数理解析研究所講究録
第 1909 巻 2014 年 200-206

On Preparing Lecture Notes ... augmenting the body of knowledge

Haiduke Sarafian
University College
Pennsylvania State University
York, USA
has2@psu.edu

Abstract

It is the objective of this note to demonstrate how the author prepares his classroom lecture notes with two distinct and yet correlated objectives. 1) From a pedagogical point of view with some modifications that meets the understanding of the students the author duplicates the known work in the text and 2) from a research point of view the author points out the shortcomings of the "known". Concerning the former by way of example in this note he presents the features of a mechanical linear oscillator and for the latter he considers its nonlinear generalization in three different areas of mechanics, electro and magneto-dynamics. As a byproduct of this generalization he also presents a fresh semi-analytic mathematical method of solving differential equations. The success of this style of preparing the lecture notes is measured by the thousands of downloads of the author's articles. This approach makes the lectures thought provoking and paves the discovery road for future generations.

Motivations and Objectives

We begin with a classic problem, A Simple Harmonic Motion. In mechanics this is composed of a linear spring and a mass. On a horizontal plane and the absence of gravity the released mass initially displaced from the equilibrium oscillates sinusoidally. The equation describing the motion is a second order differential equation. This equation trivially is solved conducive to the position of the bob as a function of time. The solution of this equation reveals the characteristics of the oscillations -- noticeably the fact that the period of oscillation is independent of the initial displacement; this is one of the features of the linear force. Details of this problem is discussed in introductory physics and engineering texts [1]. The issue is how can this classic problem be modified? One may suggest considering to replacing the linear spring with a nonlinear one. This replacement brings to the fore the mathematical challenges such as the solution of the associated equation of motion, it poses the issue of how such nonlinear spring should be fabricated! These issues are not discussed in traditional text books and the author was unable locating researched-based journal articles. This is a thought provoking issue. Here we discuss a potential solution. To address the fabrication issue instead of considering a single spring with nonlinear properties that might come about by reshaping a spring and or considering a spring with a nonuniform mass density, we consider a pair of identical linear springs and arrange them as shown in Fig 1. Displacing the mass from its initial equilibrium along the horizontal exerts a net force along the horizontal. For a scenario where the initial displacement is shorter than the un-stretched length of the spring, the functional form of the force maybe replaced with its expanded format. The expanded function is a polynomial of odd powers of displacement. This shows the force is nonlinear and its leading term is a cubic function of the position variable. From a mathematical point of view, the associated equation of motion is a nonlinear DE. It has a closed form analytic solution -- these are the Jacobi Elliptic Functions [2].
With the objective of exploring a fresh method of analyzing the problem we utilize a Computer Algebra System (CAS) -- the author's choice is being *Mathematica* [3]. Utilizing this software, we solve the equation of motion symbolically -- this is conducive the aforementioned functions. As an alternative, for a chosen initial conditions we solve the same equation numerically. Applying a CAS is quick and efficient. With minimal efforts it generates the needed output. With the output at hand we concentrate on the characteristics issues of the problem. While taking these evolutionary steps the author stumbled on a fresh method of analyzing the problem. This

method is based on 1) simulating the function of the proposed design shown in Fig 1. The simulation is carried out utilizing "An Interactive Geometry software" Cinderella [4] and 2) solving the equation of motion *semi-analytically*. The latter is a hybrid method that is composed of a mixture of analytic and numeric approach.
To address these issues in a systematic fashion we crafted this note that is composed of five sections. In addition to Motivation and Objectives in Section 2 we discuss the Physics of the problem and its formulation. In Section 3, Analysis we present the results. In Section 4 we present A Semi-Analytic method of solving DE of motion. In Section 5 we passively demonstrate two similar scenarios; one in electro and the other one in magneto dynamics. In Section 6 under Conclusions and Remarks we summarize our thoughts and finally in section 7 we suggest A Potential Extension for the Future Investigation.

2. Physics of the problem and its formulation

As shown in Fig 1, the marble is pulled from its initial equilibrium position to a point such as G. The differential length elongation of each spring is $\Delta\ell = \sqrt{L^2 + x^2} - L$, yielding to the net force,

$$F_x = 2\,k\,x\left(1 - \frac{L}{\sqrt{L^2 + x^2}}\right). \tag{1}$$

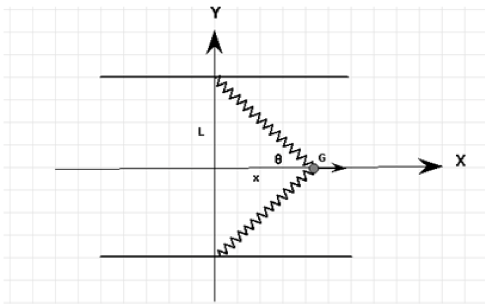

Figure 1. Two-spring arrangement leading to a combination of a cubic and quintic oscillations.

For displacements $x < L$ the quantity in the parentheses in eq (1) maybe replaced with $\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{3}{8}\left(\frac{x}{L}\right)^4 + \ldots$ yielding,

$$F_x = \frac{k}{L^2}x^3 - \frac{3}{4}\frac{k}{L^4}x^5 + \ldots \tag{2}$$

One realizes this simple mechanical device is capable of performing nonlinear oscillations; the leading terms are combination of cubic and quintic coordinate-dependent terms. With this mechanical device in hand we analyze the kinematic characteristics of the mobile marble.

3. Analysis

According to the scenario shown in Fig 1 and its accompanied force, eq (2), the equation of motion is,

$$\ddot{x} + \frac{k}{m\,L^2}x^3 - \frac{3}{4}\frac{k}{m\,L^4}x^5 = 0. \tag{3}$$

First we consider a case assuming $x < L$ i.e. the initial displacement of the marble from equilibrium is less than the length of the spring, this drops the quintic term. Sustaining the cubic term only the equation of motion is an analytically solvable DE; solutions are the Jacobi Elliptic Functions [2]. Alternatively, the same equation can be solved numerically. Assigning parameters to the stiffness, length and the mass of the marble we solve the equation numerically; output graphs of these two approach are indistinguishable. To establish the basis for the *semi-analytic* method we focus on the numeric schematic. The **values** lists a typical set of parameters in MKS units,

$$\textbf{values} = \{k \to 3.0,\ \ell \to 6.0 \times 10^{-2},\ m \to 10.0 \times 10^{-3}\}$$

The 4 x2 graphic matrix shown in Fig 2 displays the position of the marble vs. time. The differences between the graphs are due only to various initial displacements of the marble. From the top left to the bottom right these correspond to $x[0] = 0.1\,\alpha\,\ell$ for $\alpha = 1, 2, 3 \ldots$ 8 respectively. The common global feature of these plots indicates irrespective of the initial condition the marble under the influence of the cubic force does oscillate. The period of oscillation is not constant; it depends to the initial value of the displacement. This is one the distinct characteristics of a non-linear force. An inspection of these plots reveals, the absence of retarding forces such as friction enforces the constant amplitude. However, the shorter the amplitude the longer the period. Figure 3 puts the observation in perspective. This graph vividly shows reciprocal relationship between the amplitude and the corresponding period.

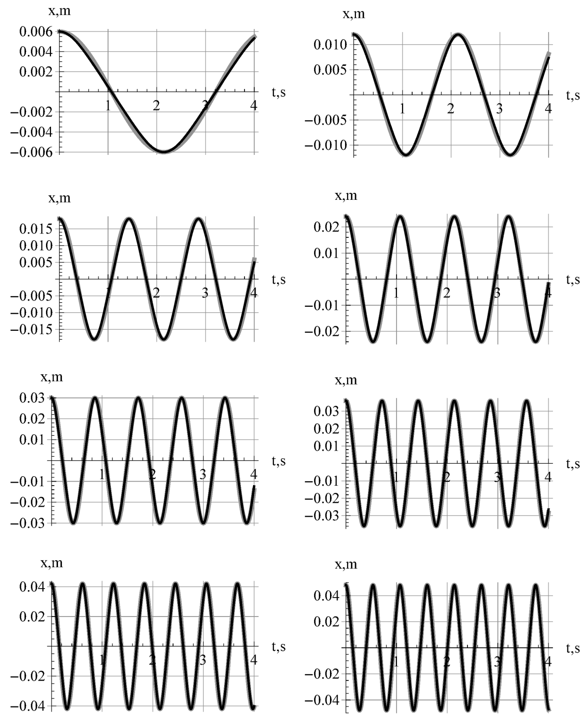

Figure 2. Output of numeric solution of equation of motion. From the top left to the bottom right graphs correspond to the initial values $x[0] = 0.1\,\alpha\,\ell$ for $\alpha = 1, 2, 3, \ldots 8$, respectively.

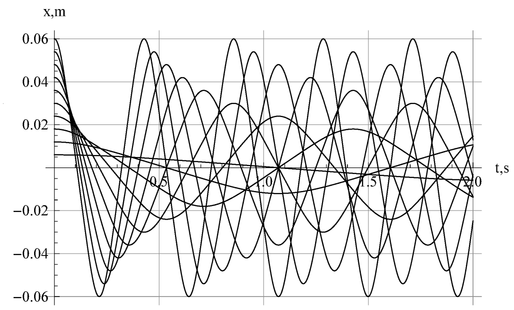

Figure 3. Display of the overlaid oscillations shown in Fig 2. It shows the longer the amplitude the shorter the period.

It is instructive to compare the oscillations emanating by a cubic force vs. the linear one. This can be done either by a direct comparison of the amplitudes or more elegantly by comparing their respective phase diagrams shown in Fig 4.

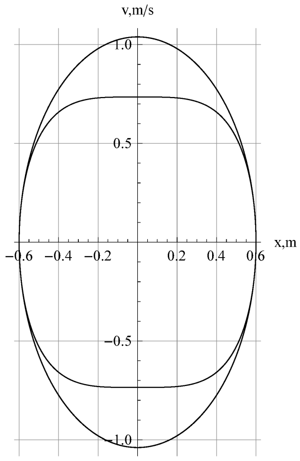

Figure 4. Phase diagrams: a) the ellipse corresponds to a linear spring, and b) the depressed ellipse corresponds to a cubic spring.

For the sake of clarity the horizontal scale of Fig 4 is magnified by a factor of 10. Figure 4 shows the distinct differences between the oscillation characteristics of a linear vs. a cubic spring. As one may easily verify the phase diagram of a linear spring is a perfect ellipse. The phase diagrams of a cubic spring is a depressed "ellipse" along the velocity axis. Depression severity of the latter is a function of the initial condition. A family of such curves is shown in Fig 5.

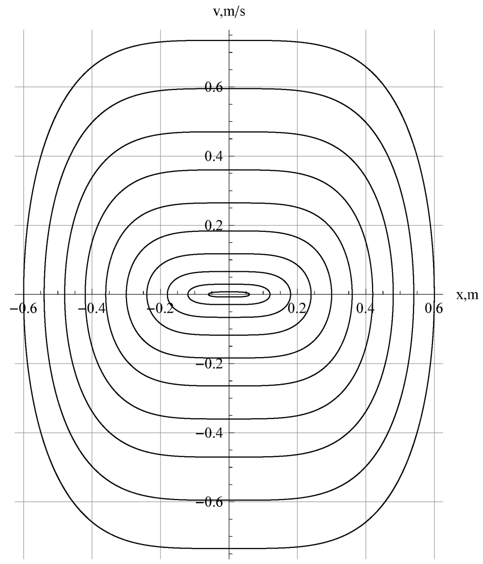

Figure 5. A family of phase diagrams of the cubic spring. The inner curve corresponds to $x[0] = 0.1\,\ell$ while the outer one is due $x[0] = 1\,\ell$, respectively.

4. A *Semi-Analytic* method of solving DE of motion

As we discussed in Section 3 the solution of equation of motion of a cubic force is an oscillating function. Accordingly, the value of initial condition impacts the period. With these two observations we propose a compatible alternative solution. This solution is neither purely numeric nor purely symbolic; it is somewhere in between, it is *semi-analytic*. From analytic point of view, first we consider a solution such as, $x(t) = \text{amp Cos}[\omega\, t]$, with $\omega = \omega(\text{amp})$. Applying $\omega = \frac{2\pi}{T(\text{amp})}$, from a numeric point of view utilizing either Figs 2 or 3 we hunt for $T(\text{amp})$. For instance, for the latter utilizing Fig 2 for the chosen initial amplitude we evaluate the maximum ordinate of the corresponding oscillation. A set of one such coordinates are shown with dots in Fig 6. Then we fit the data with an appropriate continuous function. The fitted curve is shown on
the same graph as well. Utilizing this function, e.g. $T(\text{amp})$ we compare the *semi-analytic* solution $x(t) = \text{amp Cos}\left[\frac{2\pi}{T(\text{amp})}\, t\right]$ vs. the numeric solution. A 4 x2 graphic matrix shown in Fig 7, qualifies the accuracy of our proposed method.

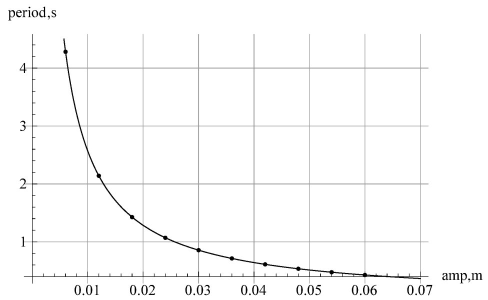

Figure 6. The dots are the periods and the corresponding amplitudes deduced from Fig 2. The solid line is the

fitted curve $T(\text{amp}) = 0.0256907\,\text{amp}^{-1}$.

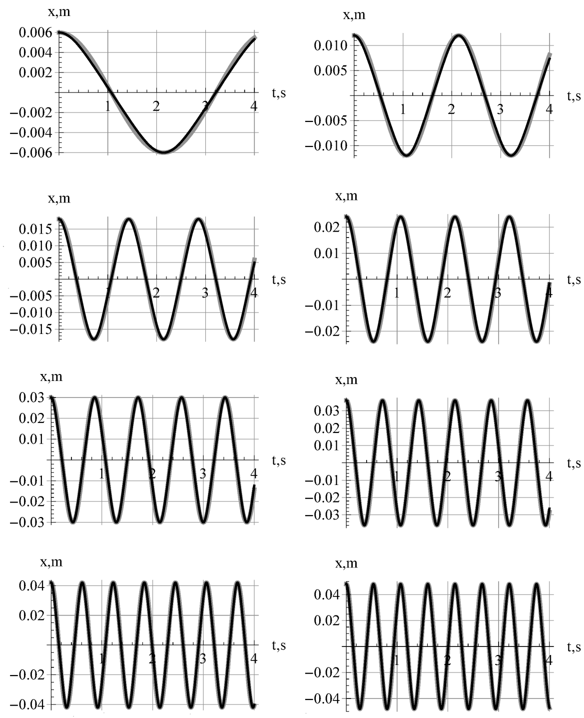

Figure 7. Comparison of the numeric solutions (black curves) vs. *semi-analytic* solution (gray curves).

These two solutions are indistinguishable. Therefore, as an option, rather than utilizing the numeric solution that in general lacks the physics insight, alternatively, one may objectively apply the *semi-analytic* method to obtain the same result.

5. Two "similar" nonlinear Oscillators

For the sake of completeness we depict two similar scenarios conducive to nonlinear oscillations in electric and magnetic fields. The left plot of the Fig 8 describes a situation where an electric point-like charge particle in a vertical plane oscillates non-linearly [5]. The right plot of Fig 8 is the case where a permanent magnetic dipole also non-linearly oscillates within the magnetic field of a DC looping current [6]. An interested reader is encouraged to download the open source articles [5-6].

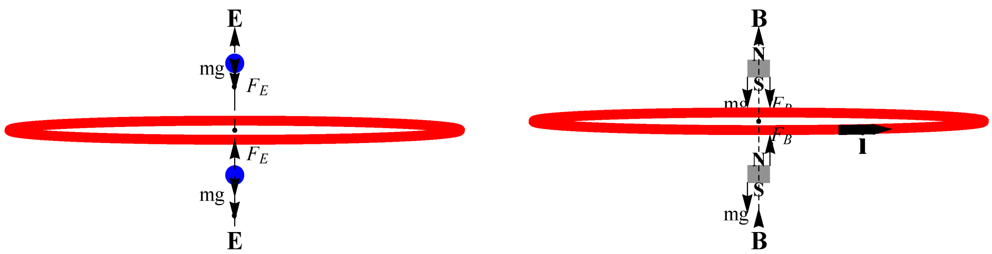

Figure 8. Oscillations of an electric monopole within the electric field of a charged ring (left). Oscillations of a

magnetic dipole within the magnetic field of a looping current (right)

6. Conclusions and Remarks

The objective of this note is to demonstrate by a way of example how the author prepares his lecture notes. As an example we discuss about the features of a mechanical system and show how it is generalized. Utilizing a set of two linear springs we design one such device; many more can be devised. We extend the analysis introducing a Computer Algebra System (CAS) and specifically demonstrate the applications of *Mathematica*. These are complementary to our previous studies [5-7] completing the scope of nonlinear oscillations in three different areas of physics: mechanics, electro and magneto-dynamics. Beyond discussing the physics of the problem, we introduce a *semi-analytic* method to objectively obtaining analytic solutions for the equations of motion. From mathematics point of view the proposed method for solving DE relies on numeric solution of DE. The output of the *semi-analytic* method is an analytic function embodying the desired objective properties. To distinguish the differences between the linear vs. non-linear oscillations their phase diagrams are compared. However, for visual understanding (not included here), the author utilizing *Mathematica* has simulated the motion. Watching the movement of the marble under the influence of nonlinear forces gives a valuable experience. Alternatively, one may simulate (as the author has done) the motion utilizing Cinderella [4]. The latter is less objective bypassing the need of numeric input.

7. A Potential Extension for the Future Investigation Conducive Publications

1) Repeat the analysis in a vertical plane and include gravity.
2) Repeat the scenario depicted in Fig 1 and consider two springs with different stiffness. Analysis the issues with and without gravity.
3) Compare the oscillations of the cubic spring vs. the non-approximated force given by eq (2).

Acknowledgement

The author is thankful to Professor Nakamura-san for inviting the author to present the work at the RIMS's conference This is the author's third RIMS participation and he is looking forward for the future collaborations. He also appreciates Professors, Takato-san, Yamashita-san and Kaneko-san's hospitality.

References

[1] David Halliday, Robert Risnick and Jerry Walker, "Fundamentals of Physics", 9th Edition, John Wiley and Sons, Inc., 2011.
Wolfgang Bauer and Gary Westfall, "University Physics with Modern Physics", McGraw Hill, 2011.
[2] Jacobi Elliptic Functions: "http://mathworld.wolfram.com/JacobiEllipticFunctions.html"
[3] *Mathematica* "A general computer software system and language intended for mathematical and other applications", V9.0, Wolfram Research, 2013.
[4] Jurgen Richter-Gebert and Ulrich H. Kortenkamp, "The Interactive Geometry Software", Cinderella 2.0, Springer, 2012.
[5] Haiduke Sarafian, "Static Electric-Spring and Nonlinear Oscillations", Journal of Electromagnetic Analysis & Applications, 2010, 2: 75-81
[6] Haiduke Sarafian, "Nonlinear Oscillations of a Magneto Static Spring-Mass", Journal of Electromagnetic Analysis and Applications, 2011, 3, 133-139.
[7] Haiduke Sarafian,13th International Conference on Computer Science and its Applications, (ICCSA2013), Springer-Verlag, pg 46-52, 2013.