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The topology of box complexes and the chromatic numbers of graphs

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1 Introduction

The existence problem of graph homomorphisms is a classical problem of graph theory. The first application of algebraic topology to the problem is Lovász' proof of the Kneser conjecture. Lovász introduced the neighborhood complex $N(G)$ of a graph G, and showed that its connectivity gives a lower bound for the chromatic number $\chi(G)$ of the graph G and the chromatic number of $K_{n,k}$ is equal to $(n-2k+2)$, which was conjectured by Kneser in 1955.

The Hom complex $\mathrm{Hom}(G, H)$ of graphs G and H was defined by Lovász after introducing the neighborhood complex, and were first mainly investigated by Babson and Kozlov in [1] and [2]. The Hom complex $\mathrm{Hom}(K_2, G)$ from the complete graph K_2 with 2 vertices to a graph G is homotopy equivalent to the neighborhood complex $N(G)$. The box complex can be naturally regarded as a $\mathbb{Z}_2$-space, and its $\mathbb{Z}_2$-homotopy type is also closely related to the chromatic number. In [2], Babson and Kozlov proved the conjecture by Lovász which states that the connectivity of $\mathrm{Hom}(C_{2m+1}, G)$ gives another lower bound for the chromatic number of G as is the case of the neighborhood complex. After that, many works on neighborhood complexes, box complexes, and Hom complexes have been done, for example, in [5], [15], or [16].

The purpose of this article is to introduce the author's works [11], [12], and [13] which are relevant to neighborhood complexes and box complexes. The first part we deal with in Section 3, we introduce some examples of graphs whose chromatic numbers are different, but their box complexes are quite similar. More precisely, we show the followings :

- There are graphs G and H such that their box complexes are homeomorphic, their neighborhood complexes are homeomorphic, but $\chi(G) \neq \chi(H)$. (Theorem 3.1)
- There are graphs X and Y such that their box complexes are $\mathbb{Z}_2$-homotopy equivalent, but $\chi(X) \neq \chi(Y)$. (Theorem 3.3)

These examples are important to see how the topology of box complex influences the chromatic number.

The second part we deal with in Section 4 is an introduction to the theory of r-fundamental groups for a positive integer r. The r-fundamental group $\pi_1^r(G, v)$ is the group associated to a graph G with a vertex v of G. Theorem 4.3 implies that 2-fundamental group gives a combinatorial description of the fundamental group of the neighborhood complex. One of the most interesting phenomena is an analogy to the covering space theory of topology. There is a notion which corresponds to the covering space called the r-covering map. Theorem 4.4 states that there is a 1-1 correspondence between subgroups of the r-fundamental group of (G, v) and based r-coverings over (G, v) whose domain is connected, as is the case of fundamental groups of based topological spaces.

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2 Preliminaries

In this section, we review definitions and known results related to box complexes. We recommend the book [8] as a reference of this section.

A *graph* is a pair (V, E) where V is a set and E is a symmetric subset of $V \times V$, that is, $(x, y) \in E$ implies $(y, x) \in E$ for any $(x, y) \in V \times V$. Hence the graphs which we deal with in this report are non-directed, have no parallel edges, but may have loops. For a graph $G = (V, E)$, V is denoted by $V(G)$ and E is denoted by $E(G)$. For given two graphs G and H, a graph homomorphism f from G to H is a map $f : V(G) \to V(H)$ such that $(f \times f)(E(G)) \subset E(H)$.

In this report, we assume that the $\mathbb{N}$ denotes the set of all non-negative integers. For a non-negative integer $n \in \mathbb{N}$, the complete graph K_n with n-vertices is the graph defined by

$$V(K_n) = \{1, \cdots, n\},$$
$$E(K_n) = \{(x, y) \mid x \neq y\}.$$

The chromatic number $\chi(G)$ of a graph G is the number

$$\chi(G) = \inf\{n \in \mathbb{N} \mid \text{There is a graph homomorphism from } G \text{ to } K_n.\}.$$

To compute the chromatic number of a given graph is called the *graph coloring problem* which has been investigated in graph theory for a long time.

An *(abstract) simplicial complex* is a pair (V, Δ) where V is a set and Δ is a family of finite subsets of V satisfying the following conditions :

- For each $v \in V$, we have $\{v\} \in \Delta$.
- For $\sigma \in \Delta$ and $\tau \in 2^V$, if $\tau \subset \sigma$, then $\tau \in \Delta$.

We often abbreviate the vertex set V, and write "Δ is a simplicial complex". In this terminology, we write $V(\Delta)$ for the vertex set of the simplicial complex Δ.

Let Δ and Δ' be simplicial complexes. A *simplicial map* from Δ to Δ' is a map $f : V(\Delta) \to V(\Delta')$ such that $f(\sigma) \in \Delta'$ for each $\sigma \in \Delta$.

There is a functor called *geometric realization*, which associates a simplicial complex to a topological space. Let Δ be a simplicial complex. Let $\mathbb{R}^{(V(\Delta))}$ denote the free $\mathbb{R}$-module generated by the set $V(\Delta)$. We regard $\mathbb{R}^{(V(\Delta))}$ as a topological space with the direct limit topology of finite dimensional $\mathbb{R}$-submodules of $\mathbb{R}^{(V(\Delta))}$. For a finite subset $S \subset V(\Delta)$, we write Δ_S for the subspace

$$\Delta_S = \{\sum_{x \in S} t_x x \in \mathbb{R}^{(V(\Delta))} \mid t_x \geq 0, \sum_{x \in S} t_x = 1.\}$$

of $\mathbb{R}^{(V(\Delta))}$. The geometric realization $|\Delta|$ of the simplicial complex Δ is the topological subspace

$$|\Delta| = \bigcup_{\sigma \in \Delta} \Delta_\sigma \subset \mathbb{R}^{(V(\Delta))}.$$

A partially ordered set is often called a *poset*. A subset c of a poset P is called a *chain* of P if the restriction of the ordering of P to c is a total ordering. The simplicial complex of all finite chains of the poset P is called the *order complex* of P, and is denoted by $\Delta(P)$. We write $|P|$ for the geometric realization of the order complex $|\Delta(P)|$ of the poset P.

We use the topological terminologies to simplicial complexes and posets via taking geometric realizations. For example, a poset P is said to be n-connected if the geometric realization $|P|$ is n-connected.

Let G be a graph. For a vertex v of G, we write $N(v)$ for the set $\{w \in V(G) \mid (v, w) \in E(G)\}$. The *neighborhood complex* of the graph G is the simplicial complex defined by

$$V(N(G)) = \{v \in V(G) \mid N(v) \neq \emptyset\},$$

$$N(G) = \{\sigma \subset V(G) \mid \#\sigma < \infty \text{ and there is } v \in V(G) \text{ such that } \sigma \subset N(v).\}.$$

The box complex $B(G)$ is the poset

$$\{(\sigma, \tau) \mid \sigma \text{ and } \tau \text{ are non-empty subsets of } V(G) \text{ and } \sigma \times \tau \subset E(G).\}$$

with the ordering $(\sigma, \tau) \leq (\sigma', \tau') \Leftrightarrow \sigma \subset \sigma'$ and $\tau \subset \tau'$.[1]

[1]The definition of box complexes is not unique. In [4] or [10], another definition of box complexes is employed, and the proposition associated to Theorem 3.2 in their definition is not yet proved. The box complex $B(G)$ we say in this article is isomorphic to the Hom complex $\mathrm{Hom}(K_2, G)$ from K_2 to G.

Theorem 2.1 (Babson-Kozlov [1]). *There is a homotopy equivalence* $|B(G)| \to |N(G)|$ *which is natural with respect to a graph* G.

Remark that $B(G)$ has a natural involution $(\sigma, \tau) \leftrightarrow (\tau, \sigma)$, and from now on, we regard $B(G)$ as a $\mathbb{Z}_2$-poset. The criterion to show that there is no graph homomorphism by using box complexes is as follows : for given graphs G and H, if there is no $\mathbb{Z}_2$-equivariant map from $B(G)$ to $B(H)$, then we have that there is no graph homomorphism from G to H. By such a criterion, we have the following.

Theorem 2.2 (Lovász [9]). *Let* n *be an integer such that* $n \geq -1$. *If* $N(G)$ *is* n*-connected, then* $\chi(G) \geq n+3$. *Here "*(-1)*-connected" means "non-empty."*

The following outline of the proof is given by Babson and Kozlov in [1] which is a little modification of the original proof by Lovász.

Proof. We can assume that G has no loops. Hence $B(G)$ is free $\mathbb{Z}_2$-poset. Suppose $N(G)$ is n-connected. By Theorem 2.1, $B(G)$ is n-connected. By the Gysin sequence, we have $w_1(B(G))^{n+1} \neq 0$. If there is a graph homomorphism $G \to K_m$, then there is a $\mathbb{Z}_2$-map $B(G) \to B(K_m) \approx S^{m-2}$. Hence $w_1(B(G))^{m-1} = 0$. Therefore we have $n+1 < m-1$, and hence we have $\chi(G) \geq n+3$. □

Lovász applied this theorem to determine the chromatic numbers of the Kneser graphs. Let n and k be positive integers such that $n \geq 2k$. The Kneser graph $K_{n,k}$ is the graph defined by

$$V(K_{n,k}) = \{\sigma \subset \{1, \cdots, n\} \mid \#\sigma = k.\},$$
$$E(K_{n,k}) = \{(\sigma, \tau) \mid \sigma \cap \tau = \emptyset\}.$$

Kneser proved that $\chi(K_{n,k}) \leq n - 2k + 2$, and conjectured that the equality holds in [6]. Lovász proved that $N(K_{n,k})$ is $(n-2k-1)$-connected. By Theorem 2.2, he proved the following called the Kneser conjecture.

Theorem 2.3 (Lovász [9]). $\chi(K_{n,k}) = n - 2k + 2$.

After the proof of the Kneser conjecture, Schriver obtained a stronger result. The stable Kneser graph $SK_{n,k}$ for positive integer n and k with $n \geq 2k$ is the graph defined by

$$V(SK_{n,k}) = \{\sigma \subset \mathbb{Z}/n\mathbb{Z} \mid \#\sigma = k. \text{ And if } x \in \sigma, \text{ then } x+1 \notin \sigma.\}$$

$$E(SK_{n,k}) = \{(\sigma, \tau) \mid \sigma \cap \tau = \emptyset.\}.$$

Schriver showed that $N(SK_{n,k})$ is $(n-2k-1)$-connected as is the case of Kneser graphs. Moreover, Björner and Longueville showed the following result.

Theorem 2.4 (Björner-Longueville [3]). *The neighborhood complex of the stable Kneser graph $SK_{n,k}$ is homotopy equivalent to the $(n-2k)$-sphere.*

Corollary 2.5 (Schriver [14]). $\chi(SK_{n,k}) = n - 2k + 2$.

Moreover, Schriver showed that the chromatic number of the subgraph of $SK_{n,k}$ deleted one vertex from $SK_{n,k}$ is lower than $n - 2k + 2$.

3 The topology of box complexes and the chromatic number

As is mentioned in the previous sections, there is a relation between the topological invariant of the box complex $B(G)$ (or the neighborhood complex $N(G)$) and the chromatic number $\chi(G)$. Then a natural question arises : how effective is to detemine the homeomorphism type of the box complex to compute the chromatic number? For example, does there exist a topological invariant of $B(G)$ or $N(G)$ which is equivalent to the chromatic number $\chi(G)$ of the graph G? In this section, we deal with such questions following [12] and [13]. Main results stated here are Theorem 3.1 and Theorem 3.3 which state that there are no non-equivariant topological invariant and $\mathbb{Z}_2$-homotopy invariant of the box complex which are equivalent to the chromatic number.

First we review the known results about these questions. In [9], Lovász proved that for a connected graph G, $\chi(G) \leq 2$ if and only if $N(G)$ is not connected. So he expected that $H_{\chi(G)-2}(N(G))$ or $\pi_{\chi(G)-2}(N(G))$ is non-trivial for every graph G. But this was negatively solved by Walker in [17]. Walker practically showed that there are finite connected graphs G and H such that $N(G)$ and $N(H)$ are homotopy equivalent, but have different chromatic numbers. This implies that any non-equivariant homotopy invariant of the neighborhood complex is not equivalent to the chromatic number.

The first result we mention here is the following, which asserts that any *non-equivariant topological* invariant is not equivalent to the chromatic number.

Theorem 3.1 (M [13]). *Let m and n be integers greater than 2. Then there are finite connected graphs G and H such that $B(G) \cong B(H)$ as posets, $N(G) \cong N(H)$ as simplicial complexes, and $\chi(G) = m$ and $\chi(H) = n$.*

To construct such examples, the key observation is the following.

Theorem 3.2 (M [13]). *Let G and H be graphs with no isolated vertices. Then the followings hold :*

(1) $K_2 \times G \cong K_2 \times H$ if and only if $B(G) \cong B(H)$ as posets.

(2) $G \cong H$ if and only if $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets.

(3) If $K_2 \times G \cong K_2 \times H$, then we have $N(G) \cong N(H)$ as simplicial complexes. If G and H are stiff, then $N(G) \cong N(H)$ as simplicial complexes implies $K_2 \times G \cong K_2 \times H$.

Here a graph G is said to be *stiff* if there are no two distinct vertices $v, w \in V(G)$ such that $N(v) \subset N(w)$.

Hence to prove Theorem 2.1, we need to construct connected graphs G and H such that $K_2 \times G \cong K_2 \times H$ but $\chi(G) = m$ and $\chi(H) = n$.

Let us observe the graph $K_2 \times G$. First we give the precise definition of the (categorical) product of graphs. For graphs G and H, the product graph $G \times H$ of G and H is the graph defined by

$$V(G \times H) = V(G) \times V(H),$$

$$E(G \times H) = \{((x,y),(x',y')) \mid (x,x') \in E(G), (y,y') \in E(H)\}.$$

Indeed, the categorical product of graphs is very complicated. For example, the long standing conjecture of Hedetniemi states that $\chi(G \times H) = \inf\{\chi(G), \chi(H)\}$ for finite graphs G and H. But fortunately, the graph $K_2 \times G$ can be rather easily understood since it has a geometric description as follows.

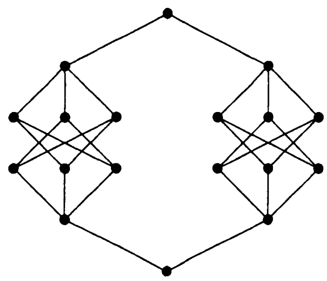
A bipartite graph is a graph G such that there is a graph homomorphism from G to K_2. Remark that for any graph G, the graph $K_2 \times G$ is bipartite since the projection $V(K_2) \times V(G) \to V(K_2)$ is a graph homomorphism to K_2. An odd involution of a bipartite graph G is a graph homomorphism $\tau : G \to G$ such that $\tau^2 = \mathrm{id}_G$ and for any vertex v, if v and $\tau(v)$ are in the same component of G, then the length of a path from v to $\tau(v)$ is odd.[2] For a bipartite graph G with an odd involution τ, we write G/τ for the quotient graph of G by the $(\mathbb{Z}/2\mathbb{Z})$-action determined by τ.

The central example of odd involutions of bipartite graphs is the involution of $K_2 \times G$ defined by $(1, v) \leftrightarrow (2, v)$ for a graph G and $v \in V(G)$. Conversely, any odd involution is written in such a way. In fact, for a bipartite graph G with an odd involution τ, we can show that $K_2 \times (G/\tau) \cong G$.

Hence to prove Theorem 3.1, it suffices to construct a connected bipartite graph K with two odd involutions τ_1 and τ_2 such that $\chi(G/\tau_1) = m$ and $\chi(G/\tau_2) = n$.

In this report, we only give the example of the case $m = 3$ and $n = 4$ in Theorem 3.1. In this case, the bipartite graph G is given as follows :

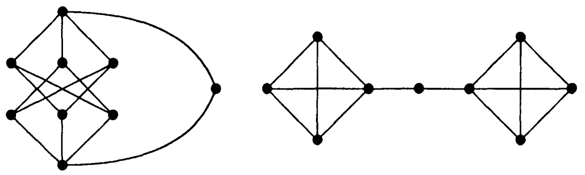
[2]The parity of the length of such a path is independent of the choice of a path since G is bipartite.



The odd involutions τ_1 and τ_2 of G is defined as follows :

- The involution τ_1 is the (180°)-rotation around the central point.
- The involution τ_2 is the reflection with respect to the horizontal line.

Then by taking quotients, we obtain the following graphs G/τ_1 and G/τ_2. It is easy to see that $\chi(G/\tau_1) = 3$ but $\chi(G/\tau_2) = 4$.



Next we consider the $\mathbb{Z}_2$-topological type of $B(G)$. As was mentioned in Theorem 3.2.(2), the $\mathbb{Z}_2$-poset structure of the box complex $B(G)$ completely determines the graph G. Hence we can expect that the chromatic number of a graph is equivalent to some invariants of $\mathbb{Z}_2$-posets. But the following example implies that the chromatic number is not equivalent to any $\mathbb{Z}_2$-*homotopy* invariant of the box complex.

Theorem 3.3 (M [12]). *There is a graph homomorphism $f : Y \to X$ between finite connected graphs such that f induces a $\mathbb{Z}_2$-homotopy equivalence $B(Y) \to B(X)$, but $\chi(Y) \neq \chi(X)$.*

The graphs X and Y are described as follows. First we define the graph Z by

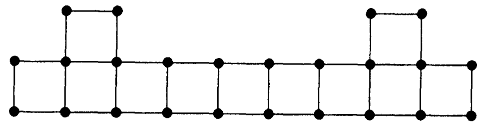
$$V(Z) = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq 9, 0 \leq y \leq 1\} \cup \{(1,2), (2,2), (7,2), (8,2)\},$$

$$E(Z) = \{((x, y), (x', y')) \mid |x - x'| + |y - y'| = 1\}.$$

The graph X is obtained by identifying vertices of Z as follows :

- The vertex $(x, 0)$ $(x = 0, 1, 2, 3)$ is identified with the vertices $(x+3, 0)$ and $(x+6, 0)$.
- The vertex $(0, y)$ $(y = 0, 1)$ is identified with the vertex $(9, y)$.

- The vertex $(x, 2)$ $(x = 2, 3)$ is identified with the vertex $(9 - x, 2)$.



The graph Z.

The graph Y is the induced subgraph of X whose vertex set is $\{[(0,0)], [(1,0)], [(2,0)]\}$. Then we have that $Y \cong K_3$ and hence $\chi(Y) = 3$. It is easy to show that $\chi(X) = 4$. To prove $B(X) \simeq B(Y)$, we need the following two lemmas.

Proposition 3.4 (M[12]). *Let G be a graph, and $e = \{(x, w), (w, x)\}$ an edge of G. Suppose that the graph G satisfies the followings :*

- *$x \neq w$ and either x or w has no loops.*
- *There is a unique graph homomorphism from L_3 (the definition is found in Section 4) to the graph $G \setminus e$ mapping 0 to x and 3 to w, where the graph $G \setminus e$ is the graph deleted the edge e from the graph G.*

Then the inclusion $G \setminus e \hookrightarrow G$ induces a homotopy equivalence $B(G \setminus e) \hookrightarrow B(G)$ between box complexes.

Proposition 3.5 (Kozlov [7]). *Let G be a graph and x a vertex of G. If there is a vertex w of G such that $x \neq w$ and $N(x) \subset N(w)$, then the inclusion $G \setminus x \hookrightarrow G$ induces a homotopy equivalence $B(G \setminus x) \hookrightarrow B(G)$.*

From the above two propositions, we can easily show that the $\mathbb{Z}_2$-map $B(Y) \hookrightarrow B(X)$ induced by the inclusion $Y \hookrightarrow X$ is a homotopy equivalence. This is in fact a $\mathbb{Z}_2$-homotopy equivalence because of the following fact of equivariant homotopy theory : a Γ-map $f : A \to B$ between free Γ-complexes is a homotopy equivalence if and only if f is a Γ-homotopy equivalence.

It is still open whether there are graphs G and H such that $B(G)$ and $B(H)$ are $\mathbb{Z}_2$-homeomorphic but $\chi(G) \neq \chi(H)$. I expect that such graphs exist, but have no idea to prove it.

4 r-fundamental groups

Here we introduce the theory of r-fundamental groups for a positive integer r which were introduced by the author in [11]. The r-fundamental group $\pi_1^r(G, v)$ is the group associated to a based graph (G, v) which is similarly defined as the fundamental groups of based topological spaces. The r-fundamental groups are applied to the existence problem of graph homomorphisms, and interesting phenomena which are similar to the covering space theory of topology are found. We give the definition of the r-neighborhood complex $N_r(G)$ which is a natural generalization of the neighborhood complex defined by Lovász, and the fundamental group of the r-neighborhood complex $\pi_1(N_r(G), v)$ is "almost" isomorphic to the $(2r)$-fundamental group $\pi_1^{2r}(G, v)$.

Let us begin to define the r-fundamental groups. From now on, we assume that r is a fixed positive integer. A based graph is a pair (G, v) where G is a graph and v is a vertex of G. For a non-negative integer n, the graph L_n is defined by

$$V(L_n) = \{0, 1, \cdots, n\},$$
$$E(L_n) = \{(x, y) \mid |x - y| = 1\}.$$

The graph L_4.

Let (G, v) be a based graph. A graph homomorphism from L_n to G mapping 0 and n to v is called a loop of (G, v) with length n. We write $L(G, v)$ for the set of all loops of (G, v). For a loop $\varphi : L_n \to G$, we write $l(\varphi)$ for the length n of φ.

For loops $\varphi, \psi \in L(G, v)$ of (G, v), consider the following two conditions :

(I) $l(\psi) = l(\varphi) + 2$ and there is $x \in \{0, 1, \cdots, l(\varphi)\}$ such that

$$\varphi(i) = \begin{cases} \psi(i) & (i \leq x) \\ \psi(i+2) & (i \geq x). \end{cases}$$

(II)$_r$ $l(\varphi) = l(\psi)$ and $\#\{i \in \{0, 1, \cdots, l(\varphi)\} \mid \varphi(i) \neq \psi(i)\} < r$.

We write $\simeq_r$ for the equivalence relation of $L(G, v)$ generated by the above two conditions. The quotient set $L(G, v)/ \simeq_r$ is called the *r-fundamental group* $\pi_1^r(G, v)$ of the based graph (G, v). It can be seen that $\pi_1^r(G, v)$ becomes a group by the composition of loops as a multiplication.

For a loop φ of a based graph (G, v), we write $[\varphi]_r \in \pi_1^r(G, v)$ for the equivalence class of $\simeq_r$ which is represented by φ. By the definition of $\simeq_r$, the parity of the length of a representative of $\alpha \in \pi_1^r(G, v)$ is independent of the choice of the representative of α. This implies that the map

$$\pi_1^r(G, v) \to \mathbb{Z}/2\mathbb{Z}, [\varphi]_r \mapsto l(\varphi)$$

is a well-defined group homomorphism. The kernel of the above group homomorphism is written by $\pi_1^r(G, v)_{ev}$, and is called the *even part of* $\pi_1^r(G, v)$. $\alpha \in \pi_1^r(G, v)$ is said to be *even* if $\alpha \in \pi_1^r(G, v)_{ev}$, and *odd* if α is not even.

For an element α of $\pi_1^r(G, v)$, define the *length of* α by

$$l(\alpha) = \inf\{l(\varphi) \mid \varphi \text{ is a representative of } \alpha.\}.$$

Here we recall the following well-known lemma, whose proof is found in Section 4.4 in [18] for example.

Lemma 4.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $a_{n+m} \leq a_n + a_m$ for $n, m \in \mathbb{N}$. Then the limit*

$$\lim_{n\to\infty} \frac{a_n}{n}$$

exists, and coincides with $\inf\{a_n/n \mid n > 0\}$.

Since $l(\alpha \cdot \beta) \leq l(\alpha) + l(\beta)$ for $\alpha, \beta \in \pi_1^r(G, v)$, we have that the limit

$$\lim_{n\to\infty} \frac{l(\alpha^n)}{n}$$

exists for any $\alpha \in \pi_1^r(G, v)$. We write $l_s(\alpha)$ for this limit, and call this the *stable length* of α.

As is mentioned in the beginning of this section, r-fundamental groups can be applied to the existence problem of graph homomorphisms as follows. Let $f : (G, v) \to (H, w)$ be a graph homomorphism preserving base points. Then f induces the map $f_* : \pi_1^r(G, v) \to \pi_1^r(H, w)$, $[\varphi]_r \mapsto [f \circ \varphi]_r$. f_* satisfies the followings.

- f_* is a group homomorphism.
- f_* preserves the parities. Hence if $\alpha \in \pi_1^r(G, v)$ is odd, then $f_*(\alpha)$ is also odd, and is non-trivial.
- f_* does not increase lengths and stable lengths.

As an application, we consider the (non-)existence of the graph homomorphism from a given graph G to an odd cycle. For a positive integer n, the graph C_n is defined by

$$V(C_n) = \mathbb{Z}/n\mathbb{Z},$$
$$E(C_n) = \{(x, x \pm 1) \mid x \in \mathbb{Z}/n\mathbb{Z}\}.$$

Then the followings hold :

- If n is positive odd integer, then we have

$$\pi_1^r(C_n) \cong \begin{cases} \mathbb{Z} & (r < n) \\ \mathbb{Z}/2\mathbb{Z} & (r \geq n). \end{cases}$$

  And in the case $r < n$, the stable length of the generator is equal to n.

- If n is positive even integer, then we have

$$\pi_1^r(C_n) \cong \begin{cases} \mathbb{Z} & (r < n/2) \\ 1 & (r \geq n/2). \end{cases}$$

And in the case of $r < n/2$, the stable length of the generator is equal to n.

Then we have the followings.

Theorem 4.2 (M [11]). *Let G be a graph and n a positive integer. If there is a graph homomorphism from G to C_n, then for any $r < n$ and for any odd element α of $\pi_1^r(G)$, we have that $l_s(\alpha) \geq n$.*

The reader may notice that in the notation $\pi_1^r(G)$ in Theorem 4.2, the base point is abbreviated. But as is the case of the fundamental groups of topological spaces, given two base points v, w in the same connected component of G, then the path connecting v with w gives an isomorphism $\pi_1^r(G,v) \to \pi_1^r(G,w)$. This isomorphism preserves parities, and stable lengths. Because of such a reason, we did not verify the base point in Theorem 4.2.

Proof of Theorem 4.2. Let α be an odd element of $\pi_1^r(G,v)$ and β a generator $\pi_1^r(C_n) \cong \mathbb{Z}$. Since $f_*(\alpha)$ is odd, there is $k \in \mathbb{Z}$ such that $f_*(\alpha) = \beta^{2k+1}$. Hence we have that

$$l_s(\alpha) \geq l_s(f_*(\alpha)) = l_s(\beta^{2k+1}) = |2k+1| l_s(\beta) = |2k+1| n \geq n.$$

□

As an application, we consider the case of the Kneser graph $K_{2k+1,k}$. Remark that since $\chi(K_{n,k}) = n - 2k + 2$ as was mentioned in Section 2, there is a graph homomorphism from $K_{2k+1,k}$ to $K_3 \cong C_3$. But we can show that $\pi_1^3(K_{2k+1,k}) \cong \mathbb{Z}/2\mathbb{Z}$, and the generator is odd. Since the stable length of an element of a finite order of $\pi_1^r(G,v)$ is obviously zero, we have that there is no graph homomorphism from $K_{2k+1,k}$ to C_5.

The graph X introduced in Section 3 has an interesting property of 2-fundamental groups. The 2-fundamental group $\pi_1^2(X)$ is isomorphic to $\mathbb{Z}$, and the stable length of the generator is equal to 7/3. Hence by the above theorem, we have that there is no graph homomorphism from X to $K_3 \cong C_3$.

Next we define the r-neighborhood complex $N_r(G)$ of a graph G. Let G be a graph and v a vertex of G. The *s-neighborhood* ($s \geq 1$) is inductively defined as follows :

$$N_1(v) = N(v), \ N_{s+1}(v) = \bigcup_{w \in N_s(v)} N(w).$$

The *r-neighborhood complex* $N_r(G)$ of a graph G is the simplicial complex associated to a graph G defined as follows :

$$V(N_r(G)) = \{v \in V(G) \mid N(v) \neq \emptyset\},$$

$$N_r(G) = \{\sigma \subset V(G) \mid \text{There is a vertex } v \text{ of } G \text{ such that } \sigma \subset N_r(v).\}.$$

In the case $r = 1$, the 1-neighborhood complex is nothing but the neighborhood complex defined by Lovász. Then the following holds.

Theorem 4.3 (M[11]). *Let (G, v) be a based graph. Suppose $N(v) \neq \emptyset$. Then there is a natural group isomorphism*

$$\pi_1(N_r(G), v) \cong \pi_1^{2r}(G, v)_{ev}.$$

Next we introduce the notion of r-covering maps. A graph homomorphism $p : G \to H$ is called an *r-covering map* if for any $v \in V(G)$ and for any s with $1 \leq s \leq r$, the map

$$p|_{N_s(v)} : N_s(v) \to N_s(p(v))$$

is bijective. A base point preserving graph homomorphism $p : (G, v) \to (H, w)$ is called an r-covering map if $p : G \to H$ is an r-covering in the non-based sense. Then there are close relations between r-fundamental groups and r-covering maps which is similar to the covering space theory in topology as follows.

Theorem 4.4. *The followings hold.*

- *If $p : (G, v) \to (H, w)$ is an r-covering map, then the group homomorphism $p_* : \pi_1^r(G, v) \to \pi_1^r(H, w)$ induced by p is injective.*
- *Let $p : (G, v) \to (H, w)$ be a based r-covering map, (T, x) a connected based graph, and $f : (T, x) \to (H, w)$ a based graph homomorphism. Then there is a based graph homomorphism $g : (T, x) \to (G, v)$ such that $p \circ g = f$ if and only if $f_*(\pi_1^r(T, x)) \subset p_*(\pi_1^r(G, v))$.*
- *Let (G, v) be a based graph and Γ a subgroup of $\pi_1^r(G, v)$. Then there is an r-covering map $p : (G_\Gamma, v_\Gamma) \to (G, v)$ such that G_Γ is connected and the image of $p_* : \pi_1^r(G_\Gamma, v_\Gamma) \to \pi_1^r(G, v)$ is equal to Γ. Moreover, such (G_Γ, v_Γ) is unique up to isomorphisms.*

Let (G, v) be a connected based graph. Recall that there is a canonical subgroup $\pi_1^r(G, v)_{ev}$ of $\pi_1^r(G, v)$ called the even part. Let us consider the associated covering of $\pi_1^r(G, v)_{ev}$ in the sense of the above correspondence. If $\chi(G) \leq 2$, then $\pi_1^r(G, v)_{ev}$ is equal to $\pi_1^r(G, v)$, and the associated r-covering is the identity $(G, v) \to (G, v)$. Suppose

$\chi(G) \geq 3$. Then $\pi_1^r(G,v)_{ev} \neq \pi_1^r(G,v)$, and the associated r-covering is the second projection $K_2 \times G \to G$.

It is an interesting problem to consider for a given graph G, how many r-coverings over G exist. The above properties of r-coverings show that the r-coverings over G can be classified by the subgroups of $\pi_1^r(G)$ (more precisely, the conjugate classes of subgroups).

First we should mention that if G has no loops, then a 1-covering over G can be regarded as a covering space over G in the usual sense of topology. Hence there are many 1-coverings over G unless G is a tree. But the case $r \geq 2$ is different.

Recall that $\pi_1^3(K_{2k+1,k})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This implies that the 3-coverings over $K_{2k+1,k}$ are only $K_{2k+1,k}$ and $K_2 \times K_{2k+1,k}$. On the other hand, $\pi_1^2(K_{2k+1,k})$ is isomorphic to $\pi_1^1(K_{2k+1,k})$ which is isomorphic to the fundamental group of the graph G if we regard G as a 1-dimensional CW-complex in the usual way. And hence many 2-coverings over $K_{2k+1,k}$ exist.

Recall that in Section 2, $N(K_{n,k})$ and $N(SK_{n,k})$ are $(n-2k-1)$-connected. Hence in the case $n \geq 2k+2$, by Theorem 4.3, we have that $\pi_1^2(K_{n,k}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_1^2(SK_{n,k}) \cong \mathbb{Z}/2\mathbb{Z}$. Hence if $n \geq 2k+2$, then the 2-coverings over $K_{n,k}$ (or $SK_{n,k}$) are only $K_{n,k}$ and $K_2 \times K_{n,k}$ ($SK_{n,k}$ and $K_2 \times SK_{n,k}$, respectively).

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