数理解析研究所講究録
第 1913 巻 2014 年 67-80

Leaf posets and multivariate hook length property

Masao ISHIKAWA
Faculty of Education, University of the Ryukyus
Nishihara, Okinawa, Japan
ishikawa@edu.u-ryukyu.ac.jp

Hiroyuki TAGAWA*
Faculty of Education, Wakayama University
Sakaedani, Wakayama, Japan
tagawa@math.edu.wakayama-u.ac.jp

1 Introduction

In [5] Robert A. Proctor defined d-complete posets, which include shapes, shifted shapes and trees, by certain local structural conditions and showed that arbitrary connected d-complete poset is decomposed into a slant sum of irreducible ones. He also classified 15 exhaustive classes of irreducible d-complete components and described all of the members of each class. In this article we define 6 types of posets called basic leaf posets which generalize the irreducible d-complete posets. By using an operation of posets called joint sums, which is a slightly generalization of slant sums, we define general leaf posets which include all d-complete posets.

Dale Peterson and Proctor [9] and Kento Nakada [4] proved that the multivariable generating function of P-partitions for any d-complete poset P has nice product formula independently. The purpose of this article is to define multivariate hook length property as a multivariate analogue of hook length property and to show that any leaf poset has multivariate hook length property which is an extension of their result.

Throughout this article, let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_{>0}$ denote the set of integers, non-negative integers and positive integers, respectively. For a set S, we denote the cardinality of S by $|S|$. From now on, P is a partially ordered set (poset) and is assumed to be finite. If $x, y\in P$, then we say x is covered by y (or y covers x) if $x<y$ and no $z\in P$ satisfies $x<z<y$. When x is covered by y, we denote $x\lessdot y$. A chain of length m is a totally ordered set with m elements, we denote a chain of length m by c_m. A tree T is a finite connected poset with a maximum element such that every element except the maximum element is covered by exactly one element.

Let P and Q be posets such that P is non-adjacent to Q, i.e. P shares no element with Q and there is no order relation between the elements of P and the elements of Q. Set the three conditions for elements x, y of $P\cup Q$ as follows: (i) $x, y\in P$ and $x\leq y$ in P, (ii) $x, y\in Q$ and $x\leq y$ in Q, (iii) $x\in P$ and $y\in Q$. Set R_1 and R_2 to be $P\cup Q$ as a set, and we define the order relation $x\leq y$ in R_1 (resp. R_2) if x, y satisfies the above conditions (i) or (ii) (resp. (i),(ii) or (iii)). We use $P+Q$ (resp. $P\oplus Q$) to denote this new poset R_1 (resp. R_2), and call it the direct sum of P and Q (resp. the ordinal sum of P and Q).

A P-partition is an order reversing map from P to the set of non-negative integers $\mathbb{N}$, i.e. a P-partition φ satisfies that $\varphi(x)\geq\varphi(y)$ if $x\leq y$ in P, and we denote the set of all P-partitions by $\mathscr{A}(P)$. We write

$$F(P;q):=\sum_{\varphi\in\mathscr{A}(P)}q^{\sum_{x\in P}\varphi(x)},$$

which called the one variable generating function of P-partitions. We say that P has hook length

*Partially supported by Grant-in-Aid for Scientific Research (C) No.23540017, Japan Society for the Promotion of Science.

property if there exists a map h from P to $\mathbb{Z}_{>0}$ satisfying

$$F(P;q) = \prod_{x\in P} \frac{1}{1-q^{h(x)}}. \tag{1}$$

If P has hook length property, then $h(x)$ is called the hook length of x, and h is called a hook length function of P. A hook length poset is a poset which has hook length property. Note that the order in the above definition of hook length posets is the dual of the order in the original definition by B. Sagan in [11]. In this article, for a hook length poset P, let h_P be a hook length function of P, i.e. h_P is a map from P to $\mathbb{Z}_{>0}$ satisfying the equality (1). Any tree T is known to be a hook length poset with the hook length $h(x)$ defined by $h(x) = |\{y \in T; y \le x\}|$. It is easy to see that the direct sum of hook length posets is also a hook length poset. More general, Peterson and Proctor [8] proved that any d-complete poset is a hook length poset. As an multivariate analogue of their result, Peterson-Proctor [9] and Nakada [4] obtained the following.

Theorem 1.1 (Peterson and Proctor [9], Nakada [4]). Let P be a d-complete poset, T be the top tree of P and let c be a d-complete coloring, i.e. c is a map from P to $\{1,2,\dots,|T|\}$ satisfying the following three conditions: (i) If x and y are incomparable, then $c(x) \neq c(y)$, (ii) If $[x,y]$ is a chain, then all $c(z)$ $(z \in [x,y])$ are distinct, (iii) If $[x,y]$ is a d_k-interval, then $c(x) = c(y)$, where $[x,y]$ is the interval between x and y, i.e. $[x,y] = \{z \in P; x \le z \le y\}$. We define a map H from P to the set of the monomials of elements $q_1, q_2, \dots, q_{|T|}$ as follows:

$$H(x) := \begin{cases} H(a)H(b)H(y)^{-1} & \text{if } [y,x] \text{ is a } d_k\text{-interval and } a,b \text{ are incomparable in } [y,x], \\ \prod_{y\le x} q_{c(y)} & \text{otherwise.} \end{cases}$$

Then, we have

$$\sum_{\varphi\in\mathscr{A}(P)} \prod_{x\in P} q_{c(x)}^{\varphi(x)} = \prod_{x\in P} \frac{1}{1-H(x)}.$$

This paper is organized as follows. In Section 2, we define 6 types of posets called basic leaf posets, which is an extension of the irreducible d-complete posets, and define general leaf posets includes all d-complete posets by using an operation called joint sums. In Section 3, we define multivariate hook length property and multivariable hook length posets as an extension of hook length property and hook length posets, and show that any basic leaf poset has multivariate hook length property. In Section 4, we describe four kinds of methods in order to make a new multivariable hook length poset (or a new hook length poset) from known multivariable hook length posets (or hook length posets) and to show that any leaf poset is a multivariable hook length poset which is a generalization of Theorem 1.1. In addition, we define extended leaf posets and multivariable extended leaf posets by using these four kinds of methods. Also, by using a list of hook length posets described in Proctor's web page [1], we check whether there exists a hook length poset which is not extended leaf poset. In the last section, we introduce a new class of multivariable hook length posets.

Note that, by the restriction of the number of pages, we define basic leaf posets by using diagrams and omit all the proofs in this article. In [2], we described the detailed definition of basic leaf posets and in our future paper [3], we will give all the proofs.

2 Leaf posets

In this section, we define leaf posets and describe our previous result. Before the definition of general leaf posets, we will define 6 types of basic leaf posets, i.e. ginkgoes, bamboos, ivies, wisterias, firs, chrysanthemums, which generalize 15 classes of irreducible d-complete posets.

Definition 2.1 (Basic leaf posets). (i) Let $m \geq 2$ be an integer, and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ be strictly decreasing sequences of non-negative integers of length m. Let γ be a non-negative integer. Then, a ginkgo $G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ is a poset defined by Diagram 1. In order to see this diagram as a Hasse diagram, 45 clockwise rotations of this diagram are required.

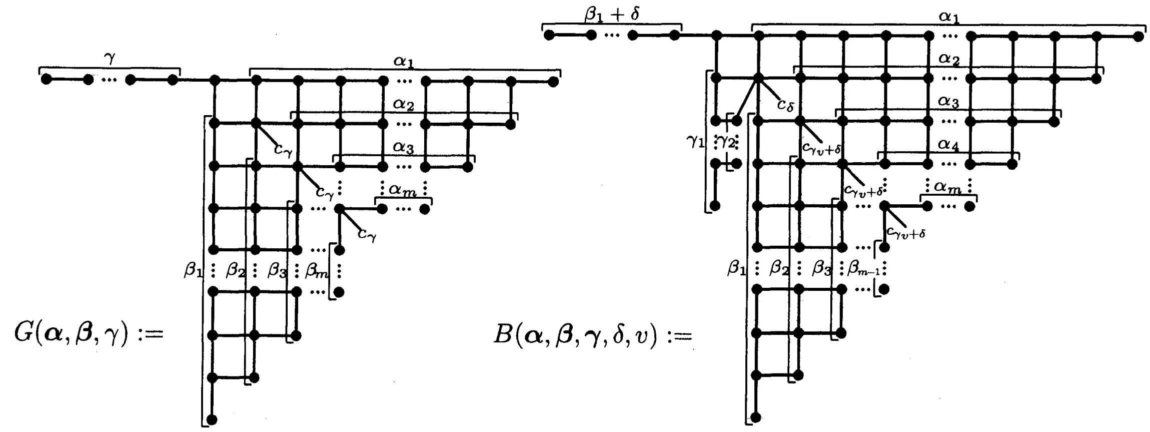


Diagram 1: Ginkgo and Bamboo

(ii) Let $m \geq 2$ be an integer, and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{m-1})$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be strictly decreasing sequences of non-negative integers. Let δ be a non-negative integer. Fix $v = 1$ or 2. Then, a bamboo $B(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v)$ is a poset defined by Diagram 1.

(iii) Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be strictly decreasing sequences of non-negative integers. Let δ be a non-negative integer. Fix $v = 1$ or 2. Then, an ivy $I(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v)$ is a poset defined by Diagram 2.

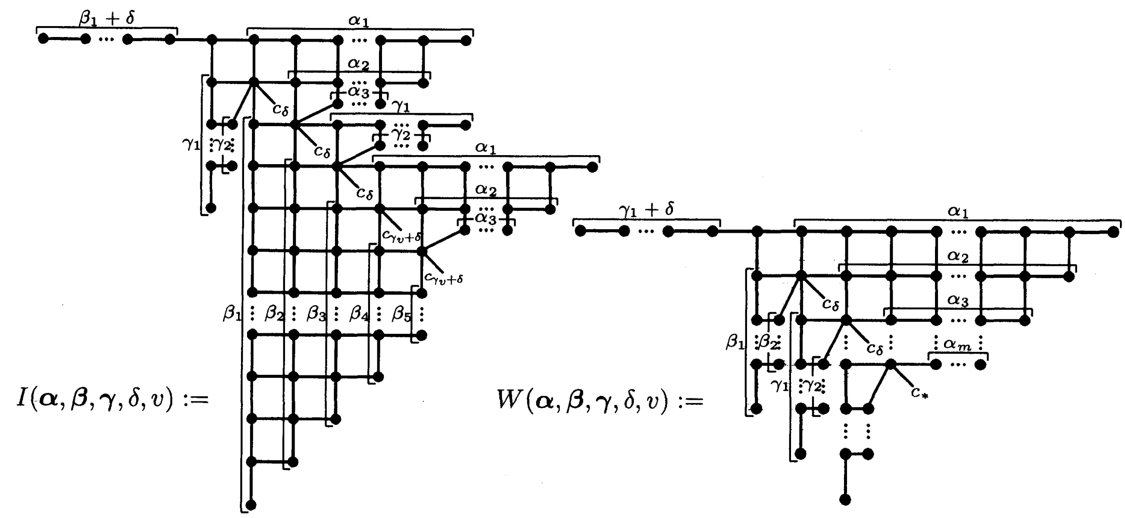


Diagram 2: Ivy and Wisteria

(iv) Let $m \geq 2$ be a positive integer, and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be strictly decreasing sequences of non-negative integers. Let δ be a non-negative integer. Fix $v = 1$ or 2. Then, a wisteria $W(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v)$ is a poset defined by Diagram 2. In the diagram, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ appear alternatively in the place under the left and c_* equals $c_{\gamma_v+\delta}$ (resp. $c_{\beta_v+\delta}$) if m is even (resp. m is odd).

(v) Let $m \geq 3$ be a positive integer, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{m-1})$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be strictly decreasing sequences of non-negative integers. Let δ be a non-negative integer. Fix positive integers s, t which satisfy $1 \leq s < t \leq 3$. Let $v \in \{s, t\}$ if m is even, or let $v \in \{1, 2\}$ if m is odd. Then, a fir $F(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, s, t, v)$ is a poset defined by Diagram 3. In the diagram, $\boldsymbol{\gamma}$

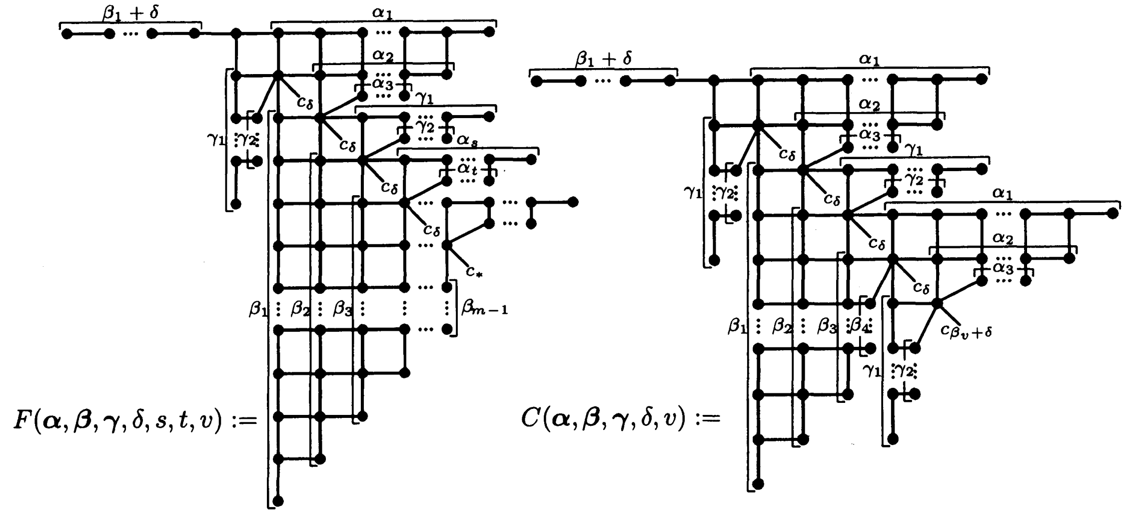


Diagram 3: Fir and Chrysanthemum

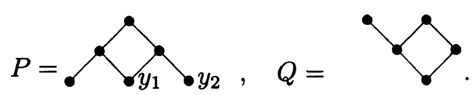
and (α_s, α_t) appear alternatively in the place upper the right and c_* equals $c_{\alpha_v+\delta}$ (resp. $c_{\gamma_v+\delta}$) if m is even (resp. m is odd).

(vi) Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be strictly decreasing sequences of non-negative integers. Let δ be a non-negative integer. Fix $v = 1, 2, 3$ or 4. Then, a chrysanthemum $C(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v)$ is a poset defined by Diagram 3.

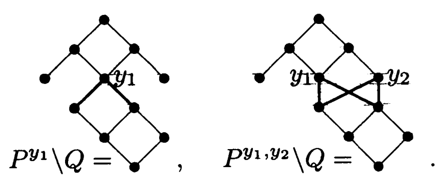
Next, we explain how to compose a general leaf poset from the basic ones. An operation called slant sum which combines two posets in order to generate a new poset was introduced by Proctor. Here we slightly generalize the definition, and call it a joint sum to distinguish from the slant sum.

Definition 2.2 (Joint sums). Let P be a finite poset and let $y_1, y_2, \ldots, y_r$ be any elements of P. Let Q be a finite poset which is non-adjacent to P. Let $x_1, x_2, \cdots, x_m$ be all the maximal elements of Q. Set R to be $P \cup Q$ as a set, and make it a partially ordered set by inserting the additional covering relations $x_i < y_j$, where $1 \leq i \leq m$ and $1 \leq j \leq r$, besides the order relations among the elements of P or Q. We use $P^{y_1, y_2, \ldots, y_r} \backslash Q$ to denote this new poset R, and call it the joint sum of P with Q at $y_1, y_2, \ldots, y_r$.

For example, let P and Q be finite posets defined by the following diagrams:



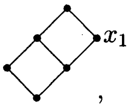
Then, we obtain the joint sum of P with Q at y_1 and the joint sum of P with Q at y_1, y_2 as follows:

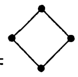
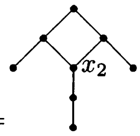


In the above diagrams, the thick lines are the additional covering relations.

An order ideal of a poset P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. The order ideal $\langle x \rangle = \{y \in P \mid y \leq x\}$ is the principal order ideal generated by x.

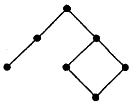
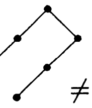
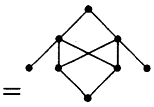
We define one more notation. Let P and Q be finite posets and x be an element of P, and let $y_1, y_2, \ldots, y_r$ be all elements of P which covers x. Then we denote a poset $(P - \langle x \rangle)^{y_1, y_2, \ldots, y_r} \backslash Q$ by $P(x, Q)$. For example, if P_1, Q_1, P_2 and Q_2 are posets defined by

$P_1 =$ 

, $Q_1 =$  , $P_2 =$ 

, $Q_2 =$ 

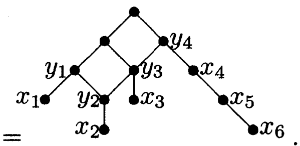
then, we have

$P_1(x_1, Q_1) =$  , $P_1(x_1, \langle x_1 \rangle) =$  $\neq P$, $P_2(x_2, Q_2) =$ 

and $P_2(x_2, \langle x_2 \rangle) = P_2$.

Definition 2.3 (Joint elements and Joint pairs). Let P be a poset and x be an element of P. If x is not a maximal element of P, $\langle x \rangle$ is a chain and P is equal to $P(x, \langle x \rangle)$, i.e. removing $\langle x \rangle$ from P and making the joint sum of $P - \langle x \rangle$ with $\langle x \rangle$ at $y_1, y_2, \ldots, y_r$, which are all elements covers x, recovers P, then we say that x is a joint element of P. Also if x is a joint element and x is covered by only one element y, then we call a pair (x, y) a joint pair of P.

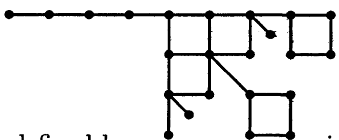
For example, in the following diagram, $x_1, x_2, y_2, x_3, x_4, x_5, x_6$ are joint elements and (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, x_4), (x_6, x_5) are all the joint pairs of P.

$P =$ 

.

Now we are in position to define the notion of general leaf posets as follows.

Definition 2.4. First we inductively define k-level leaf posets for a positive integer k. A poset P is said to be a 1-level leaf poset if it is a basic leaf poset, a tree, or obtained as a direct sum of several basic leaf posets and trees. Let LP_1 denote the set of 1-level leaf posets. For $k \geq 2$, let Q be a $(k-1)$-level leaf poset and (x, y) is a joint pair of Q and let P_1 be a 1-level leaf poset such that $|P_1| = |\langle x \rangle|$. If a poset P satisfies $P = Q(x, P_1)$, then we say that P is a k-level leaf poset. Let LP_k denote the set of k-level leaf posets, and put $\mathrm{LP} := \bigcup_{k \geq 1} \mathrm{LP}_k$. We call an element of LP a leaf poset.

Note that a poset can be k-level leaf poset for several k, i.e. there exist some $i \neq j$ such that $\mathrm{LP}_i \cap \mathrm{LP}_j \neq \emptyset$. For example, the poset defined by  is not a 1-level leaf poset, but it belongs to both LP_4 and LP_5.

By the definition of irreducible d-complete posets and basic leaf posets, we can realize each irreducible d-complete poset as a basic leaf poset. The important fact is that the notion of "slant sum" defined in [5] is included in that of "joint sum". Hence, by the definition of general leaf posets and d-complete posets, we can say that any d-complete poset is a leaf poset. Our previous result is the following (cf. [2]), which is obtained as a corollary of our main result in this article.

Theorem 2.5. Any leaf poset is a hook length poset.

As a corollary of this theorem, we can also obtain the following.

Corollary 2.6 (Peterson and Proctor [8]). Any d-complete poset is a hook length poset.

3 Multivariate hook length property

In this section, as a multivariate analogue of hook length property, we define multivariate hook length property and multivariable hook length posets. Also, we show that any basic leaf poset has multivariate hook length property.

Let P be a finite poset, $\mathscr{Q}$ be a set of variables and let $\tilde{\mathscr{Q}}$ be the set of all monomials of elements of $\mathscr{Q}$. In this article, we set $\mathscr{Q} = \{p_i, q_i, r_i; i \in \mathbb{Z}\}$. Moreover, let w be a map from P to $\mathscr{Q}$ which is called a weight of P. For a P-partition $\varphi \in \mathscr{A}(P)$, we write

$$w^{\varphi} := \prod_{x \in P} w(x)^{\varphi(x)}, \quad F(P; w) := \sum_{\varphi \in \mathscr{A}(P)} w^{\varphi},$$

which called the multivariable generating function of P-partitions. For example, let P be a chain of length 2 with the maximum element x_0 and the minimum element x_1, and let w be a weight of P defined by $w(x_1) = q_1$ and $w(x_0) = q_0$. For a P-partition φ defined by $\varphi(x_1) = b$ and $\varphi(x_0) = a$, we obtain that $w^{\varphi} = q_0^a q_1^b$. Hence, we have

$$F(P; w) = \sum_{\varphi \in \mathscr{A}(P)} w^{\varphi} = \sum_{0 \leq a \leq b} q_0^a q_1^b = \frac{1}{(1 - q_1)(1 - q_0 q_1)}.$$

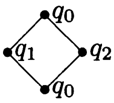
Definition 3.1 (multivariate hook length property). Let P be a finite poset and w be a weight of P, i.e. w is a map from P to $\mathscr{Q}$. We say that P (or (P, w)) has multivariate hook length property if there exists a map H from P to $\tilde{\mathscr{Q}}$ satisfying

$$F(P; w) = \prod_{x \in P} \frac{1}{1 - H(x)}, \tag{2}$$

where $|H(P)| \geq 2$ if $|P| \geq 2$. If (P, w) has multivariate hook length property, then H is called a hook function of (P, w). A multivariable hook length poset is a poset which has multivariate hook length property. For a multivariable hook length poset (P, w), let $H_{P,w}$ be a hook function of (P, w), i.e. $H_{P,w}$ is a map from P to $\tilde{\mathscr{Q}}$ satisfying the equality (2).

By the definition of multivariable hook length posets, it is easy to see that a multivariable hook length poset is a hook length poset.

Let's see some examples. By our previous example, a chain of length 2 is a multivariable hook length poset. In general, for a tree T, if a weight w of T satisfies that $|w(T)| \geq 2$, then (T, w) is a multivariable hook length poset with the hook function H defined to be $H(x) = \prod_{y \leq x} w(y)$. Let P be a d_3-interval and we define a weight w of P by

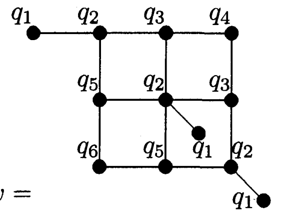
$w =$ 

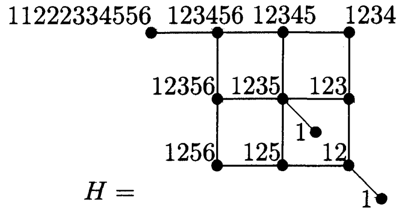
,

where the labeling means the image of the corresponding element by w. Then, we can calculate that

$$F(P; w) = \frac{1}{(1 - q_0 q_1 q_2)(1 - q_0 q_1)(1 - q_0 q_2)(1 - q_0)},$$

hence (P, w) is a multivariable hook length poset. Let P be a ginkgo $G((2,1),(2,1),1)$ and we define a weight w of P and a map H from P to $\tilde{\mathscr{Q}}$ as follows:

$w =$ 

, $H =$ 

,

where in the definition of H we put $k_1k_2\ldots k_m := q_{k_1}q_{k_2}\ldots q_{k_m}$. Then, H is a hook function of (P, w) and hence (P, w) is a multivariable hook length poset.

In general, we can obtain the following.

Proposition 3.2. Any basic leaf poset is a multivariable hook length poset.

Here, we will describe the sketch of the proof of this proposition in the case that P is a ginkgo. Before the calculation of the multivariable generating function of P-partitions, we prepare two notations. For a sequence of variables $\mathbf{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)$ and integers a and b, we write

$$\mathbf{u}^{(a,b)} := \prod_{i=a}^{b} u_i, \quad (a,b)_{\mathbf{u}}! := \prod_{i=a}^{b}(1 - u^{(i,b)}).$$

Also, we write $\mathbf{p} := (\ldots, p_{-1}, p_0, p_1, \ldots)$, $\mathbf{q} := (\ldots, q_{-1}, q_0, q_1, \ldots)$ and $\mathbf{r} := (\ldots, r_{-1}, r_0, r_1, \ldots)$.

Sketch of the proof: Let $m \geq 2$ be an integer, γ be a non-negative integer, and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ be strictly decreasing sequences of non-negative integers of length m. Let P be a ginkgo $G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ and let w be a weight of P defined by Diagram 4. Then,

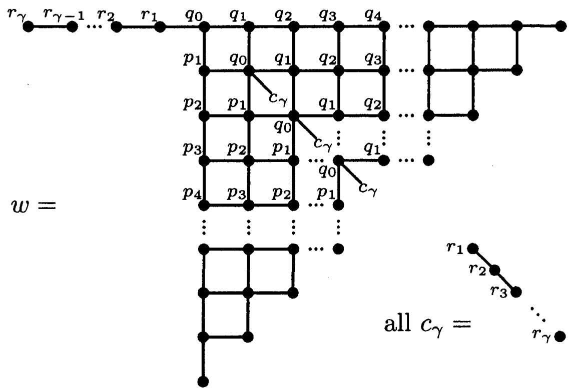
$w =$ 

Diagram 4: Weight of $G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$

by using lattice path method, the multivariable generating function of P-partitions is calculated as follows:

$$F(P;w) = \frac{\prod_{1\leq i<j\leq m}(1-\mathbf{q}^{(\alpha_j+1,\alpha_i)})(1-\mathbf{p}^{(\beta_j+1,\beta_i)})}{((1,\gamma)_{\mathbf{r}}!)^{m-1}\prod_{i=1}^{\gamma}(1-\mathbf{r}^{(1,i)}\mathbf{V})\prod_{i=1}^{m}(1,\alpha_i)_{\mathbf{q}}!(1,\beta_i)_{\mathbf{p}}!}$$
$$\times \sum_{\lambda=(x_m,x_{m-1},\ldots,x_1)\in\mathscr{P}} (\mathbf{r}^{(1,\gamma)})^{-x_1} s_\lambda(\mathbf{q}^{(0,\alpha_1)}\mathbf{r}^{(1,\gamma)}, \ldots, \mathbf{q}^{(0,\alpha_m)}\mathbf{r}^{(1,\gamma)}) s_\lambda(\mathbf{p}^{(1,\beta_1)}, \ldots, \mathbf{p}^{(1,\beta_m)}),$$

where $\mathscr{P}$ is the set of the partitions and $s_\lambda(y_1, y_2, \ldots, y_m)$ is the Schur function, and we put $\mathbf{V} = (\mathbf{r}^{(1,\gamma)})^{m-1}\prod_{i=1}^{m}\mathbf{q}^{(0,\alpha_i)}\mathbf{p}^{(1,\beta_i)}$. By an extension of Cauchy identity:

$$\sum_{\lambda=(\lambda_1,\ldots,\lambda_m)\in\mathscr{P}} w^{\lambda_m} s_\lambda(x_1, x_2, \ldots, x_m) s_\lambda(y_1, y_2, \ldots, y_m) = \frac{1-\prod_{i=1}^{m}x_iy_i}{(1-w\prod_{i=1}^{m}x_iy_i)\prod_{i,j=1}^{m}(1-x_iy_j)},$$

we can obtain that

$$F(P;w)=\frac{\prod_{1\le i<j\le m}(1-\mathbf{q}^{(\alpha_j+1,\alpha_i)})(1-\mathbf{p}^{(\beta_j+1,\beta_i)})}{(1,\gamma)_{\mathbf{r}}!^{m-1}\prod_{i=1}^{m}(1,\alpha_i)_{\mathbf{q}}!(1,\beta_i)_{\mathbf{p}}!\prod_{i=0}^{\gamma-1}(1-\mathbf{r}^{(1,i)}\mathbf{V})\prod_{i,j=1}^{m}(1-\mathbf{q}^{(0,\alpha_j)}\mathbf{p}^{(1,\beta_i)}\mathbf{r}^{(1,\gamma)})}.$$

Hence, ginkgo is a multivariable hook length poset. By almost same method, we can show that other basic leaf posets are also multivariable hook length posets. □

As a corollary of Proposition 3.2, we can easily see the following.

Lemma 3.3. Let P be a basic leaf poset or a tree with $|P|\geq 2$. There exists a weight w of P satisfying the following two conditions. (i) If x is a joint element of P, then $w(x)=w(y)$ for any element y of $\langle x\rangle$. (ii) (P,w) is a multivariable hook length poset.

4 How to make a new multivariable hook length poset

In this section we will explain four kinds of methods in order to make a new multivariable hook length poset (or a hook length poset) from known multivariable hook length posets (or hook length posets) and we define extended leaf posets and multivariable extended leaf posets by using our methods.

Let f be a map from a set A to a set C and g be a map from a set B to C. If A is non-adjacent to B, then, we define a map $f+g$ from $A\cup B$ to C by $(f+g)(x):=f(x)$ if $x\in A$ and $(f+g)(x)=g(x)$ if $x\in B$.

We can obtain three kinds of methods as follows:

Proposition 4.1. Let P and Q be finite posets which are mutually non-adjacent, and let w_P (resp. w_Q) be a weight of P (resp. Q).

(i) We have

$$F(P+Q,w_P+w_Q)=F(P;w_P)F(Q;w_Q).$$

In particular, (P,w_P) and (Q,w_Q) are multivariable hook length posets, then $(P+Q,w_P+w_Q)$ is a multivariable hook length poset.

(ii) Let Q be a chain. Then, we have

$$F(P\oplus Q;w_P+w_Q)=\frac{F(P;w_P)}{\prod_{y\in Q}(1-\prod_{z\in P}w_P(z)\prod_{z\in Q,z\leq y}w_Q(z))}.$$

In particular, (P,w_P) is a multivariable hook length poset if and only if $(P\oplus Q,w_P+w_Q)$ is a multivariable hook length poset and the monomial $\prod_{z\in P}w_P(z)\prod_{z\in Q,z\leq y}w_Q(z)$ is an element of $H_{P\oplus Q,w_P+w_Q}(P\oplus Q)$ for any element y of Q.

(iii) Let $\tilde{Q}$ and R be posets satisfying "$Q=\tilde{Q}$ and $R=\emptyset$" or "$Q=\tilde{Q}\oplus R$ and R is a chain". Let x be a joint element of P satisfying $|\langle x\rangle|=|\tilde{Q}|$ and $\prod_{z\in\langle x\rangle}w_P(z)=\prod_{z\in\tilde{Q}}w_Q(z)$. We denote the weight $w_P|_{P-\langle x\rangle}+w_Q|_{\tilde{Q}}$ by $w_{P(x,\tilde{Q})}$. Then, we have

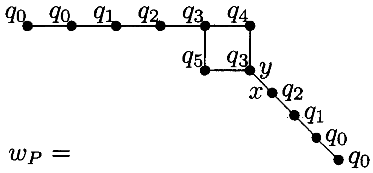
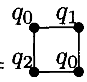
$$F(P(x,\tilde{Q});w_{P(x,\tilde{Q})})=F(P;w_P)F(Q;w_Q)\prod_{y\in\langle x\rangle}(1-\prod_{z\leq y}w_P(z))\prod_{y\in R}(1-\prod_{z\in\tilde{Q}}w_Q(z)\prod_{z\in R,z\leq y}w_Q(z)).$$

In particular, if (P,w_P) and (Q,w_Q) are multivariable hook length posets and all $\prod_{z\leq y}w_P(z)$ ($y\in\langle x\rangle$) and all $\prod_{z\in\tilde{Q}}w_Q(z)\prod_{z\leq R,z\leq y}w_Q(z)$ ($y\in R$) are elements of $H_{P,w_P}(P)\cup H_{Q,w_Q}(Q)$, then $(P(x,\tilde{Q}),w_{P(x,\tilde{Q})})$ is a multivariable hook length poset. Also, if $(P(x,\tilde{Q}),w_{P(x,\tilde{Q})})$ and (Q,w_Q) are multivariable hook length posets and $\prod_{y\in Q}(1-H_{(Q,w_Q)}(y))$ divides

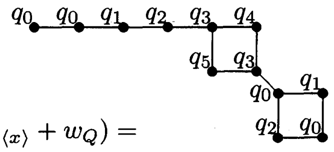
$$\prod_{y\in P(x,\tilde{Q})}(1-H_{(P(x,\tilde{Q}),w_{P(x,\tilde{Q})})}(y))\prod_{y\in\langle x\rangle}(1-\prod_{z\leq y}w_P(z))\prod_{y\in R}(1-\prod_{z\in\tilde{Q}}w_Q(z)\prod_{z\in R,z\leq y}w_Q(z)),$$

then (P, w_P) is a multivariable hook length poset.

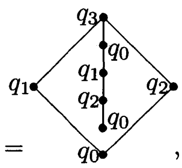
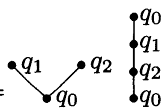
For example, let P (resp. Q) be a d_7-interval (resp. d_3-interval). And let w_P (resp. w_Q) be a weight of P (resp. Q) and x, y be elements of P defined as follows:

$w_P =$  , $w_Q =$  .

Then, we can see that (x, y) is a joint pair of P and $(P, w_P), (Q, w_Q)$ are multivariable hook length posets satisfying $|\langle x\rangle| = 4 = |Q|$, $\prod_{z\in\langle x\rangle} w_P(z) = q_0^2 q_1 q_2 = \prod_{z\in Q} w_Q(z)$ and $\{\prod_{u\leq z} w_P(u); z \in \langle x\rangle\} = \{q_0, q_0^2, q_0^2 q_1, q_0^2 q_1 q_2\} \subseteq H_{P,w_P}(P)$. Then, we have

$(P(x, Q), w_P|_{P-\langle x\rangle} + w_Q) =$ 

and, by Proposition 4.1 (iii), this poset is a multivariable hook length poset. Also, we define posets P_1, P_2 and their weights w_{P_1}, w_{P_2} as follows:

$(P_1, w_{P_1}) =$  , $(P_2, w_{P_2}) =$ 

Then, by applying Proposition 4.1 (iii) to the posets $(c_3+c_4)\oplus c_1$ and d_3-interval (resp. Proposition 4.1 (ii) to the above poset (P_1, w_{P_1})), then we can see that (P_1, w_{P_1}) (resp. (P_2, w_{P_2})) is a multivariable hook length poset. Note that P_2 is a poset described in Proctor's web page [10] as a Stanley's example that $P + Q$ is a hook length poset but P or Q is not a hook length poset.

Note that for two multivariable hook length posets (P, w_P) and (Q, w_Q) such that P is non-adjacent to Q, we can define a weight w of $P + Q$ satisfying that $w(x) \neq w(y)$ for any pair of elements (x, y) in $P \times Q$ and $(P+Q, w)$ is a multivariable hook length poset. Therefore, by Proposition 3.2, Lemma 3.3, Proposition 4.1 and the definition of general leaf posets, we can conclude the following.

Theorem 4.2. Any leaf poset is a multivariable hook length poset.

As a corollary of Proposition 4.1, we can obtain the following.

Corollary 4.3. Let P and Q be finite posets which are mutually non-adjacent.

(i) We have
$$F(P + Q; q) = F(P; q)F(Q; q).$$
In particular, P and Q are hook length posets, then $P + Q$ is a hook length poset.

(ii) Let Q be a chain. Then, we have
$$F(P \oplus Q; q) = \frac{F(P; q)}{\prod_{i=1}^{|Q|}(1 - q^{|P|+i})}.$$
In particular, P is a hook length poset if and only if $P \oplus Q$ is a hook length poset satisfying $\{|P| + i; 1 \leq i \leq |Q|\} \subseteq h_{P\oplus Q}(P \oplus Q)$.

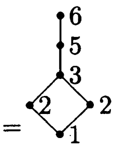
(iii) Let $\tilde{Q}$ and R be posets satisfying "$Q = \tilde{Q}$ and $R = \emptyset$" or "$Q = \tilde{Q} \oplus R$ and R is a chain". Let x be a joint element of P satisfying $|\langle x \rangle| = |\tilde{Q}|$. Then, we have

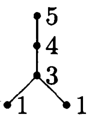
$$F(P(x,\tilde{Q});q) = F(P;q)F(Q;q)\prod_{i=1}^{|Q|}(1-q^i).$$

In particular, if P and Q are hook length posets satisfying $\{1,2,\ldots,|Q|\} \subseteq h_P(P) \cup h_Q(Q)$, then $P(x,\tilde{Q})$ is a hook length poset. Also, if $P(x,\tilde{Q})$ and Q are hook length posets and $\prod_{y\in Q}(1-q^{h_Q(y)})$ divides $\prod_{y\in P(x,\tilde{Q})}(1-q^{h_{P(x,\tilde{Q})}(y)})\prod_{i=1}^{|Q|}(1-q^i)$, then P is a hook length poset.

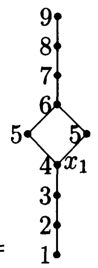
Note that Corollary 4.3 (i) is well known (cf. [12]).

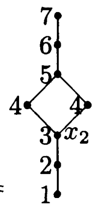
For example, if P is a d_3-interval, Q is a poset removing the minimum element from P and c_2 is a chain of length 2, then by Corollary 4.3 (ii), $P \oplus c_2$ and $Q \oplus c_2$ are also hook length posets, i.e.

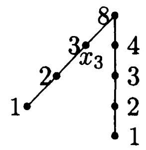
$P \oplus c_2 =$ 

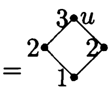
, $Q \oplus c_2 =$ 

are hook length posets, where the labeling is the hook lengthes of corresponding element. Let P_1, P_2, P_3 and Q be posets defined by

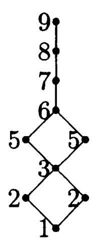
$P_1 =$ 

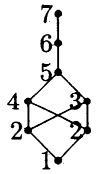
, $P_2 =$ 

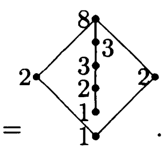
, $P_3 =$ 

, $Q =$ 

and let $\tilde{Q} = Q - \{u\}$. We can easily see that integers $1,2,3,4$, where 4 is the cardinality of a poset Q, appear as hook lengthes of P_1, P_2 and P_3. Hence, by Corollary 4.3 (iii), three posets $P_1(x_1,Q)$, $P_2(x_2,\tilde{Q})$ and $P_3(x_3,\tilde{Q})$ are hook length posets and their hook lengthes are the following.

$P_1(x_1,Q) =$ 

, $P_2(x_2,\tilde{Q}) =$ 

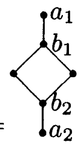
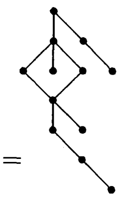
, $P_3(x_3,\tilde{Q}) =$ 

.

Note that the cardinality of the poset $P_3(x_3,\tilde{Q}) =: R$ is 8 and 8 appears as a hook length of R. Hence, by Corollary 4.3 (ii), we can see that $R - \{x_0\}$ is also a hook length poset, where x_0 is the maximum element of R. The poset $R - \{x_0\}$ was already described in this article as Stanley's example.

In order to explain our fourth method, we define one more notation. Let P be a poset, $A = \{a_1, a_2, \ldots, a_r\}$ be a subset of P with r elements and $m, m_1, m_2, \ldots, m_k$ be positive integers, and let $A_1, A_2, \ldots, A_k$ be subsets of P. We write

$$P\binom{A}{m} := (\ldots(P^{a_1}\backslash c_m)^{a_2}\backslash c_m)\ldots)^{a_r}\backslash c_m,\ P\binom{A_1, A_2, \ldots, A_k}{m_1, m_2, \ldots, m_k} := P\binom{A_1, \ldots, A_{k-1}}{m_1, \ldots, m_{k-1}}\binom{A_k}{m_k}.$$

For example, let P be a d_4-interval and let a_1 (resp. a_2) be the maximum element (resp. the minimum element) of P, and let b_1 (resp. b_2) be the maximum element (resp. the minimum element) of $P - \{a_1, a_2\}$ as follows: $P =$ . Then, we have $P\begin{pmatrix}\{a_1,a_2\}, \{b_1,b_2\} \\ 2 \quad , \quad 1\end{pmatrix} =$ .

Then, we can obtain the following fourth method.

Lemma 4.4. Let (P, w_P) be a multivariable hook length poset, $p_1, p_2, \ldots, p_r$ be r elements of $w_P(P)$, and let $m_1, m_2, \ldots, m_r$ be positive integers. We write

$$\tilde{P} := P\begin{pmatrix} w_P^{-1}(\{p_1\}), w_P^{-1}(\{p_2\}), \ldots, w_P^{-1}(\{p_r\}) \\ m_1, m_2, \ldots, m_r \end{pmatrix}.$$

For $1 \le i \le r$, we denote the cardinality of $w_P^{-1}(\{p_i\})$ by n_i.

(i) For $1 \le i \le r$, let $z_1^{(i)}, z_2^{(i)}, \ldots, z_{n_i}^{(i)}$ be all elements of $w_P^{-1}(\{p_i\})$. For $1 \le i \le r$ and $1 \le j \le n_i$, let $u_j^{(i)}$ be an element of $\tilde{P} - P$ satisfying $u_j^{(i)} \lessdot z_j^{(i)}$ in $\tilde{P}$, and let $w_j^{(i)}$ be a weight of $\langle u_j^{(i)} \rangle$. If

$$\prod_{y \in \langle u_j^{(i)} \rangle} w_j^{(i)}(y) = \prod_{y \in \langle u_k^{(i)} \rangle} w_k^{(i)}(y) (=: V_i) \quad (1 \le i \le r, 1 \le j, k \le n_i),$$

then for a weight w of P defined by

$$w(x) = \begin{cases} p_i V_i & \text{if } x \in w_P^{-1}(\{p_i\}) \ \ (i = 1, 2, \ldots, r), \\ w_P(x) & \text{otherwise,} \end{cases}$$

we have

$$F(\tilde{P}; w_P + \sum_{i=1}^{r} \sum_{j=1}^{n_i} w_j^{(i)}) = F(P; w) \prod_{i=1}^{r} \prod_{j=1}^{n_i} F(\langle u_j^{(i)} \rangle; w_j^{(i)})$$

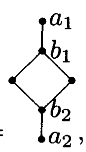
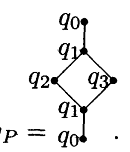
and $(\tilde{P}, w_P + \sum_{i=1}^{r} \sum_{j=1}^{n_i} w_j^{(i)})$ is a multivariable hook length poset.

(ii) Let w be a weight of P defined to be $w(x) := q^{m_i+1}$ if $x \in w_P^{-1}(\{p_i\})$ for some $i \in \{1, 2, \ldots, r\}$ and $w(x) := q$ otherwise. Then, we have

$$F(\tilde{P}; q) = F(P; w) \prod_{i=1}^{r} F(c_{m_i}; q)^{n_i}$$

and $\tilde{P}$ is a hook length poset.

For example, let P, a_1, a_2, b_1, b_2 be the same ones in our previous example, and let w_P be a weight of P defined by the following.

$P =$ , $w_P =$ .

We write a poset $P\binom{\{a_1,a_2\}, \{b_1,b_2\}}{2 \quad , \quad 1}$ by R and we define a weight w_R of R by Diagram 5. Then, we can easily see that (P, w_P) is a multivariable hook length poset, $\{a_1, a_2\} = w_P^{-1}(\{q_0\})$ and $\{b_1, b_2\} = w_P^{-1}(\{q_1\})$. Hence, by Lemma 4.4, (R, w_R) is a multivariable hook length poset and R is a hook length poset with hook length function h_R defined by Diagram 5.

As a corollary of Proposition 4.1 (iii) and Lemma 4.4 (i), we can also obtain the following.

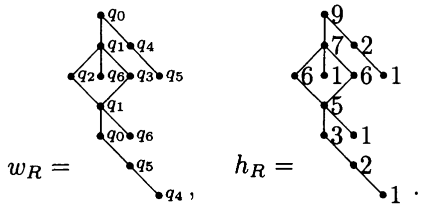
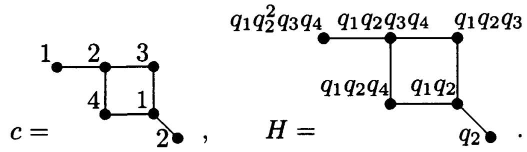


Diagram 5: w_R and h_R

Corollary 4.5. Let $(P, w_P), (Q, w_Q)$ be multivariable hook length posets and x be an element of P satisfying that $|w_P^{-1}(\{w_P(x)\})| = 1$. We define a weight w of P by $w(x) := w_P(x) \prod_{z \in Q} w_Q(z)$ and $w(y) := w_P(y)$ if $y \in P - \{x\}$. Then, we have $F(P^y \backslash Q; w_P + w_Q) = F(P; w)F(Q; w_Q)$ and $(P^x \backslash Q, w_P + w_Q)$ is a multivariable hook length poset.

By using Proposition 3.2, we can see that Theorem 1.1 is correct for irreducible d-complete posets. Therefore, by Corollary 4.5, we can obtain that Theorem 1.1 is true for all d-complete posets, and it follows that any d-complete poset P is also a multivariable hook length poset with the weight w_c defined by $w_c(x) = q_{c(x)}$, where c is a d-complete coloring of P.

Remark 4.6. Let P be a d-complete poset with top tree T, c be a map from P to $\{1, 2, \ldots, |T|\}$, and let w_c be a weight of P defined by $w_c(x) = q_{c(x)}$. Then, it is not true that (P, w_c) is a multivariable hook length poset if and only if c is a d-complete coloring. For example, let P be a d_4-interval, c be a map from P to $\{1, 2, 3, 4\}$ and H be a map from P to $\tilde{\mathscr{Q}}$ defined by

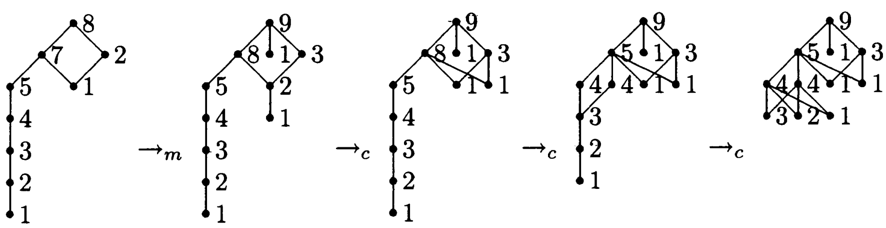


Then, c is not d-complete coloring, but we can see that H is a hook function of (P, w_c) and hence (P, w_c) is a multivariable hook length poset.

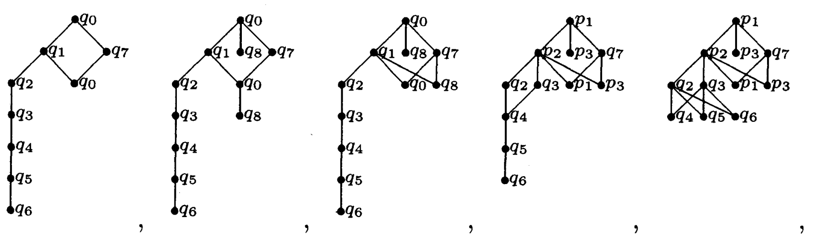
Now, we are in position to define extended leaf posets and multivariable extended leaf posets.

Definition 4.7 ((Multivariable) extended leaf posets). If P is a leaf poset, we also call P a 0-level extended leaf poset (resp. a 0-level multivariable extended leaf poset). For a positive integer k, we call P a k-level extended leaf poset (resp. a k-level multivariable extended leaf poset) if P is a $(k-1)$-level extended leaf poset (resp. a $(k-1)$-level multivariable extended leaf poset) or P is a hook length poset (resp. a multivariable hook length poset) made from $(k-1)$-level extended leaf posets (resp. $(k-1)$-level multivariable extended leaf posets) by using Corollary 4.3 and Lemma 4.4 (ii) (resp. Proposition 4.1 and Lemma 4.4 (i)). We call P an extended leaf poset (resp. a multivariable extended leaf poset) if P is a k-level extended leaf poset (resp. a k-level multivariable extended leaf poset) for some non-negative integer k. We denote the set of the extended leaf posets (resp. the set of the multivariable extended leaf posets) by ELP (resp. MELP).

For example, the following posets are all extended leaf posets.



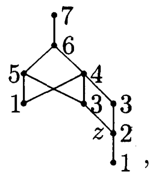
Here, the labeling of the above diagram is the hook length of the corresponding element, and $P \rightarrow_m Q$ (resp. $P \rightarrow_c Q$) means that Q is obtained from P by using Lemma 4.4 (ii) (resp. Corollary 4.3). Note that we can check that all the above posets are also multivariable extended leaf posets with weights as follows:

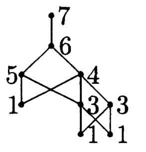


where $\{p_1, p_2, p_3\} = \{q_4, q_5, q_6\}$.

Remark 4.8. It is easy to see that any multivariable extended leaf poset is an extended leaf poset, but we cannot find an extended leaf poset which is not a multivariable extended leaf poset yet.

In Proctor's web page [1], there exists a list of hook length posets with k elements, where k is from 1 to 9. We checked his list in order to know what hook length poset is not an extended leaf poset. In order to describe the result of our investigation, we denote the set of the connected hook length posets with k elements by CHLP_k and the set of all d-complete posets by DCP. Then, by using his list, we can see that $\cup_{i=1}^{5}\mathrm{CHLP}_i \subseteq \mathrm{DCP}$, $|\mathrm{CHLP}_6 - \mathrm{DCP}| = 3$, $\mathrm{CHLP}_6 \subseteq \mathrm{ELP}$, $|\mathrm{CHLP}_7 - \mathrm{DCP}| = 6$, $\mathrm{CHLP}_7 \subseteq \mathrm{ELP}$, $|\mathrm{CHLP}_8 - \mathrm{DCP}| = 51$, $\mathrm{CHLP}_8 \subseteq \mathrm{ELP}$, $|\mathrm{CHLP}_9 - \mathrm{DCP}| = 133$ and $\mathrm{CHLP}_9 - \mathrm{ELP} = \{R_1, R_2\}$, where R_1 and R_2 are the following posets.

$R_1 =$ 

, $R_2 =$ 

,

where the labelings are also hook lengthes. Note that we can obtain the hook length poset R_2 from R_1 by using Corollary 4.3, in particular, $R_2 = R_1(z, c_1 + c_1)$. Hence, we have one question. What is a poset R_1? At the last part of this article, we give an answer of this question.

5 A new class of multivariable hook length posets

In this section, we introduce a new class of multivariable hook length posets includes a poset R_1 which is a hook length poset but not extended leaf poset found in the previous section. From now on, let $m \geq 2$ be an integer, and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ be strictly decreasing sequences of non-negative integers of length m. We can obtain the following.

Theorem 5.1. Let $\tilde{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be a poset and let w be a weight of $\tilde{G}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ defined by Diagram 6. Then, we have

$$F(\tilde{G}(\boldsymbol{\alpha},\boldsymbol{\beta}); w) = \frac{1}{(1-r_1)^m(1-\mathbf{V})\prod_{i=0}^{\beta_1}(1-r_1\mathbf{p}^{(1,i)}\mathbf{V})\prod_{i=1}^{m}(1,\alpha_i)_{\mathbf{q}}!(1,\beta_i)_{\mathbf{p}}!} \times \frac{\prod_{1\leq i<j\leq m}(1-\mathbf{q}^{(\alpha_j+1,\alpha_i)})(1-\mathbf{p}^{(\beta_j+1,\beta_i)})\prod_{k=1}^{m}(1-r_1\mathbf{p}^{(1,\beta_k)}\mathbf{V})}{\prod_{i,j=1}^{m}(1-r_1\mathbf{q}^{(0,\alpha_i)}\mathbf{p}^{(1,\beta_j)})\prod_{k=1}^{m}(1-(\mathbf{q}^{(0,\alpha_k)})^{-1}\mathbf{V})},$$

where $\mathbf{V} = r_1^m r_2 \prod_{i=1}^{m}\mathbf{q}^{(0,\alpha_i)}\mathbf{p}^{(1,\beta_i)}$, and $(\tilde{G}(\boldsymbol{\alpha},\boldsymbol{\beta}), w)$ is a multivariable hook length poset.

Key equality of the proof of Theorem 5.1 is the following infinite sum of Schur functions.

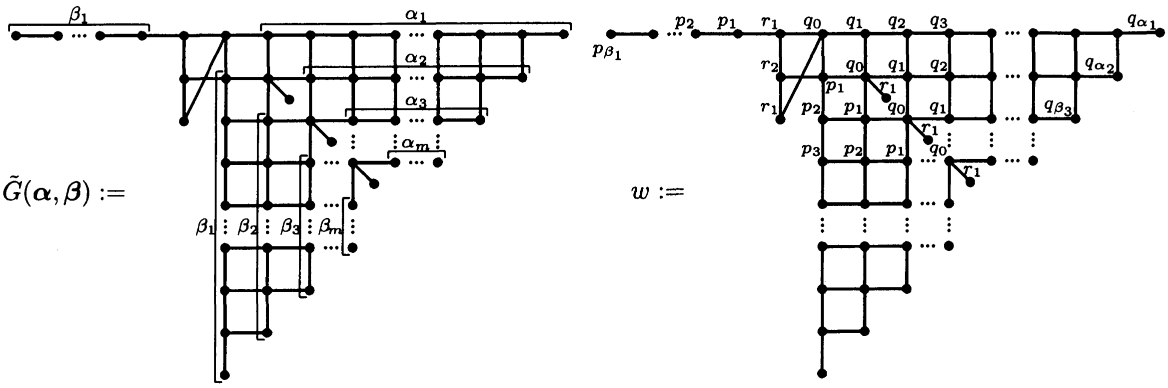


Diagram 6: $\tilde{G}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and w

Lemma 5.2. Let m be a positive integer and u, w be variables. Then, we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2m+1})\in\mathscr{P}} u^{\lambda_{2m}} w^{\lambda_{2m+1}} s_{(\lambda_1,\lambda_3,\dots,\lambda_{2m-1})}(X_m) s_{(\lambda_2,\lambda_4,\dots,\lambda_{2m})}(Y_m) s_{(\lambda_{2m-1},\lambda_{2m+1})}(1,z)$$

$$= \frac{1}{\prod_{i=0}^{1}(1-uw^i z\prod_{k=1}^m x_k y_k)\prod_{i=1}^m(1-x_i)\prod_{i,j=1}^m(1-x_i y_j)}$$

$$\times\left(\frac{\prod_{i=1}^m(1-x_i z\prod_{k=1}^m x_k y_k)}{\prod_{i=1}^m(1-y_i^{-1} z\prod_{k=1}^m x_k y_k)} + \frac{(u-1)\prod_{k=1}^m x_k y_k}{1-u\prod_{k=1}^m x_k y_k}\right),$$

where $X_m = (x_1, x_2, \dots, x_m)$ and $Y_m = (y_1, y_2, \dots, y_m)$ are sequences of variables.

References

[1] C. A. Gann and R. A. Proctor, "Chapel Hill Poset Atlas", http://www.unc.edu/~rap/Posets/.

[2] M. Ishikawa and H. Tagawa, "Schur Function Identities and Hook Length Posets", 19th International Conference on Formal Power Series and Algebraic Combinatorics, July 2-6, 2007, Nankai University, Tianjin, China, http://www.fpsac.cn/PDF-Proceedings/Posters/55.pdf.

[3] M. Ishikawa and H. Tagawa, "Schur Function Identities and Multivariate Hook Length Property", in preparation.

[4] K. Nakada, "q-hook formula of Gansner type for a generalized Young diagram", 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), 685-696, *Discrete Math. Theor. Comput. Sci. Proc.*, AK.

[5] R. A. Proctor, "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d -complete posets", *J. Algebraic Combin.* **9** (1999), 61 – 94.

[6] R. A. Proctor, "Minuscule elements of Weyl groups, the numbers game, and d-complete posets", *J. Algebra* **213** (1999), 272–303.

[7] R. A. Proctor, "d-Complete Posets Generalize Young Diagrams for the Jeu de Taquin Property", arXiv:0905.3716.

[8] R. A. Proctor, "d-Complete posets generalize Young diagrams for the hook product formula", in preparation.

[9] R. A. Proctor, "Informal Description of the Hook Length Property for Posets", http://math.unc.edu/Faculty/rap/Hook.html.

[10] R. A. Proctor, "Do d-Complete Posets Account for All Jeu de Taquin and Hook Length Posets?", http://math.unc.edu/Faculty/rap/AllAcctd.html.

[11] B. Sagan, "Enumeration of partitions with hooklengths", *European J. Combin.* **3** (1982), 85-94.

[12] R. P. Stanley, "Ordered structure and partitions", *Mem. Amer. Math. Soc.* **119** (1972), 1-104.