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On a continuity of quantum statistical models in the infinite-dimensional Hilbert space

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Abstract

Let us consider functions from a locally compact metric space to trace-class operators on a separable Hilbert space. It is a basic framework in constructing quantum statistical models. In theoretical development, Holevo gave one definition of continuity of those functions in a more general setting. A regularity condition of quantum statistical models is derived from this definition but it seems complicated. We show that it is rewritten in a simple form.

1 Introduction

In the workshop, many statistical methods including sparse modeling, Lasso, etc, have been introduced to non-statistical audience (mainly experts in quantum physics). Some of them could be applied to quantum state tomography and others require some nontrivial developments. Quantum state tomography in a finite-dimensional Hilbert space has been intensively investigated by many authors. Almost of them except for statisticians do not care about statistical modeling or a general parametric model. However, such a naive approach is no longer available if we deal with the state tomography in the infinite-dimensional Hilbert space, which is the main arena for quantum optics.

The author believes that in infinite-dimensional Hilbert spaces statistical modeling of density operators including construction of finite-dimensional parametric models, model selection, Bayesian analysis, becomes much important. However, much more technical difficulties also appear. Even regularity conditions of quantum statistical models have not been fully investigated.

Although some readers may refer to works by Holevo [2, 4], he clarified only a tiny part of statistical theory following the classical path by Wald [5]. One of reasons is that his motivation of theoretical development is *not* practical application to experimental physics.

Here, in the short article, we investigate a continuity of quantum statistical models. Usually, we often write a parametric family of density operators as $\{\rho(\theta) : \theta \in \Theta\}$ but this naive treatise is troublesome in theoretical development. For example, it is known that some proofs in statistical decision theory heavily depend on its continuity.

In finite-dimensional Hilbert spaces, which is often identified with a complex vector space, we do not have to take the continuity of a parametric model of density operators seriously. However, in infinite-dimensional Hilbert spaces, we have many possibilities in the definition of the continuity, which are no longer the same.

In functional analysis, the existence of a limit point is naturally required and thus we usually adopt a kind of weak topology mainly through linear functionals. However, as Holevo [3] mentioned, we need much stronger topology if we introduce the operator-valued integral. He introduced the class of operator-valued functions which are approximated by a finite sum of the form $\sum_{j=1}^{N} f_j(\theta)X_j$, where X_j is self-adjoint operator with finite norm and $f_j(\theta)$ is a continuous function over Θ. Thus, we are able to discuss whether a parametric family of density operators $\{T(\theta)\}_{\theta\in\Theta}$ is included in this class or not by investigating the regularity condition he gave. However, his condition seems complicated and difficult to understand.

We here emphasize that main difficulties come from 1) infinite-dimensionality and 2) noncommutativity. Both of them are essential. If we consider finite-dimensional cases, then the condition is easily rewritten in other simple terms. If we consider infinite-dimensional cases but commutative parts, the same holds.

In the present article, we show that his condition has become simple in a separable Hilbert space. In Section 2, we review Holevo's definition of the continuity in our setting, which we call regular in order to distinguish other definitions of the continuity. In Section 3, we obtain a more familiar equivalent condition, the uniform continuity with respect to the trace norm on every compact set. Our proof requires a simple lemma (Lemma 6). In Section 4, we introduce a similar quantity based on the operator norm and compare it with that based on the trace norm. We also present a one-dimensional quantum statistical model that is not regular in our sense but seems intuitively continuous. Concluding remarks follow in Section 5.

2 Continuity of Quantum Statistical Models

2.1 Preliminary

Let $\mathcal{H}$ denote a separable Hilbert space with $\dim \mathcal{H} = \infty$. We mainly deal with the trace-class operator, i.e.,

$$\mathcal{L}^1(\mathcal{H}) := \{X \in \mathcal{L}(\mathcal{H}) : \ \|X\|_1 := \mathrm{Tr}|X| < \infty\}$$

and its self-adjoint subspace,

$$\mathcal{L}^1_h(\mathcal{H}) := \{X \in \mathcal{L}^1(\mathcal{H}) : \ X = X^*\},$$

where $\mathcal{L}(\mathcal{H})$ denotes all of the linear operators.

Let Θ be a locally compact metric space, where its metric is denoted as $d(\theta_1, \theta_2)$, $\forall \theta_1, \theta_2 \in \Theta$. From a practical viewpoint, readers may consider Θ as a domain of a finite-dimensional Euclidean space. Now a (premature) quantum statistical model is given by any.map denoted as $T(\theta)$ satisfying $T(\theta) \geq 0$ and $\mathrm{Tr}T(\theta) = 1$ for every $\theta \in \Theta$.

However this naive definition is not enough to develop statistical decision theory. Decades ago Holevo [2, 3] developed statistical decision theory in the quantum setting based on Wald's classical counterpart [5]. His first general framework is written in terms of Banach algebra and general topology, which is far beyond our familiar statistical models. Thus, we restrict his theory to some class of operators in a separable Hilbert space and clarify the meaning of a continuity of quantum statistical models defined by Holevo [3].

2.2 Continuity of quantum statistical models

Definition 1.

Suppose that a self-adjoint operator-valued function $T:\ \Theta \to \mathcal{L}_h^1(\mathcal{H})$ is given. Let K be a compact subset of Θ. For every $\delta > 0$, we set

$$K_\delta := \{(\theta, \eta) \in K \times K :\ d(\theta, \eta) < \delta\}.$$

Then, a *variation norm* on a set K_δ of T is defined by

$$\omega_{T,1}(K_\delta) := \inf\{\|X\|_1 :\ -X \leq T(\theta) - T(\eta) \leq X,\ \forall(\theta, \eta) \in K_\delta\}.$$

An operator-valued function $T(\theta)$ is called a *quantum statistical model* if it satisfies

$$T(\theta) \geq 0,\ \operatorname{Tr}T(\theta) = 1,\ \forall\theta \in \Theta.$$

Although it is very formal definition, it is enough in the following argument. By definition, it is included in $\mathcal{L}_h^1(\mathcal{H})$.

Definition 2.

Suppose that a quantum statistical model $T(\theta)\ :\ \Theta \to \mathcal{L}_h^1(\mathcal{H})$ is given. The quantum statistical model is said to be *regular* if

$$\lim_{\delta \to 0} \omega_{T,1}(K_\delta) = 0 \tag{1}$$

holds for every compact set $K \subseteq \Theta$.

Roughly speaking, the above condition (1) requires that a variation $T(\theta) - T(\eta)$ be uniformly bounded by a trace-class self-adjoint operator and that it goes zero when the distance between θ and η goes to zero. Holevo [3] imposes this condition (or an equivalent condition) on $T(\theta)$. He regarded $T(\theta)$ as a continuous function. It was enough in order to show some theorems like complete class theorem, the existence theorem of Bayes solution and so on.

However, the condition, $-X \leq T(\theta) - T(\eta) \leq X$, is not so easy to understand due to noncommutativity. For example, it is unclear how some other quantities based on a norm, say,

$$\sup_{(\theta,\eta)\in K_\delta} \|T(\theta) - T(\eta)\|_\infty$$

is related to the above definition.

In this short article, we give a simple equivalent condition and compare other possible regularity conditions.

Definition 3.
Let us consider a self-adjoint operator-valued function $T:\ \Theta \to \mathcal{L}_h^1(\mathcal{H})$ in a Hilbert space. The variation of T is evaluated by the following:

$$\begin{aligned}
\widetilde{\omega}_{T,1}(K_\delta) &:= \inf\{\mathrm{Tr}|Z| :\ Z \in \mathcal{L}_h^1(\mathcal{H}), |T(\theta) - T(\eta)| \le Z,\ \forall(\theta,\eta) \in K_\delta\}, \\
\omega_{T,\pm}(K_\delta) &:= \inf\{\mathrm{Tr}|Z| :\ Z \in \mathcal{L}_h^1(\mathcal{H}), (T(\theta) - T(\eta))_\pm \le Z,\ \forall(\theta,\eta) \in K_\delta\}, \\
M_{T,1}(K_\delta) &:= \sup\{\mathrm{Tr}|T(\theta) - T(\eta)| :\ \ \forall(\theta,\eta) \in K_\delta\},
\end{aligned}$$

where $X = X_+ - X_-$ is the Jordan decomposition and X_+ is called a *positive part* while X_- is called a *negative part.* By definition $X_+X_- = 0$ holds.

3 Main Result

Proposition 4.
Let us consider a self-adjoint operator-valued function $T:\ \Theta \to \mathcal{L}_h^1(\mathcal{H})$ in a Hilbert space. For every $\delta > 0$ and every compact set $K \subseteq \Theta$, $\omega_{T,1}(K_\delta) = \omega_{T,\pm}(K_\delta) = M_{T,1}(K_\delta)$ and $\omega_{T,1}(K_\delta) \le \widetilde{\omega}_{T,1}(K_\delta) \le 2\omega_{T,1}(K_\delta)$ hold.

Before proof, we present a simple lemma. First we give an elementary version in a matrix form.

Lemma 5.
Suppose that a $2n \times 2n$ matrix Y is given by

$$Y = \begin{pmatrix} Y_+ & O \\ O & -Y_- \end{pmatrix},\ Y_+ \ge 0,\ Y_- \ge 0,$$

where $Y_{\pm}$ are $n \times n$ square matrices. In addition, another $2n \times 2n$ square matrix Z satisfies

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \geq \begin{pmatrix} Y_+ & O \\ O & -Y_- \end{pmatrix} \geq - \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}.$$

Then the following holds.

(i) $Z \geq 0$,

(ii) $\mathrm{Tr}|Z| = \mathrm{Tr}Z = \mathrm{Tr}Z_{11} + \mathrm{Tr}Z_{22}$,

(iii) $\mathrm{Tr}|Z| \geq \mathrm{Tr}Y_+ + \mathrm{Tr}Y_- = \mathrm{Tr}|Y|$.

Remark

Note that $X \geq Y$ does not necessarily imply $|X| \geq |Y|$!

Proof.

The first assertion is trivial because $Z \geq -Z$. Note that $Z_{11} \geq 0$ and $Z_{22} \geq 0$ due to the positivity condition. The second assertion is shown from the first one. The conditions imply $Z_{11} \geq Y_+$ and $Z_{22} \geq Y_-$. Thus,

$$\begin{aligned} \mathrm{Tr}|Z| = \mathrm{Tr}Z_{11} + \mathrm{Tr}Z_{22} &\geq \mathrm{Tr}Y_{11} + \mathrm{Tr}Y_{22} \\ &\geq \mathrm{Tr}|Y| \end{aligned}$$

holds. *Q.E.D.*

Inspired by this elementary result, we show the following for trace-class self-adjoint operators.

Lemma 6.

$\forall Y,\ \forall Z \in \mathcal{L}^1_h(\mathcal{H})$, the following holds.

$$Z \geq Y \geq -Z \quad \Rightarrow \quad Z \geq 0 \ \&\ \mathrm{Tr}|Z| \geq \mathrm{Tr}|Y|.$$

Proof.

Trivially $Z \geq 0$ holds. The Jordan decomposition of Y is given by

$$Y = Y_+ - Y_-,\ Y_+ \geq 0,\ Y_- \geq 0,\ Y_+Y_- = 0.$$

Using Nagaoka's notation (See, e.g., section 1.5 in the textbook by Hayashi [1]), we introduce two mutually orthogonal projections

$$P := \{Y \geq 0\},\ Q := \{Y < 0\} = I - P.$$

Then,

$$PZP \geq PYP = Y_+, \quad -Y_- = QYQ \geq Q(-Z)Q.$$

Since the pinching map by P and Q, i.e., $X \mapsto PXP + QXQ$, does not change the trace of a positive operator,

$$\mathrm{Tr}|Z| = \mathrm{Tr}Z = \mathrm{Tr}PZP + \mathrm{Tr}QZQ \geq \mathrm{Tr}Y_+ + \mathrm{Tr}Y_- = \mathrm{Tr}|Y|$$

holds. *Q.E.D.*

Now we have the above lemma and show Proposition 4.

Proof.
(proof of Proposition 4)

From now on we write ω_1 instead of $\omega_{T,1}(K_\delta)$ and so on. First we note that $\omega_+ = \omega_-$. Indeed, due to Jordan decomposition, $T(\theta) - T(\eta) = (T(\theta) - T(\eta))_+ - (T(\theta) - T(\eta))_-$ holds. We may write $|T(\theta) - T(\eta)| = (T(\theta) - T(\eta))_+ + (T(\theta) - T(\eta))_-$. In particular, the above condition is symmetric with respect to θ and η if the parameters cover K_δ. Thus, $\omega_+ = \omega_-$.

Then, we show that $\omega_1 \geq M_1 \geq \omega_\pm$. From the lemma,

$$Z \geq T(\theta) - T(\eta) \geq -Z \tag{2}$$

implies that

$$\mathrm{Tr}|Z| \geq \mathrm{Tr}|T(\theta) - T(\eta)| \geq \mathrm{Tr}(T(\theta) - T(\eta))_\pm.$$

When the inequalities (2) hold for every $(\theta, \eta) \in K_\delta$, Z satisfies

$$\begin{aligned}\mathrm{Tr}|Z| &\geq \sup_{(\theta,\eta)\in K_\delta} \mathrm{Tr}|T(\theta) - T(\eta)| \\ &\geq \sup_{(\theta,\eta)\in K_\delta} \mathrm{Tr}(T(\theta) - T(\eta))_\pm.\end{aligned}$$

Taking the infimum of the lefthand side, we obtain

$$\omega_1 = \inf \mathrm{Tr}|Z| \geq M_1 \geq \omega_{\pm}.$$

Next, we show the inequality $\omega_1 \leq \omega_{\pm}$. Note that

$$\begin{aligned} & T(\theta) - T(\eta) \leq (T(\theta) - T(\eta))_+ \leq Z,\ \forall(\theta,\eta) \in K_\delta, \\ \Leftrightarrow & -Z \leq -(T(\theta) - T(\eta))_- \leq T(\theta) - T(\eta), \forall(\theta,\eta) \in K_\delta. \end{aligned}$$

Thus, the above condition implies

$$-Z \leq T(\theta) - T(\eta) \leq Z,\ \forall(\theta,\eta) \in K_\delta.$$

Since the condition of Z in the definition of ω_+ is slightly more strict than that in the definition of ω_1,

$$\begin{aligned} \omega_1 &= \inf\{\mathrm{Tr}|Z| :\ \text{cond. of } \omega_1\} \\ &\leq \inf\{\mathrm{Tr}|Z| :\ \text{cond. of } \omega_+\} = \omega_+ \end{aligned}$$

holds. Thus, $\omega_1 = \omega_{\pm} = M_1$.

In the latter half, we use the same reasoning. Since $(T(\theta) - T(\eta))_+ \leq |T(\theta) - T(\eta)|$, it is easily seen that $\omega_+ \leq \tilde{\omega}_1$. Since $|T(\theta) - T(\eta)| = (T(\theta) - T(\eta))_+ + (T(\theta) - T(\eta))_-$, it is easily seen that $\tilde{\omega}_1 \leq \omega_+ + \omega_- = 2\omega_+$. *Q.E.D.*

In Holevo [3], the variation norm $\omega_{T,1}(K_\delta)$ is defined in a more general setting in order to define the integral of the operator like $\int f(\theta)T(\theta)\mathrm{d}\theta$, where $f(\theta)$ is a measurable function on Θ. However, in our problem, a Hilbert space $\mathcal{H}$ and the sets of the trace-class operator on $\mathcal{H}$ is enough. Then, as shown above, $M_{T,1}(K_\delta) = \omega_{T,1}(K_\delta)$. As a consequence, we can easily interpret the regularity condition of quantum statistical models.

Theorem 7.
Let us consider a quantum statistical model $T:\ \Theta \to \mathcal{L}^1_h(\mathcal{H})$. Then, the following conditions are equivalent.

(i) the quantum statistical model is regular, i.e.,

$\lim_{\delta \to 0} \omega_{T,1}(K_\delta) = 0$, for every compact set $K \subseteq \Theta$.

(ii) $\lim_{\delta \to 0} M_{T,1}(K_\delta) = 0$, for every compact set $K \subseteq \Theta$.

(iii) When $T(\theta)$ is regarded as an operator-valued function on Θ, it is in the set $\mathcal{C}(\Theta; \mathcal{L}^1_h(\mathcal{H}))$.

For the third condition including the definition of $\mathcal{C}(\Theta; \mathcal{L}^1_h(\mathcal{H}))$, see Holevo [3].

Due to condition (ii) in Theorem 7, for a compact metric space Θ, a quantum statistical model $\{T(\theta) : \theta \in \Theta\}$ is regular if and only if

$$\|T(\theta) - T(\theta_0)\|_1 \to 0, \quad \text{as } \theta \to \theta_0$$

for every $\theta_0 \in \Theta$ holds. This kind of definition is easily understood compared to the original definition by Holevo. (By standard technique, it is easily shown that it is equivalent to the uniform continuity over Θ.)

4 Discussion

For comparison, let us consider some quantities similar to the above $\omega_T(K_\delta)$.

Lemma 8.
Let us consider a self-adjoint bounded-operator valued function $T(\theta)$ in a Hilbert space. Let us define

$$\omega_{T,\infty}(K_\delta) := \inf\{\|X\|_\infty : -X \le T(\theta) - T(\eta) \le X, \ \forall(\theta, \eta) \in K_\delta\},$$
$$M_{T,\infty}(K_\delta) := \sup\{\|T(\theta) - T(\eta)\|_\infty : \forall(\theta, \eta) \in K_\delta\}$$

for every $\delta > 0$ and every compact set $K \subseteq \Theta$. Then,

$$\omega_{T,\infty}(K_\delta) = M_{T,\infty}(K_\delta)$$

holds.

Proof.

Again we omit T and write $\omega_\infty(K_\delta)$ instead of $\omega_{T,\infty}(K_\delta)$ and so on.

For every $(\theta,\eta) \in K_\delta$, $T(\theta) - T(\eta) \le \|T(\theta) - T(\eta)\|_\infty I \le M_\infty(K_\delta)I$, where I denotes the identity operator, holds. Thus, $X = M_\infty(K_\delta)I$ satisfies the condition in the definition of $\omega_\infty(K_\delta)$. We obtain

$$\omega_\infty(K_\delta) \le \|M_\infty(K_\delta)I\|_\infty = M_\infty(K_\delta).$$

On the other hand, for every $\epsilon > 0$, there exists X_ϵ such that

$$-X_\epsilon \le T(\theta) - T(\eta) \le X_\epsilon, \ \forall(\theta,\eta) \in K_\delta.$$

and

$$\|X_\epsilon\|_\infty \le \omega_\infty(K_\delta) + \epsilon.$$

If we take a unit vector $|\varphi_{\eta,\theta}\rangle$ satisfying $\langle\varphi_{\eta,\theta}|(T(\theta)-T(\eta))|\varphi_{\eta,\theta}\rangle = \|T(\theta) - T(\eta)\|_\infty$, then

$$\begin{aligned}\|T(\theta) - T(\eta)\|_\infty &= \langle\varphi_{\eta,\theta}|(T(\theta) - T(\eta))|\varphi_{\eta,\theta}\rangle \\ &\le \langle\varphi_{\eta,\theta}|X_\epsilon|\varphi_{\eta,\theta}\rangle \\ &\le \|X_\epsilon\|_\infty\end{aligned}$$

holds for every $(\theta,\eta) \in K_\delta$. Therefore,

$$M_\infty(K_\delta) \le \|X_\epsilon\|_\infty \le \omega_\infty(K_\delta) + \epsilon, \forall\epsilon > 0$$

holds, which implies $M_\infty(K_\delta) \le \omega_\infty(K_\delta)$. We finally obtain

$$M_\infty(K_\delta) = \omega_\infty(K_\delta).$$

Q.E.D.

From the above lemma, we immediately obtain the following result.

Theorem 9.

Let us consider a self-adjoint bounded-operator valued function $T(\theta)$ in a Hilbert space. Then, the following conditions are equivalent.

(i) $\lim_{\delta\to 0} \omega_{T,\infty}(K_\delta) = 0$, for every compact set $K \subseteq \Theta$.

(ii) $\lim_{\delta\to 0} M_{T,\infty}(K_\delta) = 0$, for every compact set $K \subseteq \Theta$.

Unfortunately, this condition seems meaningless as a regularity condition of quantum statistical models.

Remark

Let $\{X_\alpha\} \subseteq \mathcal{L}_h^1(\mathcal{H})$ be a net (a map from a directed set to a set is called a *net*). For $1 \le p \le q \le \infty$,

$$\begin{aligned}\|X_\alpha - X\|_1 \to 0 &\Rightarrow \|X_\alpha - X\|_p \to 0\\ &\Rightarrow \|X_\alpha - X\|_q \to 0\\ &\Rightarrow \|X_\alpha - X\|_\infty \to 0\\ &\Rightarrow \forall\psi \in \mathcal{H},\ \|X_\alpha\psi - X\psi\| \to 0,\\ &\Rightarrow \forall\psi,\phi \in \mathcal{H},\ \langle\psi|X_\alpha|\phi\rangle - \langle\psi|X|\phi\rangle \to 0\end{aligned}$$

holds.

From the above remark, we see that the regularity condition in Theorem 7 seems strong. Let us see one artificial example that is not regular but continuous in the following sense.

An operator-valued function $X(\eta)$ is said to be *continuous with respect to the operator norm at* η_0 if it satisfies

$$d(\eta,\eta_0) \to 0 \Rightarrow \|X(\eta) - X(\eta_0)\|_\infty \to 0.$$

This definition coincides with our intuition of continuity. However, this continuity does not imply the regularity in infinite-dimensional Hilbert spaces. We present an explicit example.

Lemma 10.
There exists a quantum statistical model $\{X(\eta) : 0 \le \eta \le 1\}$ satisfying the following conditions.

(i) $X(\eta)$ is continuous with respect to the operator norm at every $\eta \in [0,1]$.

(ii) $\lim_{\delta \to 0} \omega_{X,\infty}([0,1]_\delta) = 0$

(iii) $\forall \delta > 0, \omega_{X,1}([0,1]_\delta) = \infty$.

Proof.
We construct an example explicitly. First let us define a $n \times n$ square matrix as

$$X_n(\eta) := \frac{C}{n^2} \begin{pmatrix} q_1^{(n)}(\eta) & 0 & \cdots & \\ 0 & q_2^{(n)}(\eta) & \cdots & \\ & & \ddots & \\ & & & q_n^{(n)}(\eta) \end{pmatrix},$$

where C is a positive constant independent of n. (More explicitly $C = (\pi^2/6)^{-1}$ but it is not important for the following argument.) The n-dimensional vector

$$(q_1^{(n)}(\eta), q_2^{(n)}(\eta), \dots, q_n^{(n)}(\eta))$$

is a continuous probability vector on $0 \le \eta \le 1$ defined in the following manner.

When $n \ge 2$, the n-dimensional vector $q^{(n)}$ passes each extremal point, $(1,0,\dots.0), \dots, (0,0,\dots,1)$ at least once for $0 \le \eta < 1/n$. As a whole, the n-dimensional vector $q^{(n)}$ is continuous for $0 \le \eta \le 1$.

In particular, we see

$$\begin{aligned} \mathrm{Tr} X_n(\eta) &= \frac{C}{n^2} \left\{ q_1^{(n)}(\eta) + q_2^{(n)}(\eta) + \cdots + q_n^{(n)}(\eta) \right\} \\ &= \frac{C}{n^2}. \end{aligned}$$

Now we write n unit vectors as $|e_j^{(n)}\rangle$. When $X_n(\eta) \le A_n$, $\forall \eta \in [0,1]$ holds true, it is necessary that the following holds:

$$\begin{aligned} \frac{C}{n^2} \sup_\eta q_j^{(n)}(\eta) &= \frac{C}{n^2} \sup_\eta \langle e_j^{(n)} | X_n(\eta) | e_j^{(n)} \rangle \\ &\le \langle e_j^{(n)} | A_n | e_j^{(n)} \rangle, \end{aligned}$$

where the lefthand side coincides with $\frac{C}{n^2}$. Thus, taking sum of both terms over $j = 1, \ldots, n$, we obtain

$$\frac{C}{n} \leq \mathrm{Tr} A_n \tag{3}$$

for every A_n satisfying $X_n(\eta) \leq A_n$, $\forall \eta \in [0,1]$.

Now we show that the lefthand side of (3) is the lower bound. We take A_n as

$$A_n := \frac{C}{n^2} \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & 1 & \cdots & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Clearly $X_n(\eta) \leq A_n$, $\forall \eta \in [0,1]$ holds.

Putting these matrices together, we define

$$X(\eta) = \begin{pmatrix} X_1(\eta) & O & O & \\ O & X_2(\eta) & O & \\ O & O & \ddots & \\ & & & \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & O & O & \\ O & A_2 & O & \\ O & O & \ddots & \\ & & & \end{pmatrix}.$$

It is easily seen that the above definition makes sense because the infinite sum of matrices converges in the trace norm. Again $X(\eta) \leq A$, $\forall \eta \in [0,1]$ holds.

Now we easily see that $\{X(\eta) : \eta \in [0,1]\}$ is a quantum statistical model. (i.e., $X(\eta) \geq 0, \mathrm{Tr} X(\eta) = 1$, $\forall \eta \in [0,1]$.)

It is also continuous with respect to the operator norm. We deal with the condition (ii) in the next lemma (Lemma 11).

Now we set $K := [0,1]$ and fix $\delta > 0$. We show the condition (iii), $\omega_{X,1}(K_\delta) = \infty$. First, there exists $N \geq 2$ such that $0 < 1/N < \delta$. By definition of $\omega_{X,1}(K_\delta)$, we may assume that there exists a self-adjoint trace-class operator B satisfying

$$-B \leq X(\eta_1) - X(0) \leq B, \ |\eta_1 - 0| < \delta.$$

(Otherwise $\omega_{X,1}(K_\delta) = \infty$ holds true.)

For fixed $X(0)$, let us consider the condition

$$X(\eta_1) \leq B + X(0) \equiv D, \ 0 \leq \eta_1 < \delta.$$

By Jordan decomposition, we may write $D = D_+ - D_-$, where $D_\pm \geq 0$ and $D_+D_- = 0$. Since $D \leq D_+$,

$$X(\eta_1) \leq D_+,\ 0 \leq \eta_1 < \delta.$$

holds.

Now we take $\bigcup_{n=1}^{\infty}\{|e_j^{(n)}\rangle; 1 \leq j \leq n\}$ as one completely orthonormal system. For each $n = 1, 2, \ldots,$

$$\mathrm{Tr}X_n(\eta_1) \leq \sum_{j=1}^{n}\langle e_j^{(n)}|D_+|e_j^{(n)}\rangle$$

necessarily holds. For every $n \geq N$, $X_n(\eta_1)$ $(0 \leq \eta_1 < 1/n)$ passes each extremal point, i.e., $\mathrm{diag}(1, 0, \ldots, 0)$, $\mathrm{diag}(0, 1, \ldots, 0)$, $\ldots$, $\mathrm{diag}(0, \ldots, 0, 1)$ by its definition. It implies

$$\sum_{j=1}^{n}\langle e_j^{(n)}|D_+|e_j^{(n)}\rangle \geq \sup_{0\leq\eta_1<\delta} \mathrm{Tr}X_n(\eta_1) = \frac{C}{n},\ n \geq N.$$

Therefore,

$$\begin{aligned}\mathrm{Tr}D_+ &= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\langle e_j^{(n)}|D_+|e_j^{(n)}\rangle \\ &\geq \sum_{n=N}^{\infty}\frac{C}{n} = \infty\end{aligned}$$

It implies $D = D_+ - D_-$ is not a trace-class operator. Neither is B since $\mathrm{Tr}X(0) = 1$. Thus $\omega_{X,1}(K_\delta) = \infty$ is proved. *Q.E.D.*

Finally we show the uniform continuity of the model in Lemma 10.

Lemma 11.

The quantum statistical model $\{X(\eta) : 0 \leq \eta \leq 1\}$ in Lemma 10 is uniformly continuous with respect to the operator norm and as a consequence $\lim_{\delta\to 0}\omega_{X,\infty}([0,1]_\delta) = 0$ (Lemma 10 condition (ii)) holds.

Proof.

Here, we describe

$$X(\eta) = \bigoplus_{n=1}^{\infty} X_n(\eta)$$

for notational convenience.

First, for any positive number $\epsilon > 0$, we choose a positive integer M such that

$$\sum_{n=M+1}^{\infty} \frac{2C}{n^2} < \frac{\epsilon}{2}.$$

(Since the infinite sum $\sum_n \frac{1}{n^2}$ converges, the above M necessarily exists.)

If we restrict $X(\eta)$ to the image of $\{X_1(\eta), X_2(\eta), \ldots, X_M(\eta)\}$, which is M-dimensional subspace, then clearly it is continuous with respect to the operator norm for every $\eta \in [0,1]$. In other words,

$$\bigoplus_{n=1}^{M} X_n(\eta)$$

is continuous with respect to the operator norm. Since the closed interval $[0,1]$ is compact, standard argument shows that it is also uniformly continuous over $[0,1]$. Therefore, we may choose $\sigma > 0$ such that

$$|\eta_0 - \eta_1| < \sigma \implies \left\| \bigoplus_{j=1}^{M} X_j(\eta_0) - \bigoplus_{j=1}^{M} X_j(\eta_1) \right\|_\infty < \frac{\epsilon}{2}.$$

Thus, when $|\eta_0 - \eta_1| < \sigma$,

$$\begin{aligned}
&\left\| \bigoplus_{j=1}^{\infty} X_j(\eta_0) - \bigoplus_{j=1}^{\infty} X_j(\eta_1) \right\|_\infty \\
&\leq \left\| \bigoplus_{j=1}^{M} X_j(\eta_0) - \bigoplus_{j=1}^{M} X_j(\eta_1) \right\|_\infty + \sum_{j=M+1}^{\infty} \|X_j(\eta_0) - X_j(\eta_1)\|_\infty \\
&\leq \frac{\epsilon}{2} + \sum_{j=M+1}^{\infty} \frac{2C}{j^2} \\
&\leq \epsilon
\end{aligned}$$

holds. In the middle, we used the following inequality: For each $n = 1, 2, \ldots$,

$$\begin{aligned}
\|X_n(\eta_0) - X_n(\eta_1)\|_\infty &= \max_{1 \leq j \leq n} \frac{C}{n^2} \left| q_j^{(n)}(\eta_0) - q_j^{(n)}(\eta_1) \right| \\
&\leq \frac{2C}{n^2}
\end{aligned}$$

holds.

The latter statement is trivial because from Theorem 8

$$\lim_{\delta\to 0}\omega_{X,\infty}(K_\delta) = \lim_{\delta\to 0} M_{X,\infty}(K_\delta)$$

and the righthand side is equal to zero because the uniform continuity. *Q.E.D.*

5 Concluding Remarks

In the present article, we investigate some regularity conditions of quantum statistical models in infinite-dimensional Hilbert spaces. Original work by Holevo [3] yields very general framework but each meaning of regularity conditions is unclear. In our specific setting, we show that the condition is equivalent to the uniform continuity over the trace norm. The uniform continuity over the trace norm is stronger than that over the operator norm. Our result will give the basis on which we investigate quantum statistical models in a general framework.

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