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Construction of Infinite Product Possibility Space

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Abstract: In this paper, we construct infinite product posssibility space. Then, we show that there exists countable independent fuzzy variables.
Keywords: Possibility measure, product possibility space, independent fuzzy variables.

1 Introduction

Dubois and Prade wrote a book on Possibility theory [1] in 1988. Then, B. Liu [5] has studied it with axiomatic foundations. K.Iwamura and M. Kageyama have studied possibility-based fuzzy linear programming problems with finitely many independent fuzzy sets [2]. They summarized axiomatic foundation of possibility measure and independence in K. Iwamura and M. Kageyama [3].

Here, we will extend K.Iwamura and M.Kageyama [3] to infinite product possibility space. In section 2, we give brief definitions. In section3, we show how to construct infinite product possibility space. In sectioin 4, we show the existence of coutable independent fuzzy variables.

2 Definitions

Let Θ be an arbitrary non-empty set. Let $\mathcal{P}(\Theta)$ be the power set of Θ. We call a real valued set function Pos on $\mathcal{P}(\Theta)$ possibility measure if it satisfies the following three axioms [5].

P1: $\mathrm{Pos}\{\Theta\} = 1.$

P2: $\mathrm{Pos}\{\emptyset\} = 0.$

P3: $\mathrm{Pos}\{\cup_i A_i\} = \sup_i \mathrm{Pos}\{A_i\}$ *for any collection* $\{A_i\} \subset \mathcal{P}(\Theta)$.

Each element in $\mathcal{P}(\Theta)$ is called an event. To an event A, a number $\mathrm{Pos}\{A\}$ which indicates the possibility that A will occur is assigned. We call $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ a possibility space. A fuzzy variable ξ is defined as a function from Θ to the set of

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real numbers $\Re$. Let $\xi_1, \ldots, \xi_n$ be fuzzy variables on Θ. Then $\xi_1, \ldots, \xi_n$ are said to be independent if they satisfy

$$\text{Pos}\{\theta \in \Theta | \xi_1(\theta) \in B_1 \ \& \cdots \& \ \xi_n(\theta) \in B_n\} = \bigwedge_{i=1}^{n} \text{Pos}\{\theta \in \Theta | \xi_i(\theta) \in B_i\}$$

for any subsets $B_1, \ldots, B_n$ of $\Re$, where $a \wedge b = \min(a, b)$.

Let T be an infinite set. Fuzzy variables $\xi_t, t \in T$ are called independent fuzzy variables if for any subsets B_t of $\Re, t \in T$,

$$\text{Pos}\{\theta \in \Theta | \xi_t(\theta) \in B_t \ \forall t \in T\} = \inf_{t \in T} \text{Pos}\{\theta \in \Theta | \xi_t(\theta) \in B_t\}$$

holds.

3 Infinite Product Possibility Space

Let $(\Theta_t, \mathcal{P}(\Theta_t), \text{Pos}_t), t \in T$ be a family of possibility spaces. For any $B_T \in \mathcal{P}(\Theta_T)$, where $\Theta_T = \prod_{t \in T} \Theta_t$, we define a real valued set function $\text{Pos}_T\{\}$ over $\mathcal{P}(\Theta_T)$ by

$$\text{Pos}_T\{B_T\} = \sup_{\theta_T \in B_T} (\inf_{t \in T} \text{Pos}_t\{\theta_t\}), \tag{3.1}$$

where $\theta_T = (\theta_t, t \in T), \theta_t \in \Theta_t, t \in T, \theta_T \in \Theta_T$.

Theorem 3.1 *Let $B_T = \prod_{t \in T} B_t, B_t \subset \Theta_t$. Then we get*

$$\text{Pos}_T\{\prod_{t \in T} B_t\} = \sup_{\theta_T \in \prod_{t \in T} B_t} (\inf_{t \in T} \text{Pos}_t\{\theta_t\}) \tag{3.2}$$

$$= \inf_{t \in T}(\sup_{\theta_t : \theta_t \in B_t} \text{Pos}_t\{\theta_t\}) \tag{3.3}$$

$$= \inf_{t \in T}(\text{Pos}_t\{B_t\}) \tag{3.4}$$

Proof; Let, for any $t \in T$,

$$\sup_{\theta_t : \theta_t \in B_t} \text{Pos}_t\{\theta_t\} = b_t. \tag{3.5}$$

Then, we get

$$\text{Pos}_t\{\theta_t\} \le b_t \text{ for any } \theta_t \text{ with } \theta_t \in B_t \tag{3.6}$$

and for any $\epsilon > 0$ there exists $\theta_t^\epsilon \in B_t$ such that

$$b_t - \epsilon < \text{Pos}_t\{\theta_t^\epsilon\}. \tag{3.7}$$

Therefore, by (3.6), for any $\theta_t \in B_t$

$$\inf_{t' \in T} \text{Pos}_{t'}\{\theta_{t'}\} \le \text{Pos}_t\{\theta_t\} \le b_t, \inf_{t' \in T} \text{Pos}_{t'}\{\theta_{t'}\} \le \inf_{t' \in T} b_{t'}. \tag{3.8}$$

Furthermore, for this $\epsilon(>0)$, there exists $\theta^\epsilon = (\theta^\epsilon_{t'}, t' \in T) \in \prod_{t' \in T} B_{t'} \subseteq \Theta_T$ such that

$$\inf_{t' \in T} b_{t'} \leq \inf_{t' \in T}(\mathrm{Pos}_{t'}\{\theta^\epsilon_{t'}\} + \epsilon) = \inf_{t' \in T} \mathrm{Pos}_{t'}\{\theta^\epsilon_{t'}\} + \epsilon. \tag{3.9}$$

And so, we further get

$$\inf_{t' \in T} b_{t'} - 2\epsilon \leq \inf_{t' \in T}(\mathrm{Pos}_{t'}\{\theta^\epsilon_{t'}\}) - \epsilon < \inf_{t' \in T} \mathrm{Pos}_{t'}\{\theta^\epsilon_{t'}\}. \tag{3.10}$$

This statement with (3.8) tells us that

$$\inf_{t' \in T} b_{t'} = \sup_{\theta_T \in \prod_{t' \in T} B_{t'}} (\inf_{t \in T} \mathrm{Pos}_t\{\theta_t\})$$

and so, through (3.5) we finally get

$$\inf_{t \in T}(\sup_{\theta_t : \theta_t \in B_t} \mathrm{Pos}_t\{\theta_t\}) = \sup_{\theta_T : \theta_T \in \prod_{t' \in T} B_{t'}} (\inf_{t \in T} \mathrm{Pos}_t\{\theta_t\}), \tag{3.11}$$

$$\sup_{\theta_t : \theta_t \in B_t} \mathrm{Pos}_t\{\theta_t\} = \mathrm{Pos}_t\{B_t\} \text{ [by P3 in the possibility measure axioms]}, \tag{3.12}$$

$$\inf_{t \in T}(\sup_{\theta_t : \theta_t \in B_t} \mathrm{Pos}_t\{\theta_t\}) = \inf_{t \in T} \mathrm{Pos}_t\{B_t\}. \tag{3.13}$$

□

Theorem 3.2 *We see that*

$$\mathrm{Pos}_T\{\Theta_T\} = 1, \tag{3.14}$$

$$\mathrm{Pos}_T\{\emptyset\} = 0, \tag{3.15}$$

$$\mathrm{Pos}_T\{\cup_i A_i\} = \sup_i \mathrm{Pos}_T\{A_i\} \text{ for any collection } \{A_i\} \text{ of } \mathcal{P}(\Theta_T) \tag{3.16}$$

holds.

Proof; By (3.4) and $\Theta_T = \prod_{t \in T} \Theta_t$, we get

$$\mathrm{Pos}_T\{\Theta_T\} = \mathrm{Pos}_T\{\prod_{t \in T} \Theta_t\} = \inf_{t \in T} \mathrm{Pos}_t\{\Theta_t\} = \inf_{t \in T} 1 = 1.$$

$$\mathrm{Pos}_T\{\emptyset\} = \mathrm{Pos}_T\{\prod_{t \in T} \emptyset\} = \inf_{t \in T} \mathrm{Pos}_t\{\emptyset\} = \inf_{t \in T} 0 = 0.$$

For any collection $\{A_i\}$ of $\mathcal{P}(\Theta_T)$, we get

$$\mathrm{Pos}_T\{\cup_i A_i\} = \sup_{\theta_T \in \cup_i A_i} (\inf_{t \in T} \mathrm{Pos}_t\{\theta_t\}) \tag{3.17}$$

and

$$\sup_i \mathrm{Pos}_T\{A_i\} = \sup_i \sup_{\theta_T \in A_i} (\inf_{t \in T} \mathrm{Pos}_t\{\theta_t\})$$

by (3.1). Let $\sup_i \sup_{\theta_T \in A_i}(\inf_{t\in T} \mathrm{Pos}_t\{\theta_t\}) = b$. Then, we get

$$\sup_{\theta_T \in A_i}(\inf_{t\in T} \mathrm{Pos}_t\{\theta_t\}) \le b \text{ for any } i \tag{3.18}$$

and for any $\epsilon > 0$ there exists $i(\epsilon)$ such that

$$b - \epsilon < \sup_{\theta_T \in A_{i(\epsilon)}}(\inf_{t\in T} \mathrm{Pos}_t\{\theta_t\}) \le b. \tag{3.19}$$

From (3.19), for this $\epsilon > 0$, there exists $\theta^d_{T,\epsilon} = (\theta^d_{t,\epsilon}, t \in T) \in A_{i(\epsilon)}(\subset \cup_k A_k)$ such that

$$\sup_{\theta_T \in A_{i(\epsilon)}}(\inf_{t\in T} \mathrm{Pos}_t\{\theta_t\}) - \epsilon < \inf_{t\in T} \mathrm{Pos}_t\{\theta^d_{t,\epsilon}\} \tag{3.20}$$

holds. So, (3.20) with (3.19) leads to

$$b - 2\epsilon < \inf_{t\in T} \mathrm{Pos}_t\{\theta^d_{t,\epsilon}\}. \tag{3.21}$$

On the other hand, for any $\theta^d_T = (\theta^d_t, t \in T) \in \cup_k A_k$, there exists k^d_0 such that $\theta^d_T \in A_{k^d_0}$. Using (3.18) with $i = k^d_0$, we get

$$\inf_{t\in T} \mathrm{Pos}_t\{\theta^d_t\} \le \sup_{\theta^d_T \in A_{k^d_0}}(\inf_{t\in T} \mathrm{Pos}_t\{\theta^d_t\}) \le b,$$

which leads to

$$\inf_{t\in T} \mathrm{Pos}_t\{\theta^d_t\} \le b. \tag{3.22}$$

Therefore, by (3.21) and (3.22) , we get

$$b = \sup_{\theta^d_T \in \cup_k A_k}(\inf_{t\in T} \mathrm{Pos}_t\{\theta^d_t\}),$$

which tells us that

$$\sup_i(\sup_{\theta_T \in A_i}(\inf_{t\in T} \mathrm{Pos}_t\{\theta_t\})) = \sup_{\theta_T \in \cup_i A_i}(\inf_{t\in T} \mathrm{Pos}_t\{\theta_t\}),$$

$$\sup_i \mathrm{Pos}_T\{A_i\} = \mathrm{Pos}_T\{\cup_i A_i\}$$

holds. □

4 Countable Independent Fuzzy Variables

Let $T = \{1, 2, \dots\}$. Let ξ_i be a fuzzy variable from Θ_i to $\Re$. Let $B_i \in \mathcal{P}(\Theta_i)$ be given for $i \in T$. Define $\tilde{\xi}_i(\theta) = \xi_i(\theta_i)$ for any $\theta = (\theta_i, i \in T)$. Then , we get

$$\{\theta \in \Theta_T | \tilde{\xi}_i(\theta) \in B_i\} = \{\theta_i \in \Theta_i | \xi_i(\theta_i) \in B_i\} \times \prod_{t \in T\setminus\{i\}} \Theta_t,$$

$$\{\theta \in \Theta_T | \tilde{\xi}_i(\theta) \in B_i, \forall i \in T\} = \{\theta \in \Theta_T | \xi_i(\theta_i) \in B_i, \forall i \in T\}$$
$$= \prod_{i \in T} \{\theta_i \in \Theta_i | \xi_i(\theta_i) \in B_i\}.$$

By (3.2) and (3.3) , we get

$$\mathrm{Pos}_T\{\tilde{\xi}_i \in B_i, \forall i \in T\} = \inf_{i \in T} \mathrm{Pos}_i\{\xi_i \in B_i\},$$

$$\mathrm{Pos}_T\{\tilde{\xi}_i \in B_i\} = \mathrm{Pos}_i\{\xi_i \in B_i\}, i \in T.$$

Hence , we get

$$\mathrm{Pos}_T\{\tilde{\xi}_i \in B_i, \forall i \in T\} = \inf_{i \in T} \mathrm{Pos}_T\{\tilde{\xi}_i \in B_i\},$$

i.e. ,

Theorem 4.1 *There exists coutable independent fuzzy variables.*

□

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