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A polynomial invariant and the forbidden move of virtual knots

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1 Preliminaries

A virtual knot diagram is presented by a knot diagram having virtual crossings as well as real crossings in Fig. 1. Two virtual knot diagrams are equivalent if one can be obtained from the other by a finite sequence of generalized Reidemeister moves in Fig. 2. The equivalence class of virtual knot diagrams modulo the generalized Reidemeister moves is called a *virtual knot.*

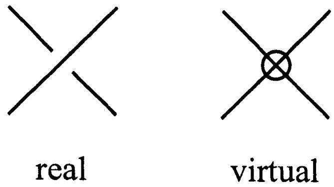

Fig. 1 Crossing types

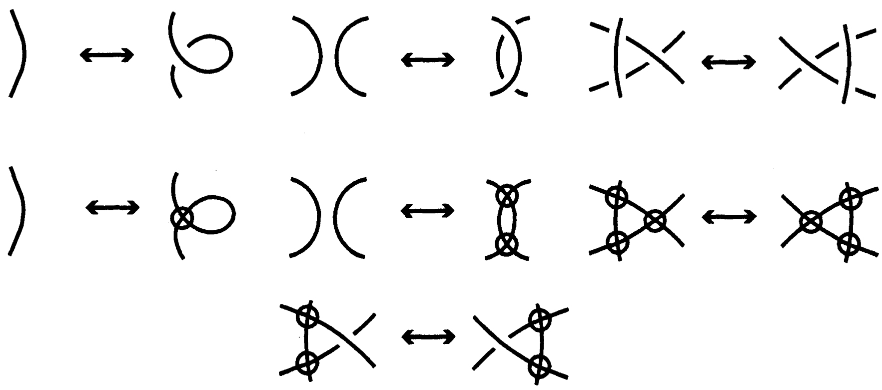

Fig. 2 Generalized Reidemeister moves

Virtual knots can be described by Gauss diagrams. The Gauss diagram of a virtual knot is an oriented circle as the preimage of the immersed circle with chords connecting the preimages of each real crossing. We specify crossing information on each chord by directing the chord toward the under crossing equipped with the sign (Fig. 3) of the crossing. The generalized Reidemeister moves can be encoded in the Gauss diagrams as in Fig. 4 (see [1]). It is well-known that there exists an one-to-one correspondence between all virtual knots and all equivalence classes of Gauss diagrams module the generalized Reidemeister moves (see [4]).

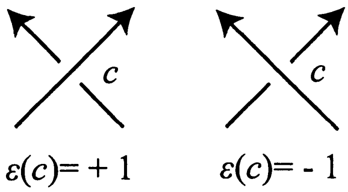

Fig. 3 The sign of a real crossing

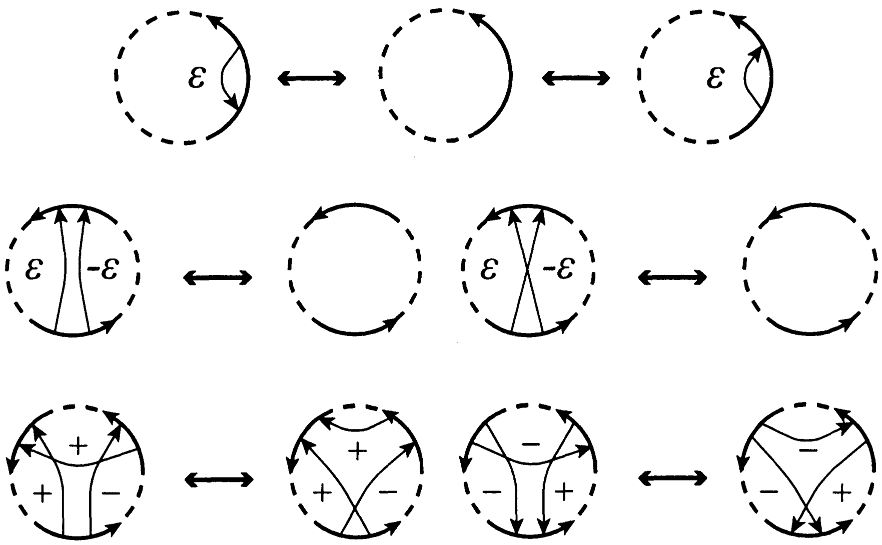

Fig. 4 Generalized Reidemeister moves of Gauss diagrams

Both of the moves depicted in Fig. 5 are called *forbidden moves*, and denoted by F. These are presented by local moves of Gauss diagrams in Fig. 6. It is well known that any virtual knot can be deformed into any other virtual knot by using forbidden moves. Let K be a virtual knot, and D a virtual knot diagram of K. If a virtual knot diagram D can be transformed into a trivial knot diagram by a set of generalized Reidemeister moves and forbidden moves, we denote the minimum number of times of forbidden moves needed to transform D into a trivial knot diagram by $u_F(K)$ and call it the *unknotting number* of K by forbidden moves.

Fig. 5

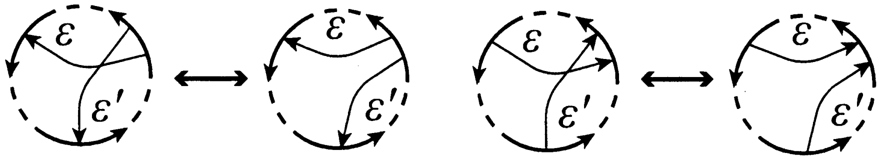

Fig. 6

To estimate unknotting numbers by forbidden moves, we define invariants for virtual knots. Let G be a Gauss diagram of a virtual knot K, and $\gamma = \overrightarrow{PQ}$ a chord in G with sign $\varepsilon(\gamma)$ such γ is oriented from P to Q. We give the signs to the endpoints P and Q, denoted by $\varepsilon(P)$ and $\varepsilon(Q)$, respectively, such that $\varepsilon(P) = -\varepsilon(\gamma)$ and $\varepsilon(Q) = \varepsilon(\gamma)$. For a chord $\gamma = \overrightarrow{PQ}$ in a Gauss diagram G, the specified arc of γ is the arc α in $\mathbb{S}^1$ with endpoints P and Q oriented from P to Q with respect to the orientation of $\mathbb{S}^1$. The index of γ is the sum of the signs of all the points on α except P and Q, and denoted by $i(\gamma)$.

Definition 1.1. Using the index $i(\gamma)$, we define the *n-writhe* $J_n(K)$, the *index polynomial* $\mathbf{p}_t(K)$ and the *odd writhe polynomial* $f_K(t)$ for a virtual knot K (see [11], [6], and [3]):

1. For a non-zero integer n, the n-writhe is given by

$$J_n(K) = \sum_{i(\gamma)=n} \varepsilon(\gamma),$$

2. the index polynomial is given by

$$\mathbf{p}_t(K) = \sum_{\gamma} \varepsilon(\gamma)(t^{|i(\gamma)|} - 1),$$

3. the odd writhe polynomial is given by

$$f_K(t) = \sum_{i(\gamma) \text{ is odd}} \varepsilon(\gamma) t^{1-i(\gamma)}.$$

These are invariants for a virtual knot K. The index polynomial $\mathbf{p}_t(K)$ and the odd writhe polynomial $f_K(t)$ is induced from the n-writhe $J_n(K)$.

Proposition 1.2 ([11]).

1. The following equation holds for $\mathbf{p}_t(K)$ and $J_n(K)$:

$$\mathbf{p}_t(K) = \sum_{n>0} \{J_n(K) + J_{-n}(K)\}(t^n - 1), \tag{1}$$

2. the following equation holds for $f_K(t)$ and $J_n(K)$:

$$f_K(t) = \sum_{n \in \mathbb{Z}} J_{1-2n}(K) t^{2n}. \tag{2}$$

We define a polynomial with coefficients of the n-writhe $J_n(K)$.

Definition 1.3. For a virtual knot K, the polynomial invariant $\mathbf{q}_t(K)$ is given by

$$\begin{aligned}\mathbf{q}_t(K) &= \sum_{n \in \mathbb{Z}} J_n(K)(t^n - 1)\\ &= \sum_{\gamma} \varepsilon(\gamma)(t^{i(\gamma)} - 1).\end{aligned}$$

The polynomial $\mathbf{q}_t(K)$ is an invariant for K, and equivalent to the affine index polynomial (see [9]) fundamentally. It induces the index polynomial $\mathbf{p}_t(K)$ and the odd writhe polynomial $f_K(t)$ from equations (1), (2).

2 Forbidden moves and $\mathbf{q}_t(K)$

We consider the relation between the polynomial invariant $\mathbf{q}_t(K)$ and forbidden moves.

Theorem 2.1. Let K and K' be two virtual knots which can be transformed into each other by a single forbidden move. Then we have

$$\mathbf{q}_t(K) - \mathbf{q}_t(K') = (t-1)(\pm t^{\ell} \pm t^m),$$

for some integers k and ℓ.

Proof. Let K and K' be virtual knots which can be transformed into each other by a single forbidden move. Virtual knots K and K' are represented by Gauss diagrams G and G' respectively in cases (I) or (II) of Fig. 7. We only show the case (I) since the other case can be treated similarly. Chords γ_i and γ_i' (i= 1, 2) of G and G' are the two chords in the part where a forbidden move is applied. For chords γ_i and γ_i', we have

$$\begin{aligned}i(\gamma_1') &= i(\gamma_1) - \varepsilon(\gamma_2),\\ i(\gamma_2') &= i(\gamma_2) + \varepsilon(\gamma_1)\end{aligned}$$

,where $\varepsilon(\gamma_i) = \varepsilon(\gamma_i')$. Therefore,

$$\begin{aligned}\mathbf{q}_t(K) - \mathbf{q}_t(K') =& \varepsilon(\gamma_1)(t^{i(\gamma_1)} - 1) + \varepsilon(\gamma_2)(t^{i(\gamma_2)} - 1) \\ & - \varepsilon(\gamma_1')(t^{i(\gamma_1')} - 1) - \varepsilon(\gamma_2')(t^{i(\gamma_2')} - 1) \\ =& \varepsilon(\gamma_1)t^{i(\gamma_1)} + \varepsilon(\gamma_2)t^{i(\gamma_2)} - \varepsilon(\gamma_1)t^{i(\gamma_1)-\varepsilon(\gamma_2)} - \varepsilon(\gamma_2)t^{i(\gamma_2)+\varepsilon(\gamma_1)} \\ =& \varepsilon(\gamma_1)t^{i(\gamma_1)}(1 - t^{-\varepsilon(\gamma_2)}) + \varepsilon(\gamma_2)t^{i(\gamma_2)}(1 - t^{\varepsilon(\gamma_1)}) \\ =& \begin{cases} (t-1)(t^{i(\gamma_1)-1} - t^{i(\gamma_2)}) & (\varepsilon(\gamma_i) = 1) \\ (t-1)(-t^{i(\gamma_1)} + t^{i(\gamma_2)}) & (\varepsilon(\gamma_1) = 1,\ \varepsilon(\gamma_2) = -1) \\ (t-1)(-t^{i(\gamma_1)-1} + t^{i(\gamma_2)-1}) & (\varepsilon(\gamma_1) = -1,\ \varepsilon(\gamma_2) = 1) \\ (t-1)(t^{i(\gamma_1)} - t^{i(\gamma_2)-1}) & (\varepsilon(\gamma_i) = -1) \end{cases}\end{aligned}$$

□

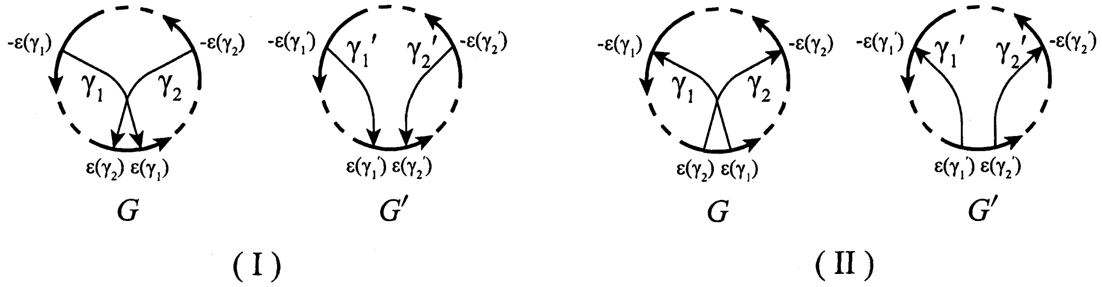

Fig. 7

Corollary 2.2. Let K be a virtual knot, and $\mathbf{q}_t(K) = (t-1)\sum_{n\in\mathbb{Z}} a_n t^n$. Then,

$$u_F(K) \geq \frac{\sum_{n\in\mathbb{Z}} |a_n|}{2}.$$

From Theorem 2.1, we have the estimate of the unknotting number for a virtual knot K by forbidden moves using the value of $J_n(K)$ as the following:

$$u_F(K) \geq \frac{\sum_{n\neq 0} |J_n(K)|}{4}.$$

We know some estimates of the unknotting number $u_F(K)$.

Theorem 2.3. We may estimate $u_F(K)$ by using values of $\mathbf{p}_t(K)$ and $f_K(t)$ (see [13], [2]):

1. Let K be a virtual knot, and $\mathbf{p}_t(K) = (t-1)\sum_{n\geq 0} b_n t^n$. Then,

$$u_F(K) \geq \frac{\sum_{n\geq 0} |b_n|}{2},$$

2. let K be a virtual knot, and $f_K(t) = \sum_{n\neq 0} c_n t^n$. Then,

$$u_F(K) \geq \frac{\sum_{n\neq 0} |c_n|}{2}.$$

Since the polynomial invariant $\mathbf{q}_t(K)$ induces the index polynomial $\mathbf{p}_t(K)$ and the odd writhe invariant $f_K(t)$, we have

$$u_F(K) \geq \frac{\sum_{n\neq 0} |a_n|}{2} \geq \frac{\sum_{n\neq 0} |J_n(K)|}{4}, \frac{\sum_{n\geq 0} |b_n|}{2}, \frac{\sum_{n\neq 0} |c_n|}{2}.$$

We see that the estimate of $u_F(K)$ by $\mathbf{q}_t(K)$ is the strongest estimate in these.

3 Examples

We calculate unknotting numbers of virtual knots by forbidden moves by Corollary 2.2.

Example 3.1. For a positive even number m, let K_m be a virtual knot with a diagram D_m in Fig. 8. The virtual knot K_m is represented by the Gauss diagram G_m in Fig. 9. Indexes of all chords of G_m are calculated as the following:

$$i(\gamma_1) = -m,\ i(\gamma_2) = -1 \cdots i(\gamma_{m+1}) = -1,$$
$$i(\gamma_{m+2}) = -1 \cdots i(\gamma_{2m+1}) = -1,\ i(\gamma_{2m+2}) = m.$$

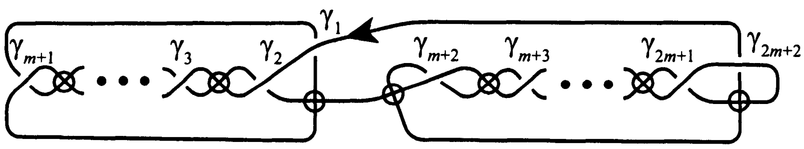

Fig. 8 The virtual knot D_m

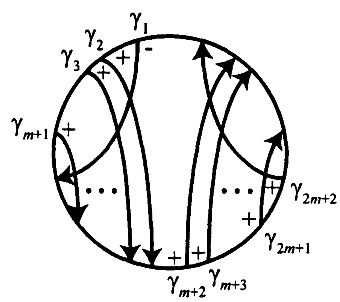

Fig. 9 The Gauss diagram G_m

Calculating values of invariants $\mathbf{p}_t$, $f(t)$, J_n and $\mathbf{q}_t$ for a virtual knot K_m,

$$\mathbf{p}_t(K_m) = 2m(t-1), \tag{3}$$

$$f_{K_m}(t) = 2mt^2, \tag{4}$$

$$J_{-m}(K_m) = -1,\ J_{-1}(K_m) = 2m,\ J_m(K_m) = 1, \tag{5}$$

$$\mathbf{q}_t(K_m) = (t-1)\{t^{m-1} + \cdots + 1 + (-2m+1)t^{-1} + \cdots + t^{-m}\}. \tag{6}$$

From equations (3), (4), (5) and (6), we have equations (7), (8), (9) and (10) respectively.

$$u_F(K_m) \geq m, \tag{7}$$

$$u_F(K_m) \geq m, \tag{8}$$

$$u_F(K_m) \geq \frac{m+1}{2}, \tag{9}$$

$$u_F(K_m) \geq 2m-1. \tag{10}$$

Since we may transform Gauss diagram G_m into a trivial knot as shown in the Fig. 10 by $2m-1$ forbidden moves, we obtain $u_F(K_m) = 2m-1$. Therefore there exist a virtual knot K_k such that the unknotting number $u_F(K_k)$ can be determined from $\mathbf{q}_t(K_k)$ though it cannot be determined from $\mathbf{p}_t(K_k)$, $f_{K_k}(t)$ and $J_n(K_k)$ for any natural number k.

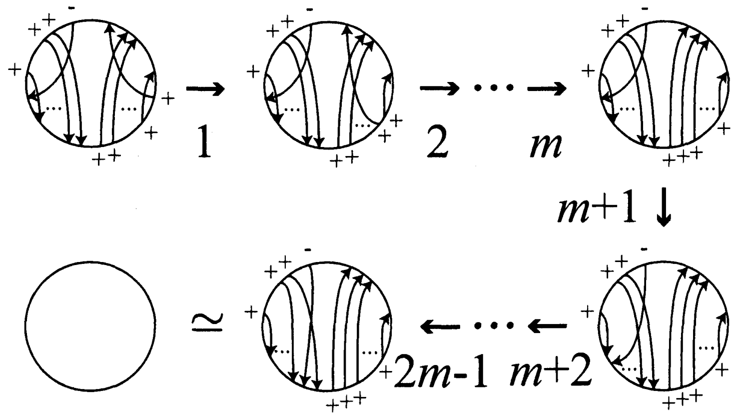

Fig. 10

Example 3.2. Table 1 shows unknotting numbers for virtual knots with up to 4 real crossing points (see [5]) by forbidden moves using the value of $\mathbf{q}_t$.

Acknowledgments

The author would like to thank Professor Yoshiyuki Ohyama for many helpful suggestions and comments. The author is also grateful to Shin Satoh for valuable advices.

K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$
0.1	0	4.20	1	4.48	3	4.76	1	4.104	3-5
2.1	1	4.21	2	4.49	1	4.77	1	4.105	1-5
3.1	1	4.22	1	4.50	1	4.78	3-4	4.106	2-3
3.2	1	4.23	1	4.51	1	4.79	1	4.107	1-4
3.3	2	4.24	2-3	4.52	1	4.80	3	4.108	1-4
3.4	1	4.25	2	4.53	2	4.81	2		
3.5	2-3	4.26	2	4.54	1	4.82	3		
3.6	1-4	4.27	1-2	4.55	1	4.83	2		
3.7	2-3	4.28	2	4.56	1	4.84	1-2		
4.1	2	4.29	2	4.57	1	4.85	2		
4.2	1-2	4.30	1-2	4.58	1	4.86	2-3		
4.3	2	4.31	1	4.59	1	4.87	4-5		
4.4	1	4.32	1	4.60	1	4.88	1		
4.5	1	4.33	1	4.61	1-4	4.89	4		
4.6	1-2	4.34	1	4.62	3-4	4.90	1-2		
4.7	2	4.35	1	4.63	2	4.91	4-5		
4.8	1-2	4.36	2	4.64	1-2	4.92	3-5		
4.9	1-2	4.37	3	4.65	2-4	4.93	2		
4.10	1-2	4.38	1	4.66	2-3	4.94	1-5		
4.11	2	4.39	1	4.67	1-2	4.95	3-5		
4.12	1-2	4.40	1	4.68	1-3	4.96	2-3		
4.13	1-2	4.41	1	4.69	1-2	4.97	1-2		
4.14	1-2	4.42	1	4.70	1-2	4.98	1-3		
4.15	2	4.43	2	4.71	1-2	4.99	1-3		
4.16	1	4.44	1-2	4.72	1-2	4.100	2-7		
4.17	1	4.45	2	4.73	2	4.101	3-4		
4.18	1	4.46	1-2	4.74	1	4.102	2-4		
4.19	1-2	4.47	2	4.75	1-2	4.103	3-6		

Tab. 1 Unknotting numbers

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