqrt{\langle f, f\rangle_{H^s}}$. For $s > 0$, it is known that the norm equivalence

$$\|f\|_{H^s} \sim \|f\|_{L^2} + \|f\|_{\dot{H}^s} \tag{2.1}$$

holds, where

$$\|f\|_{\dot{H}^s} := \|f\|_{\dot{B}^s_{2,2}} = \Big(\sum_{j\in\mathbb{Z}} 2^{2sj}\|\Delta_j f\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

We end this section by recalling several commutator estimates in the Sobolev spaces.

Lemma 2.2 (Kato–Ponce [11]). *For $s \geqslant 0$, there exists a positive constant $C = C(s)$ such that*

$$\left\|(1-\Delta)^{\frac{s}{2}}(fg) - f(1-\Delta)^{\frac{s}{2}}g\right\|_{L^2} \leqslant C\left(\|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}} + \|g\|_{L^\infty}\|f\|_{H^s}\right)$$

for all $f \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ and $g \in L^\infty(\mathbb{R}^3) \cap H^{s-1}(\mathbb{R}^3)$.

Lemma 2.3 (Chen–Miao–Zhang [7] and [19]). *For $s > 0$, there exists a positive constant $C = C(s)$ such that*

$$\Big(\sum_{j\in\mathbb{Z}} 2^{2sj}\,\|[u\cdot\nabla, \Delta_j]\,f\|_{L^2}^2\Big)^{\frac{1}{2}} \leqslant C\left(\|\nabla u\|_{L^\infty}\|f\|_{\dot{H}^s} + \|\nabla f\|_{L^\infty}\|u\|_{\dot{H}^s}\right)$$

for all $(u, f) \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)^{3+1}$ with $\operatorname{div} u = 0$.

3 Blow–up Criterion

In this section, we shall prove the blow–up criterion of the Beale–Kato–Majda type [1]. We first state the uniform local existence results of the unique classical solutions to (E'_Ω).

Theorem 3.1. *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then, for $u_0 \in H^s(\mathbb{R}^3)$ satisfying* $\operatorname{div} u_0 = 0$, *there exists a positive time $T = T(s, \|u_0\|_{H^s})$ depending only on s and $\|u_0\|_{H^s}$ such that* (E'_Ω) *possesses a unique classical solution u for all $\Omega \in \mathbb{R}$ satisfying*

$$u \in C([0,T]; H^s(\mathbb{R}^3)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3)).$$

As in the case of the original Euler equations $\Omega = 0$, Theorem 3.1 follows from the classical energy method and the Kato-Ponce commutator estimates (Lemma 2.2). Indeed, thanks to the skew-symmetry of the Coriolis force:

$$\int_{\mathbb{R}^3} \Omega e_3 \times u(t,x)\cdot u(t,x)dx = 0,$$

we can obtain the uniform H^s energy estimates with respect to $\Omega \in \mathbb{R}$. For the details, see [14].

Proposition 3.2. *Let $s > 5/2$, and let $u_0 \in H^s(\mathbb{R}^3)$ satisfy* $\operatorname{div} u_0 = 0$. *Suppose that u is a solution of* (E'_Ω) *in the class $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$. If*

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt < \infty,$$

then u can be continued to the solution in $C([0,T'); H^s(\mathbb{R}^3)) \cap C^1([0,T'); H^{s-1}(\mathbb{R}^3))$ for some $T' > T$.

An immediate consequence of the above proposition is the following.

Corollary 3.3. *Let* $s > 5/2$*, and let* $u_0 \in H^s(\mathbb{R}^3)$ *satisfy* $\operatorname{div} u_0 = 0$*. Suppose that* u *is a solution of* (E'_Ω) *in the class* $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$*. Suppose that* T *is maximal, that is,* u *cannot be continued to* $C([0,T'); H^s(\mathbb{R}^3)) \cap C^1([0,T'); H^{s-1}(\mathbb{R}^3))$ *for any* $T' > T$*. Then,*

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt = \infty.$$

Proof of Proposition 3.2. By Theorem 3.1, we see that if $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$, the time interval $[0,T)$ of the existence of solutions to (E'_Ω) in the class $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$ depends only on s and $\|u_0\|_{H^s}$. Hence by the standard argument which continues the local solutions, it suffices to establish an a priori estimate for u in $H^s(\mathbb{R}^3)$ in terms of s, $\|u_0\|_{H^s}$ and $\int_0^T \|\nabla u(t)\|_{L^\infty} dt$.

Taking the L^2–inner product of (E'_Ω) with u, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0$$

by the skew–symmetry of $e_3 \times u$ and the divergence–free condition. Hence we see that

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \tag{3.1}$$

for all $t \in [0,T)$. We next derive the $\dot{H}^s$–estimate for u. Applying the Littlewood–Paley operator Δ_j to both sides of (E'_Ω), we have

$$\partial_t \Delta_j u + \mathbb{P}\Omega e_3 \times \Delta_j u + \mathbb{P}\Delta_j (u \cdot \nabla) u = 0. \tag{3.2}$$

Taking the L^2–inner product of (3.2) with $\Delta_j u$, and using $\langle (u \cdot \nabla)\Delta_j u, \Delta_j u\rangle_{L^2} = 0$, which follows from the divergence–free condition, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u(t)\|_{L^2}^2 = -\langle \Delta_j (u(t) \cdot \nabla) u(t), \Delta_j u(t)\rangle_{L^2} = \langle [u(t) \cdot \nabla, \Delta_j] u(t), \Delta_j u(t)\rangle_{L^2}.$$

From this and the Schwartz inequality, we see that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u(t)\|_{L^2}^2 \leqslant \|[u(t) \cdot \nabla, \Delta_j] u(t)\|_{L^2} \|\Delta_j u(t)\|_{L^2}.$$

This gives

$$\frac{d}{dt}\|\Delta_j u(t)\|_{L^2} \leqslant \|[u(t) \cdot \nabla, \Delta_j] u(t)\|_{L^2}.$$

Integrating both sides on $[0,t]$, we have

$$\|\Delta_j u(t)\|_{L^2} \leqslant \|\Delta_j u_0\|_{L^2} + \int_0^t \|[u(\tau) \cdot \nabla, \Delta_j] u(\tau)\|_{L^2} d\tau. \tag{3.3}$$

Multiplying both sides of (3.3) by 2^{sj} and then taking the $\ell^2(\mathbb{Z})$–norm, we have from the Minkowski inequality that

$$\|u(t)\|_{\dot{H}^s} \leqslant \|u_0\|_{\dot{H}^s} + \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{2sj} \|[u(\tau)\cdot\nabla, \Delta_j]\, u(\tau)\|_{L^2}^2 \right)^{\frac{1}{2}} d\tau.$$

By Lemma 2.3, it follows that

$$\|u(t)\|_{\dot{H}^s} \leqslant \|u_0\|_{\dot{H}^s} + C\int_0^t \|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{H^s} d\tau$$

with some positive constant $C = C(s)$. Combining (3.1) and the above inequality, by (2.1) we see that there exists a positive constant $C = C(s)$ such that

$$\|u(t)\|_{H^s} \leqslant C\,\|u_0\|_{H^s} + C\int_0^t \|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{H^s} d\tau.$$

Hence from the Gronwall inequality we obtain

$$\|u(t)\|_{H^s} \leqslant C\|u_0\|_{H^s} \exp\left\{ C\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\} \tag{3.4}$$

for all $t \in [0, T)$. This completes the proof of Proposition 3.2. □

4 Strichartz Estimates

In this section, we shall give the sharp Strichartz estimates for the propagator generated by the Coriolis force $\Omega e_3 \times u$. As it is already observed in [5, 6, 9], the system (E'_Ω) exhibits a dispersion phenomenon which is due to the presence of the Coriolis force. This is closely related to the Strichartz estimates for the operator $e^{\pm i\Omega t\frac{D_3}{|D|}}$ defined by the Fourier integral

$$e^{\pm i\Omega t\frac{D_3}{|D|}} f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm i\Omega t\frac{\xi_3}{|\xi|}} \widehat{f}(\xi) d\xi, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3.$$

Here $\widehat{f}$ denotes the Fourier transform of f. We are interested in the space-time Strichartz estimate

$$\left\| e^{\pm it\frac{D_3}{|D|}} f \right\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s}, \tag{4.1}$$

where Ωt is replaced by $t \in \mathbb{R}$. By the scaling $f \mapsto f(\lambda\cdot)$ for $\lambda > 0$, it is easy to see that the exponents q, r and s should satisfy $s = 3/2 - 3/r$. Once this estimate is established the simple change of variables $t \mapsto \Omega t$ shows that the effect of the Coriolis parameter Ω is given by $C|\Omega|^{-\frac{1}{q}}$. Also, since the function $\xi_3/|\xi|$ is homogeneous of degree 0, by the

Littlewood-Paley decomposition and the scaling argument, the matter is reduced to the frequency localized case. Now we consider the operator

$$\mathscr{G}_\pm(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \widehat{\Phi}(\xi)\widehat{f}(\xi)d\xi, \quad (t,x) \in \mathbb{R}^{1+3},$$

where $\Phi \in \mathscr{S}(\mathbb{R}^3)$ satisfies $\operatorname{supp}\widehat{\Phi} \subset \{2^{-2} \leqslant |\xi| \leqslant 2^2\}$ and $\widehat{\Phi} = 1$ on $\{2^{-1} \leqslant |\xi| \leqslant 2\}$. The operator $\mathscr{G}_\pm$ was previously studied by Dutrifoy [9, Corollary 2], who obtained the space-time estimates

$$\|\mathscr{G}_\pm(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2} \tag{4.2}$$

for the exponents q and r which satisfy $4 < q \leqslant \infty, 2 \leqslant r \leqslant \infty$ and the admissible relation $1/q + 1/(2r) < 1/4$. We extend the estimate to the optimal range except an endpoint case. More precisely, we shall prove the following.

Theorem 4.1. *Let $2 \leqslant q, r \leqslant \infty$ with $(q,r) \neq (2,\infty)$. Then, the space-time estimate* (4.2) *holds if and only if*

$$\frac{1}{q} + \frac{1}{r} \leqslant \frac{1}{2}. \tag{4.3}$$

As it was shown by Montgomery-Smith [16], it seems likely that one can show the failure of the endpoint case $(q,r) = (2,\infty)$.

By (4.2), the Littlewood-Paley theory and the embedding $\dot{B}^0_{r,2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ for $2 \leqslant r < \infty$, we can prove the Strichartz estimates (4.1) for the original operator as a corollary of Theorem 4.1. Then by the scaling $t \mapsto \Omega t$ we have the following which shows the dispersive effect of the Coriolis forces.

Corollary 4.2. *Let $2 \leqslant q \leqslant \infty$ and $2 \leqslant r < \infty$, and let $\Omega \in \mathbb{R} \setminus \{0\}$. Then, the space-time estimate*

$$\left\| e^{\pm i\Omega t\frac{D_3}{|D|}} f \right\|_{L^q_t L^r_x} \lesssim |\Omega|^{-\frac{1}{q}} \|f\|_{\dot{H}^{\frac{3}{2}-\frac{3}{r}}}$$

holds if q and r satisfy (4.3).

Let us set

$$\Psi(\xi) := \frac{\xi_3}{|\xi|}, \quad \xi \neq 0.$$

As it is well-known, the boundedness of the estimate (4.2) is closely related to the curvature of the surface

$$\Sigma_0 := \left\{ (\xi,\rho) \in \mathbb{R}^3 \times \mathbb{R} \,\middle|\, \rho = \Psi(\xi),\ \frac{1}{4} \leqslant |\xi| \leqslant 4 \right\}.$$

A direct computation shows that the Hessian matrix $H\Psi = (\frac{\partial^2\Psi}{\partial\xi_j\partial\xi_k})_{1\leqslant j,k\leqslant 3}$ of Ψ is equal to

$$H\Psi(\xi) = \frac{1}{|\xi|^5}\begin{pmatrix} \xi_3(3\xi_1^2 - |\xi|^2) & 3\xi_1\xi_2\xi_3 & \xi_1(3\xi_3^2 - |\xi|^2) \\ 3\xi_1\xi_2\xi_3 & \xi_3(3\xi_2^2 - |\xi|^2) & \xi_2(3\xi_3^2 - |\xi|^2) \\ \xi_1(3\xi_3^2 - |\xi|^2) & \xi_2(3\xi_3^2 - |\xi|^2) & -3\xi_3(\xi_1^2 + \xi_2^2) \end{pmatrix},$$

and

$$\det H\Psi(\xi) = \frac{(\xi_1^2+\xi_2^2)\xi_3}{|\xi|^9}. \tag{4.4}$$

This shows that the surface Σ_0 has 3 non-vanishing principal curvatures (non-vanishing Gaussian curvature) unless $(\xi_1, \xi_2) = 0$ or $\xi_3 = 0$. Hence, taking a sufficiently small $c > 0$, we decompose $\mathscr{G}_\pm(t)$ as follows:

$$\mathscr{G}_\pm(t)f = \mathscr{G}_\pm^1(t)f + \mathscr{G}_\pm^2(t)f + \mathscr{G}_\pm^3(t)f,$$

where

$$\mathscr{G}_\pm^1(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} (1-\psi(|\xi_h|/c))(1-\psi(|\xi_3|/c))\widehat{\Phi}(\xi)\widehat{f}(\xi)d\xi,$$

$$\mathscr{G}_\pm^2(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \psi(|\xi_h|/c)\widehat{\Phi}(\xi)\widehat{f}(\xi)d\xi,$$

$$\mathscr{G}_\pm^3(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \psi(|\xi_3|/c)\widehat{\Phi}(\xi)\widehat{f}(\xi)d\xi.$$

Here, ψ is a compactly supported smooth function on the half line $[0,\infty)$ such that $\psi = 1$ on $[0, 1/2]$ and $\operatorname{supp}\psi \subset [0,1]$, and we denote $\xi = (\xi_h, \xi_3) \in \mathbb{R}^2 \times \mathbb{R}$. Note that $\operatorname{supp}\psi(|\xi_h|/c)\ \psi(|\xi_3|/c) \cap \operatorname{supp}\widehat{\Phi}(\xi) = \emptyset$ if c is small enough, to say $c < 2^{-3}$.

For the proof of Theorem 4.1, let us recall the following abstract Strichartz estimates by Keel and Tao [12].

Theorem 4.3 (Keel–Tao [12]). *Let $\{U(t)\}_{t\in\mathbb{R}}$ be a family of operators. Suppose that for all $t, s \in \mathbb{R}$*

$$\|U(s)(U(t))^*f\|_{L^\infty} \lesssim (1+|t-s|)^{-\sigma}\|f\|_{L^1}, \tag{4.5}$$

$$\|U(s)(U(t))^*f\|_{L^2} \lesssim \|f\|_{L^2}. \tag{4.6}$$

Then the estimate

$$\|U(t)f\|_{L_t^qL_x^r} \lesssim \|f\|_{L^2}$$

holds for all $2 \leqslant q, r \leqslant \infty$ with $(q, r, \sigma) \neq (2, \infty, 1)$ satisfying

$$\frac{1}{q} + \frac{\sigma}{r} \leqslant \frac{\sigma}{2}.$$

Therefore, for the proof of the sufficiency part of Theorem 4.1, it suffices to show the dispersive estimates (4.5) since the second property (4.6) follows from the Plancherel theorem.

Proposition 4.4 ([13]). *The dispersive estimates*

$$\left\|\mathscr{G}_\pm^1(t)f\right\|_{L^\infty} \lesssim (1+|t|)^{-\frac{3}{2}}\|f\|_{L^1}, \tag{4.7}$$

$$\left\|\mathscr{G}_\pm^2(t)f\right\|_{L^\infty} \lesssim (1+|t|)^{-1}\|f\|_{L^1}, \tag{4.8}$$

$$\left\|\mathscr{G}_\pm^3(t)f\right\|_{L^\infty} \lesssim (1+|t|)^{-1}\|f\|_{L^1} \tag{4.9}$$

hold for all $t \in \mathbb{R}$ and $f \in L^1(\mathbb{R}^3)$.

For the readers' convenience, we present the proof of Proposition 4.4. To this end, we recall the following lemma due to Littman [15].

Lemma 4.5 (Littman [15], Stein [18, page 361]). *Let $d\mu$ be a surface measure on a smooth surface S in $\mathbb{R}^n$, and let $\phi \in C_0^\infty(\mathbb{R}^n)$. Suppose that for all $x \in S$, at least k of the principal curvatures are not zero. Then, it holds*

$$\left|\widehat{\phi d\mu}(\eta)\right| \leqslant A|\eta|^{-\frac{k}{2}}.$$

For the surface Σ_0, the number of non-vanishing principal curvatures is equal to the number of the nonzero eigenvalues, that is, the rank of the symmetric matrix $H\Psi$.

Proof of Proposition 4.4. The desired estimates (4.7), (4.8) and (4.9) follow from the estimates

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \left(1-\psi(|\xi_h|/c)\right)\left(1-\psi(|\xi_3|/c)\right)\widehat{\Phi}(\xi)d\xi\right| \leqslant C(1+|t|)^{-\frac{3}{2}}, \tag{4.10}$$

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \psi(|\xi_h|/c)\widehat{\Phi}(\xi)d\xi\right| \leqslant C(1+|t|)^{-1}, \tag{4.11}$$

and

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \psi(|\xi_3|/c)\widehat{\Phi}(\xi)d\xi\right| \leqslant C(1+|t|)^{-1}. \tag{4.12}$$

Since $\det H\Psi \neq 0$ on the support of $\left(1-\psi(|\xi_h|/c)\right)\left(1-\psi(|\xi_3|/c)\right)\widehat{\Phi}(\xi)$ by (4.4), the first estimates (4.10) follows from the standard stationary phase method or Lemma 4.5.

In order to show (4.11) and (4.12), we shall use Lemma 4.5. We show that on the support of $\psi(|\xi_h|/c)\widehat{\Phi}(\xi)$ and $\psi(|\xi_3|/c)\widehat{\Phi}(\xi)$ the $H\Psi$ has two eigenvalues with their absolute values near to 1. In fact, the determinant of $H\Psi$ is zero if $\xi_1 = \xi_2 = 0$ or $\xi_3 = 0$. Hence it is sufficient to consider the cases $\xi_1 = \xi_2 = 0$ and $\xi_3 = 0$ while $|\xi| \sim 1$ (more precisely, $1/4 \leqslant |\xi| \leqslant 4$), and show that there are two non-zero eigenvalues. Then choosing sufficiently small $c > 0$, we obtain the desired estimates by the continuity.

In the case $\xi_1 = \xi_2 = 0$, we see that

$$H\Psi(0,0,\xi_3) = \frac{1}{|\xi_3|^5}\begin{pmatrix} -\xi_3^3 & 0 & 0 \\ 0 & -\xi_3^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $|\xi| \sim 1$ and c is small, $|\xi_3| \sim 1$. Thus, two non-zero eigenvalues $-\xi_3^3|\xi_3|^{-5}$ and $-\xi_3^3|\xi_3|^{-5}$ are of absolute value near to 1.

In the case $\xi_3 = 0$, we have that

$$H\Psi(\xi_1,\xi_2,0) = \frac{1}{|\xi_h|^5}\begin{pmatrix} 0 & 0 & -\xi_1|\xi_h|^2 \\ 0 & 0 & -\xi_2|\xi_h|^2 \\ -\xi_1|\xi_h|^2 & -\xi_2|\xi_h|^2 & 0 \end{pmatrix}.$$

A calculation gives

$$\det\left(\lambda I - H\Psi(\xi_1,\xi_2,0)\right) = \lambda\left(\lambda - \frac{1}{|\xi_h|^2}\right)\left(\lambda + \frac{1}{|\xi_h|^2}\right).$$

Since $|\xi| \sim 1$ and c is small, we see that $|\xi_h| \sim 1$. Therefore, two non-zero eigenvalues are of absolute value near to 1. This completes the proof of Proposition 4.4. □

Proof of the necessity part of Theorem 4.1. We shall show the necessity of the condition (4.3). For $0 < \delta \ll 1$ and $N \gg 1$, let us consider

$$\mathcal{R} := \left\{\xi \in \mathbb{R}^3 \mid \xi_1, \xi_2 \in [1/2, 1],\ \xi_3 \in [\delta, 2\delta]\right\},$$
$$A := \left\{(t,x) \in \mathbb{R}^{1+3} \mid |x_1|, |x_2| \leqslant N^{-1},\ |x_3| \leqslant N^{-1}\delta^{-1},\ |t| \leqslant N^{-1}\delta^{-1}\right\}$$

and define f_δ by setting $\widehat{f_\delta} = \chi_{\mathcal{R}}$, which denotes the characteristic function on $\mathcal{R}$. Choosing sufficiently small δ, we see that $\mathcal{R} \subset \left\{\xi \in \mathbb{R}^3 \mid 1/2 \leqslant |\xi| \leqslant 2\right\}$ and

$$\left|x \cdot \xi \pm t\frac{\xi_3}{|\xi|}\right| \lesssim \frac{1}{N}$$

if $\xi \in \mathcal{R}$ and $(t,x) \in A$. Hence we have

$$|\mathscr{G}_\pm(t) f_\delta(x)| \geqslant \left|\int_{\mathcal{R}} \cos\left(x \cdot \xi \pm t\frac{\xi_3}{|\xi|}\right) d\xi\right| \gtrsim |\mathcal{R}| \sim \delta$$

for $(t,x) \in A$ with sufficiently large N. Therefore (4.2) implies that

$$\delta^{1-\frac{1}{q}-\frac{1}{r}} \lesssim \|\mathscr{G}_\pm(t) f_\delta\|_{L^q_t L^r_x(A)} \leqslant \|\mathscr{G}_\pm(t) f_\delta\|_{L^q_t L^r_x} \lesssim \|f_\delta\|_{L^2} \sim \delta^{\frac{1}{2}}.$$

The condition (4.3) follows by letting $\delta \to 0$. □

5 Proof of Theorem 1.1

We shall prove that the local solution u to (E'_Ω) constructed by Theorem 3.1 can be extended to any time interval $[0,T]$ provided the speed of rotation is high enough. To this end we adapt the ideas in [4,9].

Let $u_0 \in H^{s+1}(\mathbb{R}^3)$ satisfy $\operatorname{div} u_0 = 0$, and let u be the solution to (E'_Ω) in the class $C([0,T_\Omega); H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T_\Omega); H^s(\mathbb{R}^3)))$, where $0 < T_\Omega < \infty$ denotes the maximal time of the existence. We define the projection operator $P_\pm$ by

$$P_\pm v := \frac{1}{2}\left(\mathbb{P}v \pm i\frac{D}{|D|} \times v\right)$$

for $v \in L^2(\mathbb{R}^3)$. Note that $P_\pm \mathbb{P} = P_\pm$. Furthermore, for the divergence–free vector field v,

$$v = P_+v + P_-v, \quad \mathbb{P}e_3 \times v = -i\frac{D_3}{|D|}\left(P_+v - P_-v\right), \quad P_\pm P_\pm v = P_\pm v, \quad P_\pm P_\mp v = 0. \tag{5.1}$$

Hence applying $P_\pm$ to both sides of (E'_Ω), we have

$$\begin{cases} \partial_t P_\pm u \mp i\Omega\frac{D_3}{|D|}P_\pm u + P_\pm(u\cdot\nabla)u = 0, \\ P_\pm u(0,x) = P_\pm u_0(x). \end{cases}$$

By the Plancherel theorem and the Lebesgue dominated convergence theorem, we see that $A_\pm := \pm i\Omega\frac{D_3}{|D|}$ is the infinitesimal generator of the C_0 semigroup $e^{\pm i\Omega t\frac{D_3}{|D|}}$ on $L^2(\mathbb{R}^3)$ with the domain of the generator $D(A_\pm) = L^2(\mathbb{R}^3)$. Therefore by the Duhamel principle we obtain

$$P_\pm u(t) = e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0 - \int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}}P_\pm(u(\tau)\cdot\nabla)u(\tau)d\tau. \tag{5.2}$$

Since $s > 5/2$, we can take $\alpha = \alpha(s) \in (0,1)$ so that $s \geqslant 5/2+\alpha$. In what follows we show the $B^{1+\alpha}_{\infty,\infty}$–estimate for the solution u. By (5.1) it suffices to show the estimates for P_+u, P_-u, separately. By Theorem 4.1 and scaling, for $2 < q < \infty$ there exists a positive constant $C = C(q)$ such that

$$\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}f\right\|_{L^q(0,\infty;L^\infty)} \leqslant C|\Omega|^{-\frac{1}{q}}(2^j)^{\frac{3}{2}}\|\Delta_j f\|_{L^2} \tag{5.3}$$

for all $j \in \mathbb{Z}$ and $\Omega \in \mathbb{R}\setminus\{0\}$. For the high frequency part of the first term in the right hand side of (5.2), by the Minkowski inequality and (5.3), we see that

$$\begin{aligned} &\left\|\sup_{j\geqslant 1} 2^{(1+\alpha)j}\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^\infty}\right\|_{L^q_t(0,\infty)} \\ &\leqslant \left\|\Big(\sum_{j\geqslant 1} 2^{2(1+\alpha)j}\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|^2_{L^\infty}\Big)^{\frac{1}{2}}\right\|_{L^q_t(0,\infty)} \\ &\leqslant \Big(\sum_{j\geqslant 1} 2^{2(1+\alpha)j}\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|^2_{L^q(0,\infty;L^\infty)}\Big)^{\frac{1}{2}} \\ &\leqslant C|\Omega|^{-\frac{1}{q}}\Big(\sum_{j\geqslant 1} 2^{2(\frac{5}{2}+\alpha)j}\|\Delta_j P_\pm u_0\|^2_{L^2}\Big)^{\frac{1}{2}}. \end{aligned}$$

Hence, by the Plancherel theorem, we have

$$\left\|\sup_{j\geqslant 1} 2^{(1+\alpha)j}\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^\infty}\right\|_{L^q_t(0,\infty)} \leqslant C|\Omega|^{-\frac{1}{q}}\|u_0\|_{\dot H^{\frac{5}{2}+\alpha}}. \tag{5.4}$$

We now handle the low frequency part. Since $u_0 \in H^{s+1}(\mathbb{R}^3)$, we have $e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0 \in H^{s+1}(\mathbb{R}^3)$ and $e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0 = \sum_{j\in\mathbb{Z}}\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0$ in $L^\infty(\mathbb{R}^3)$. Hence it follows from (5.3) that

$$\left\|\chi * e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^q(0,\infty;L^\infty)} \leqslant C\left\|\sum_{j=-\infty}^{2}\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^\infty}\right\|_{L^q(0,\infty)}$$

$$\leqslant C\sum_{j=-\infty}^{2}\left\|\Delta_j e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^q(0,\infty;L^\infty)}$$

$$\leqslant C|\Omega|^{-\frac{1}{q}}\sum_{j=-\infty}^{2}2^{\frac{3}{2}j}\left\|\Delta_j u_0\right\|_{L^2}$$

$$\leqslant C|\Omega|^{-\frac{1}{q}}\Big(\sum_{j=-\infty}^{2}2^{3j}\Big)^{\frac{1}{2}}\Big(\sum_{j=-\infty}^{2}\left\|\Delta_j u_0\right\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

Hence we have

$$\left\|\chi * e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^q(0,\infty;L^\infty)} \leqslant C|\Omega|^{-\frac{1}{q}}\|u_0\|_{L^2}. \tag{5.5}$$

Combining (5.4) and (5.5), by (2.1) we obtain

$$\left\|e^{\pm i\Omega t\frac{D_3}{|D|}}P_\pm u_0\right\|_{L^q(0,\infty;B^{1+\alpha}_{\infty,\infty})} \leqslant C|\Omega|^{-\frac{1}{q}}\|u_0\|_{H^{\frac{5}{2}+\alpha}} \tag{5.6}$$

for some constant $C = C(q,\alpha) > 0$. Next we consider the nonlinear term in (5.2). It follows from the Minkowski inequality and (5.3) that

$$\begin{aligned}
&\left\|\Delta_j\int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}}P_\pm(u(\tau)\cdot\nabla)u(\tau)d\tau\right\|_{L^q(0,T;L^\infty)}\\
&\leqslant\int_0^T\left\|\Delta_j e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}}P_\pm(u(\tau)\cdot\nabla)u(\tau)\right\|_{L^q_t(\tau,T;L^\infty)}d\tau\\
&\leqslant C|\Omega|^{-\frac{1}{q}}(2^j)^{\frac{3}{2}}\int_0^T\|\Delta_j(u(\tau)\cdot\nabla)u(\tau)\|_{L^2}d\tau
\end{aligned} \tag{5.7}$$

for all $0 < T < T_\Omega$. Hence for the high frequency part, by the Minkowski inequality and the above inequality we see that

$$\begin{aligned}
&\left\|\sup_{j\geqslant1}2^{(1+\alpha)j}\left\|\Delta_j\int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}}P_\pm(u(\tau)\cdot\nabla)u(\tau)d\tau\right\|_{L^\infty}\right\|_{L^q(0,T)}\\
&\leqslant\left\|\Big(\sum_{j\geqslant1}2^{2(1+\alpha)j}\left\|\Delta_j\int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}}P_\pm(u(\tau)\cdot\nabla)u(\tau)d\tau\right\|_{L^\infty}^2\Big)^{\frac{1}{2}}\right\|_{L^q(0,T)}\\
&\leqslant\Big(\sum_{j\geqslant1}2^{2(1+\alpha)j}\left\|\Delta_j\int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}}P_\pm(u(\tau)\cdot\nabla)u(\tau)d\tau\right\|_{L^q(0,T;L^\infty)}^2\Big)^{\frac{1}{2}}\\
&\leqslant C|\Omega|^{-\frac{1}{q}}\Big\{\sum_{j\geqslant1}2^{2(\frac{5}{2}+\alpha)j}\Big(\int_0^T\|\Delta_j(u(\tau)\cdot\nabla)u(\tau)\|_{L^2}d\tau\Big)^2\Big\}^{\frac{1}{2}}\\
&\leqslant C|\Omega|^{-\frac{1}{q}}\int_0^T\Big(\sum_{j\geqslant1}2^{2(\frac{5}{2}+\alpha)j}\|\Delta_j(u(\tau)\cdot\nabla)u(\tau)\|_{L^2}^2\Big)^{\frac{1}{2}}d\tau.
\end{aligned}$$

Therefore, we obtain

$$\left\| \sup_{j\geqslant 1} 2^{(1+\alpha)j} \left\| \Delta_j \int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}} P_\pm (u(\tau)\cdot\nabla)u(\tau)d\tau \right\|_{L^\infty} \right\|_{L^q(0,T)} \leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \|(u(\tau)\cdot\nabla)u(\tau)\|_{\dot H^{\frac{5}{2}+\alpha}} d\tau. \tag{5.8}$$

For the low frequency part, we repeat the line of argument which was used to show (5.5). In fact, by the Minkowski inequality and (5.7) we see that

$$\begin{aligned}
&\left\| \chi * \int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}} P_\pm (u(\tau)\cdot\nabla)u(\tau)d\tau \right\|_{L^q(0,T;L^\infty)} \\
&\leqslant C \sum_{j=-\infty}^{2} \left\| \Delta_j \int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}} P_\pm (u(\tau)\cdot\nabla)u(\tau)d\tau \right\|_{L^q(0,T;L^\infty)} \\
&\leqslant C|\Omega|^{-\frac{1}{q}} \sum_{j=-\infty}^{2} (2^j)^{\frac{3}{2}} \int_0^T \|(u(\tau)\cdot\nabla)u(\tau)\|_{L^2} d\tau.
\end{aligned}$$

Then, by the Schwarz inequality as before, we have

$$\begin{aligned}
&\left\| \chi * \int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}} P_\pm (u(\tau)\cdot\nabla)u(\tau)d\tau \right\|_{L^q(0,T;L^\infty)} \\
&\leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \Big(\sum_{j=-\infty}^{2} \|\Delta_j (u(\tau)\cdot\nabla)u(\tau)\|_{L^2}^2 \Big)^{\frac{1}{2}} d\tau \\
&\leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \|(u(\tau)\cdot\nabla)u(\tau)\|_{L^2} d\tau.
\end{aligned}$$

From this and (5.8), we see that

$$\begin{aligned}
&\left\| \int_0^t e^{\pm i\Omega(t-\tau)\frac{D_3}{|D|}} P_\pm (u(\tau)\cdot\nabla)u(\tau)d\tau \right\|_{L^q(0,T;B^{1+\alpha}_{\infty,\infty})} \\
&\leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \Big(\|(u(\tau)\cdot\nabla)u(\tau)\|_{L^2} + \|(u(\tau)\cdot\nabla)u(\tau)\|_{\dot H^{\frac{5}{2}+\alpha}} \Big) d\tau \\
&\leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \|(u(\tau)\cdot\nabla)u(\tau)\|_{H^{\frac{5}{2}+\alpha}} d\tau
\end{aligned}$$

for all $0 < T < T_\Omega$ with some constant $C = C(q,\alpha) > 0$. Therefore, by combining this and (5.6) and by recalling (5.2) and (5.1), we see that for $2 < q < \infty$ there exists a positive constant $C = C(q,\alpha)$ such that

$$\|u\|_{L^q(0,T;B^{1+\alpha}_{\infty,\infty})} \leqslant C|\Omega|^{-\frac{1}{q}} \left(\|u_0\|_{H^{\frac{5}{2}+\alpha}} + \int_0^T \|(u(\tau)\cdot\nabla)u(\tau)\|_{H^{\frac{5}{2}+\alpha}} d\tau \right) \tag{5.9}$$

for all $0 < T < T_\Omega$.

Now, let us define

$$V(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau, \quad 0 < t < T_\Omega.$$

Since $s \geqslant 5/2 + \alpha$, by applying the continuous embeddings $B^{1+\alpha}_{\infty,\infty}(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$, the Hölder inequality, (5.9) and (3.4), successively, we see that

$$\begin{aligned} V(t) &\leqslant C \int_0^t \|u(\tau)\|_{B^{1+\alpha}_{\infty,\infty}}\, d\tau \leqslant Ct^{1-\frac{1}{q}} \|u\|_{L^q(0,t;B^{1+\alpha}_{\infty,\infty})} \\ &\leqslant Ct^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \left(\|u_0\|_{H^s} + \int_0^t \|(u(\tau)\cdot\nabla)u(\tau)\|_{H^s}\, d\tau \right) \\ &\leqslant Ct^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \left(\|u_0\|_{H^{s+1}} + \int_0^t \|u(\tau)\|^2_{H^{s+1}}\, d\tau \right) \\ &\leqslant Ct^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \left(\|u_0\|_{H^{s+1}} + \|u_0\|^2_{H^{s+1}} \int_0^t e^{CV(\tau)} d\tau \right). \end{aligned}$$

Hence, there exist positive constants $C_1 = C_1(q, s)$ and $C_2 = C_2(s)$ such that

$$V(t) \leqslant C_1 t^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \|u_0\|_{H^{s+1}} \left(1 + \|u_0\|_{H^{s+1}} t e^{C_2 V(t)}\right) \tag{5.10}$$

for all $0 < t < T_\Omega$. Now, for given $0 < T < \infty$, let us define

$$X_{T,\Omega} := \left\{ t \in [0,T] \cap [0,T_\Omega) \;\middle|\; V(t) \leqslant C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}} \right\}, \quad \widetilde{T_\Omega} := \sup X_{T,\Omega}.$$

We shall prove that $\widetilde{T_\Omega} = \min\{T, T_\Omega\}$ by contradiction. Assume that $\widetilde{T_\Omega} < \min\{T, T_\Omega\}$. Then we can take $\widetilde{T}$ so that $\widetilde{T_\Omega} < \widetilde{T} < \min\{T, T_\Omega\}$. Since $u \in C([0,\widetilde{T}]; H^{s+1}(\mathbb{R}^3))$, $V(t)$ is uniformly continuous on $[0, \widetilde{T}]$, and it holds that

$$V(\widetilde{T_\Omega}) \leqslant C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}}. \tag{5.11}$$

Take a sufficiently large $\Omega \in \mathbb{R} \setminus \{0\}$ so that

$$|\Omega|^{\frac{1}{q}} \geqslant 2 \left(1 + \|u_0\|_{H^{s+1}} T e^{C_1 C_2 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}}} \right). \tag{5.12}$$

Then, since $\widetilde{T_\Omega} < T$, by (5.10), (5.11) and (5.12) we have

$$\begin{aligned} V(\widetilde{T_\Omega}) &\leqslant C_1 (\widetilde{T_\Omega})^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \|u_0\|_{H^{s+1}} \left(1 + \|u_0\|_{H^{s+1}} \widetilde{T_\Omega} e^{C_2 V(\widetilde{T_\Omega})}\right) \\ &\leqslant C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}} |\Omega|^{-\frac{1}{q}} \left(1 + \|u_0\|_{H^{s+1}} T e^{C_1 C_2 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}}} \right) \\ &\leqslant \frac{1}{2} C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}}. \end{aligned}$$

Hence there exists S such that $\widetilde{T}_\Omega < S < \widetilde{T}$ and $V(S) \leqslant C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}}$, which contradicts the definition of $\widetilde{T}_\Omega$. Thus, we have $\widetilde{T}_\Omega = \min\{T, T_\Omega\}$ provided the Coriolis parameters Ω satisfy (5.12).

If $T_\Omega < T$, we have $T_\Omega = \widetilde{T}_\Omega = \sup X_{T,\Omega}$. Therefore, it holds that

$$V(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \leqslant C_1 T^{1-\frac{1}{q}} \|u_0\|_{H^{s+1}} < \infty$$

for all $0 \leqslant t < T_\Omega$. By Corollary 3.3, this contradicts the maximality of T_Ω. Hence we obtain that $T_\Omega \geqslant T$ if Ω is as large as in (5.12). This completes the proof of Theorem 1.1.

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