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ON OPERATOR THEORETICAL DESCRIPTION OF RELLICH IDENTITY FOR DIVERGENCE FORM ELLIPTIC OPERATORS AND ITS APPLICATIONS

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1. INTRODUCTION

This article is a resume of the recent work [31, 32, 33] by the author and Hideyuki Miura (Tokyo Institute of Technology). We consider the second order elliptic operator of divergence form in $\mathbb{R}^{d+1}=\{(x,t)\in\mathbb{R}^{d}\times\mathbb{R}\}$,

$$\mathcal{A}=-\nabla\cdot A\nabla,\quad A=A(x)=\big(a_{i,j}(x)\big)_{1\leq i,j\leq d+1}\,. \tag{1.1}$$

Here $d\in\mathbb{N}$, $\nabla=(\nabla_{x},\partial_{t})^{\top}$ with $\nabla_{x}=(\partial_{1},\cdots,\partial_{d})^{\top}$, and each $a_{i,j}$ is complex-valued and assumed to be independent of the t variable. The adjoint matrix of A will be denoted by A^{*}. We assume the standard ellipticity condition

$$\mathrm{Re}\langle A(x)\eta,\eta\rangle\geq\nu_{1}|\eta|^{2},\quad |\langle A(x)\eta,\zeta\rangle|\leq\nu_{2}|\eta||\zeta| \tag{1.2}$$

for all $\eta,\zeta\in\mathbb{C}^{d+1}$ with positive constants ν_{1},ν_{2}. Here $\langle\cdot,\cdot\rangle$ denote the inner product of $\mathbb{C}^{d+1}$, i.e., $\langle\eta,\zeta\rangle=\sum_{j=1}^{d+1}\eta_{j}\bar{\zeta}_{j}$ for $\eta,\zeta\in\mathbb{C}^{d+1}$. For later use we set

$$A'=(a_{i,j})_{1\leq i,j\leq d},\quad b=a_{d+1,d+1},$$
$$\mathbf{r_1}=(a_{1,d+1},\cdots,a_{d,d+1})^{\top},\quad \mathbf{r_2}=(a_{d+1,1},\cdots,a_{d+1,d})^{\top}.$$

We will also use the notation $\mathcal{A}'=-\nabla_{x}\cdot A'\nabla_{x}$, and we call $\mathbf{r_1}$ and $\mathbf{r_2}$ the off-block vectors of A. The domain of a linear operator T in a Banach space H will be denoted by $D_{H}(T)$. Under the condition (1.2) the standard theory of sesquilinear forms gives a realization of $\mathcal{A}$ in $L^{2}(\mathbb{R}^{d+1})$, denoted again by $\mathcal{A}$. The simplest example of $\mathcal{A}$ is the $(d+1)$-dimensional Laplacian $-\Delta=-\Delta_{x}-\partial_{t}^{2}=-\sum_{j=1}^{d}\partial_{j}^{2}-\partial_{t}^{2}$. In this case we have a factorization

$$-\Delta=-(\partial_{t}-(-\Delta_{x})^{\frac{1}{2}})(\partial_{t}+(-\Delta_{x})^{\frac{1}{2}}). \tag{1.3}$$

Clearly the factorization (1.3) is valid including the relation of domains, for we have$D_{L^{2}}((-\Delta_{x})^{1/2})=H^{1}(\mathbb{R}^{d})$, $D_{L^{2}}((\partial_{t}\pm(-\Delta_{x})^{1/2}))=H^{1}(\mathbb{R}^{d+1})$, and $D_{L^{2}}((\partial_{t}-(-\Delta_{x})^{1/2})(\partial_{t}+(-\Delta_{x})^{1/2}))=H^{2}(\mathbb{R}^{d+1})$. Another key feature of (1.3) is that it is a factorization of the operator in the t variable and the x variables. Hence, by the t-independent assumption for the coefficients of A, the factorization into the first order differential operators as in (1.3) is easily extended to the case when A is a typical block matrix, i.e., $\mathbf{r_1} = \mathbf{r_2} = \mathbf{0}$ and $b = 1$, at least in the formal level. Indeed, it suffices to replace $(-\Delta_x)^{1/2}$ by $\mathcal{A}'^{1/2}$, the square root of $\mathcal{A}'$ in $L^2(\mathbb{R}^d)$. However, in contrast to the Laplacian case, the validity of the topological factorization is far from trivial in this case, since the domain of the squre root of $\mathcal{A}$ has to be characterized as $H^1(\mathbb{R}^d)$ to achieve the identity $D_{L^2}(\mathcal{A}) = D_{L^2}\big((\partial_t - \mathcal{A}'^{1/2})(\partial_t + \mathcal{A}'^{1/2})\big)$. The characterization $D_{L^2}(\mathcal{A}'^{1/2}) = H^1(\mathbb{R}^d)$ is nothing but the Kato square root problem for divergence form elliptic operators, which was finally settled in [6]. Our first goal is to give sufficient conditions on A, which may be a full entry matrix, so that the exact topological factorization of $\mathcal{A}$ like (1.3) is verified. To this end we introduce some terminologies.

Definition 1.1. (i) *For a given $h \in \mathcal{S}'(\mathbb{R}^d)$ we denote by $M_h : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ the multiplier $M_h u = hu$.*
(ii) *We denote by $E_{\mathcal{A}} : \dot{H}^{1/2}(\mathbb{R}^d) \to \dot{H}^1(\mathbb{R}^{d+1}_+)$ the $\mathcal{A}$-extension operator, i.e., $u = E_{\mathcal{A}} g$ is the solution to the Dirichlet problem*

$$\left\{ \begin{aligned} \mathcal{A}u &= 0 \text{ in } \mathbb{R}^{d+1}_+, \\ u &= g \text{ on } \partial\mathbb{R}^{d+1}_+ = \mathbb{R}^d. \end{aligned} \right. \tag{1.4}$$

The one parameter family of linear operators $\{E_{\mathcal{A}}(t)\}_{t\ge 0}$, defined by $E_{\mathcal{A}}(t) g = (E_{\mathcal{A}} g)(\cdot, t)$ for $g \in \dot{H}^{1/2}(\mathbb{R}^d)$, is called the Poisson semigroup associated with $\mathcal{A}$.
(iii) *We denote by $\Lambda_{\mathcal{A}} : D_{L^2}(\Lambda_{\mathcal{A}}) \subset \dot{H}^{1/2}(\mathbb{R}^d) \to \dot{H}^{-1/2}(\mathbb{R}^d) = \big(\dot{H}^{1/2}(\mathbb{R}^d)\big)^*$ the Dirichlet-Neumann map associated with $\mathcal{A}$, which is defined through the sesquilinear form*

$$\langle \Lambda_{\mathcal{A}} g, \varphi \rangle_{\dot{H}^{-\frac12}, \dot{H}^{\frac12}} = \langle A \nabla E_{\mathcal{A}} g, \nabla E_{\mathcal{A}} \varphi \rangle_{L^2(\mathbb{R}^{d+1}_+)}, \quad g, \varphi \in \dot{H}^{\frac12}(\mathbb{R}^d). \tag{1.5}$$

Here $\langle \cdot, \cdot \rangle_{\dot{H}^{-1/2}, \dot{H}^{1/2}}$ denotes the duality coupling of $\dot{H}^{-1/2}(\mathbb{R}^d)$ and $\dot{H}^{1/2}(\mathbb{R}^d)$.

Remark 1.1. From the standard theory for sesquilinear forms [27], due to the ellipticity condition (1.2), the Poisson semigroup $\{E_{\mathcal{A}}(t)\}_{t\ge 0}$ is well-defined for $\dot{H}^{1/2}(\mathbb{R}^d)$ and the Dirichlet-Neumann map $\Lambda_{\mathcal{A}}$ is extended as an injective m-sectorial operator in $L^2(\mathbb{R}^d)$ satisfying $D_{L^2}(\Lambda_{\mathcal{A}}) \subset H^{1/2}(\mathbb{R}^d)$.

Our first result is Theorem (1.1) below. We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of finite Radon measures, and $L^{p,\infty}(\mathbb{R}^d)$ is the Lorentz space $L^{p,q}(\mathbb{R}^d)$ with the exponent $q = \infty$.

Theorem 1.1 ([31, 33]). *Suppose that either*

(i) *A is Lipschitz, or*

(ii) *A is Hermite, or both*

(iiia) *for* $j = 1, 2$, $\nabla_x \cdot \mathbf{r}_j$ *belongs to* $L^{d,\infty}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *if* $d \geq 2$ *(or* $\nabla_x \cdot \mathbf{r}_j$ *belongs to* $\mathcal{M}(\mathbb{R}) + L^\infty(\mathbb{R})$ *if* $d = 1$*) with small* $L^{d,\infty}(\mathbb{R}^d)$ *parts (or small* $\mathcal{M}(\mathbb{R})$ *parts resp.) and*

(iiib) $\mathrm{Im}\,(\mathbf{r_1} + \mathbf{r_2}) = \mathbf{0}$ *and* $\mathrm{Im}\, b = 0$.

Then $H^1(\mathbb{R}^d)$ *is continuously embedded in* $D_{L^2}(\Lambda_{\mathcal{A}}) \cap D_{L^2}(\Lambda_{\mathcal{A}^*})$*, and the operators* $-\mathbf{P}_{\mathcal{A}}$, $-\mathbf{P}_{\mathcal{A}^*}$ *defined by*

$$D_{L^2}(\mathbf{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d), \quad -\mathbf{P}_{\mathcal{A}} f = -M_{1/b}\Lambda_{\mathcal{A}} f - M_{\mathbf{r_2}/b} \cdot \nabla_x f\,, \tag{1.6}$$

$$D_{L^2}(\mathbf{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d), \quad -\mathbf{P}_{\mathcal{A}^*} f = -M_{1/\bar{b}}\Lambda_{\mathcal{A}^*} f - M_{\bar{\mathbf{r}}_\mathbf{1}/\bar{b}} \cdot \nabla_x f\,, \tag{1.7}$$

generate strongly continuous and analytic semigroups in $L^2(\mathbb{R}^d)$*. Moreover, the realization of* $\mathcal{A}'$ *in* $L^2(\mathbb{R}^d)$ *and the realization* $\mathcal{A}$ *in* $L^2(\mathbb{R}^{d+1})$ *are respectively factorized as*

$$\mathcal{A}' = M_b \mathcal{Q}_{\mathcal{A}} \mathbf{P}_{\mathcal{A}}, \quad \mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_{\bar{b}}\mathbf{P}_{\mathcal{A}^*})^*, \tag{1.8}$$

$$\mathcal{A} = -M_b(\partial_t - \mathcal{Q}_{\mathcal{A}})(\partial_t + \mathbf{P}_{\mathcal{A}}). \tag{1.9}$$

Here $(M_{\bar{b}}\mathbf{P}_{\mathcal{A}^*})^*$ *is the adjoint of* $M_{\bar{b}}\mathbf{P}_{\mathcal{A}^*}$ *in* $L^2(\mathbb{R}^d)$.

Remark 1.2. The operator $-\mathbf{P}_{\mathcal{A}}$ is nothing but the generator of the Poisson semigroup in $L^2(\mathbb{R}^d)$, i.e., $-\mathbf{P}_{\mathcal{A}} f = -\mathcal{P}_{\mathcal{A}} f := \lim_{t\to 0} t^{-1}(E_{\mathcal{A}}(t)f - f)$ in $L^2(\mathbb{R}^d)$. In other words, Theorem 1.1 includes the following assertion: the Poisson semigroup $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ in $H^{1/2}(\mathbb{R}^d)$ is extended as an analytic semigroup in $L^2(\mathbb{R}^d)$, and the domain of its generator is characterized as $H^1(\mathbb{R}^d)$. We note that when $\mathbf{r_1} = \mathbf{r_2} = 0$ and $b = 1$ the operator $\mathbf{P}_{\mathcal{A}}$ is the square root of $\mathcal{A}'$. Hence, the characterization $D_{L^2}(\mathbf{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ in the case (iii) of Theorem 1.1 is closely related with the Kato square root problem.

Remark 1.3. When A possesses enough regularity it is classical in the theory of pseudo-differential operators that one looks for the factorization of $\mathcal{A}$ in the form $-M_b(\partial_t - \mathcal{A}_1)(\partial_t + \mathcal{A}_2)$ for some first order operators $\mathcal{A}_1$ and $\mathcal{A}_2$ but with modulo lower order operators; e.g. [43]. On the other hand, (1.9) is just exact, i.e., any modifications by lower order operators are not required, and (1.9) holds under mild regularity assumptions on A.

Let us call the operator $-\mathbf{P}_{\mathcal{A}}$ in Theorem 1.1 the Poisson operator associated with $\mathcal{A}$. Theorem 1.1 states that if A possesses either some regularity or symmetry then the topological factorization of the type (1.3) is still valid, and $-(-\Delta_x)^{1/2}$ for the Laplacian case is replaced by the Poisson operator $-\mathbf{P}_{\mathcal{A}}$ in general case. The condition (iiia) of Theorem 1.1 imposes the regularity for the divergence of the off-block vectors. The spaces $L^{d,\infty}(\mathbb{R}^d)$ for $d \geq 2$ and $\mathcal{M}(\mathbb{R})$ for $d = 1$ in (iiia) are critical in view of scaling as a local regularity for $\nabla_x \cdot \mathbf{r}_j$. Indeed, in view of scaling the Multiplication operator $M_{\nabla_x \cdot \mathbf{r}_j}$ is comparable with the first order operator when $\nabla_x \cdot \mathbf{r}_j$belongs to these spaces. In [30] it is shown that if A is a 2×2 matrix of the form $a_{1,1} = a_{2,2} = 1$ and $a_{1,2} = -a_{2,1} = m \operatorname{sign} x$ with large $m \in \mathbb{R}$, then the Poisson semigroup $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ in $H^{1/2}(\mathbb{R})$ is not extended as a semigroup in $L^2(\mathbb{R})$. Hence, when $d = 1$, the smallness condition for $\mathcal{M}(\mathbb{R})$ part of $\nabla_x \cdot \mathbf{r}_j$ in (iiia) is optimal in this sense.

The factorizations (1.8) and (1.9) are regarded as operator theoretical descriptions of the Rellich identity. The Rellich identity is a classical tool to investigate the boundary behavior of solutions to the elliptic equations; cf. [40, 39, 24]. It is particularly well-known when A is real symmetric, and the typical version is

$$\langle A'\nabla_x g, \nabla_x g\rangle_{L^2(\mathbb{R}^d)} = \langle \gamma\partial_t E_{\mathcal{A}} g, M_b \gamma\partial_t E_{\mathcal{A}} g\rangle_{L^2(\mathbb{R}^d)}, \tag{1.10}$$

where γ is the trace operator to the boundary $\partial\mathbb{R}^{d+1}_+ \simeq \mathbb{R}^d$. The identity (1.10) is formally obtained by a simple integration by parts with the aid of the t-independence of the coefficients of A. Since $\gamma\partial_t E_{\mathcal{A}} = -\mathcal{P}_{\mathcal{A}}$, we observe from (1.10) that $\mathcal{P}_{\mathcal{A}}$ is comparable with ∇_x in $L^2(\mathbb{R}^d)$ at least when A is real symmetric. Even for a general matrix A we can formally derive the identity

$$\langle A'\nabla_x g, \nabla_x g\rangle_{L^2(\mathbb{R}^d)} = \langle \gamma\partial_t E_{\mathcal{A}} g, M_{\bar{b}} \gamma\partial_t E_{\mathcal{A}^*} g\rangle_{L^2(\mathbb{R}^d)},$$

or its more general version

$$\langle A'\nabla_x g, \nabla_x h\rangle_{L^2(\mathbb{R}^d)} = \langle \gamma\partial_t E_{\mathcal{A}} g, M_{\bar{b}} \gamma\partial_t E_{\mathcal{A}^*} h\rangle_{L^2(\mathbb{R}^d)}. \tag{1.11}$$

Replacing $\gamma\partial_t E_{\mathcal{A}}$ and $\gamma\partial_t E_{\mathcal{A}^*}$ by $-\mathcal{P}_{\mathcal{A}}$ and $-\mathcal{P}_{\mathcal{A}^*}$ respectively, and setting $\mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*$, we have from (1.11) the formal identity

$$\langle \mathcal{A}' g, h\rangle_{L^2(\mathbb{R}^d)} = \langle M_b \mathcal{Q}_{\mathcal{A}} \mathcal{P}_{\mathcal{A}} g, h\rangle_{L^2(\mathbb{R}^d)}. \tag{1.12}$$

The identity (1.12) implies (1.8) due to the formal relation $\mathbf{P}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$. The identity (1.9) is formally obtained in the similar manner. The essential difficulty here is to characterize g and h for which (1.11) is verified. When A is nonsmooth and nonsymmetric this problem is highly nontrivial. Theorem 1.1 states that the identity (1.11) holds for all $g, h \in H^1(\mathbb{R}^d)$ under the assumptions of either (i) or (ii) or (iiia)-(iiib).

As for the proof of Theorem 1.1, we need different approaches for each of (i), (ii), and (iiia)-(iiib). The proof for the case (i) is based on the calculus of principal symbols for $\mathbf{P}_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ (see [32]), while the proof for the case (ii) is based on the Rellich identity (1.10). The case (iiia)-(iiib) is related with the Kato square root problem, and the proof for this case relies on the fact $D_{L^2}(\mathcal{A}'^{1/2}) = H^1(\mathbb{R}^d)$ obtained by [6]. In each case the following four lemmas play a central role.

Lemma 1.1 ([31, Proposition 2.4]). *The one-parameter family $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ defines a strong continuous and analytic semigroup in $H^{1/2}(\mathbb{R}^d)$. Hence* *there is a unique sectorial operator* $-\mathcal{P}_{\mathcal{A}} : D_{H^{1/2}}(\mathcal{P}_{\mathcal{A}}) \to H^{1/2}(\mathbb{R}^d)$ *such that* $E_{\mathcal{A}}(t) = e^{-t\mathcal{P}_{\mathcal{A}}}$.

Lemma 1.2 ([31, Proposition 3.3]). *The following two statements are equivalent.*

(i) $D_{H^{1/2}}(\mathcal{P}_{\mathcal{A}}) \subset D_{L^2}(\Lambda_{\mathcal{A}^*})$ *and* $\|\Lambda_{\mathcal{A}^*} f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)}$ *holds for* $f \in D_{H^{1/2}}(\mathcal{P}_{\mathcal{A}})$.

(ii) $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ *is extended as a strongly continuous semigroup in* $L^2(\mathbb{R}^d)$ *and* $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ *is continuously embedded in* $H^1(\mathbb{R}^d)$.

Moreover, if the condition (ii) *(and hence,* (i)*) holds then* $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ *is continuously embedded in* $D_{L^2}(\Lambda_{\mathcal{A}})$, $H^1(\mathbb{R}^d)$ *is continuously embedded in* $D_{L^2}(\Lambda_{\mathcal{A}^*})$, *and it follows that*

$$\mathcal{P}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f + M_{\mathbf{r_2}/b} \cdot \nabla_x f,$$
$$\langle \mathcal{A}' f, g\rangle_{\dot{H}^{-1},\dot{H}^1} = \langle \mathcal{P}_{\mathcal{A}} f, \Lambda_{\mathcal{A}^*} g + M_{\bar{\mathbf{r}}_1} \cdot \nabla_x g\rangle_{L^2(\mathbb{R}^d)}$$

for $f \in D_{L^2}(\mathcal{P}_{\mathcal{A}})$ *and* $g \in H^1(\mathbb{R}^d)$.

Lemma 1.3 ([31, Corollary 3.5, Proposition 3.6]). *Assume that* $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ *and* $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\geq 0}$ *are extended as strongly continuous semigroups in* $L^2(\mathbb{R}^d)$ *and that* $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ *and* $D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$ *are continuously embedded in* $H^1(\mathbb{R}^d)$. *Then we have*

$$\langle A'\nabla_x f, \nabla_x g\rangle_{L^2(\mathbb{R}^d)} = \langle \mathcal{P}_{\mathcal{A}} f, M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*} g\rangle_{L^2(\mathbb{R}^d)}, \quad f \in D_{L^2}(\mathcal{P}_{\mathcal{A}}),\ g \in D_{L^2}(\mathcal{P}_{\mathcal{A}^*}),$$
$$C'\|f\|_{H^1(\mathbb{R}^d)} \leq \|\mathcal{P}_{\mathcal{A}} f\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)}, \quad f \in D_{L^2}(\mathcal{P}_{\mathcal{A}}).$$

If in addition that $\liminf_{t\to 0} \|\mathrm{d}/\mathrm{d}t\ e^{-t\mathcal{P}_{\mathcal{A}}} f\|_{L^2(\mathbb{R}^d)} < \infty$ *holds for all* $f \in C_0^\infty(\mathbb{R}^d)$ *then* $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ *with equivalent norms.*

Remark 1.4. A similar sufficient condition for the characterization $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ with equivalent norms is given in [2, Theorem 4.1], where he also studied the case for elliptic systems. Our approach, different from [2], is based on the Rellich type identity.

Lemma 1.4 ([31, Lemma 3.8]). *Assume that the semigroups* $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ *and* $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\geq 0}$ *in* $H^{1/2}(\mathbb{R}^d)$ *are extended as strongly continuous semigroups in* $L^2(\mathbb{R}^d)$ *and that* $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ *holds with equivalent norms. Then* $H^1(\mathbb{R}^d)$ *is continuously embedded in* $D_{L^2}(\Lambda_{\mathcal{A}}) \cap D_{L^2}(\Lambda_{\mathcal{A}^*})$ *and*

$$\mathcal{P}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f + M_{\mathbf{r_2}/b} \cdot \nabla_x f, \quad f \in H^1(\mathbb{R}^d), \tag{1.13}$$
$$\mathcal{P}_{\mathcal{A}^*} g = M_{1/\bar{b}}\Lambda_{\mathcal{A}^*} g + M_{\bar{\mathbf{r}}_1/\bar{b}} \cdot \nabla_x g, \quad g \in H^1(\mathbb{R}^d). \tag{1.14}$$

Moreover, the realization of $\mathcal{A}'$ *in* $L^2(\mathbb{R}^d)$ *and the realization of* $\mathcal{A}$ *in* $L^2(\mathbb{R}^{d+1})$ *are respectively factorized as*

$$\mathcal{A}' = M_b \mathcal{Q}_{\mathcal{A}} \mathcal{P}_{\mathcal{A}}, \qquad \mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*, \tag{1.15}$$
$$\mathcal{A} = -M_b(\partial_t - \mathcal{Q}_{\mathcal{A}})(\partial_t + \mathcal{P}_{\mathcal{A}}). \tag{1.16}$$

Here $(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*$ *is the adjoint of* $M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*}$ *in* $L^2(\mathbb{R}^d)$.

2. Applications

The factorization (1.9) is important since it provides the integral solution formula for the inhomogeneous Dirichlet problem

$$\left\{\begin{aligned} \mathcal{A}u &= F \text{ in } \mathbb{R}^{d+1}_+, \\ u &= g \text{ on } \partial\mathbb{R}^{d+1}_+, \end{aligned}\right. \tag{2.1}$$

and the inhomogeneous Neumann problem

$$\left\{\begin{aligned} \mathcal{A}u &= F \text{ in } \mathbb{R}^{d+1}_+, \\ -\mathbf{e}_{d+1}\cdot A\nabla u &= g \text{ on } \partial\mathbb{R}^{d+1}_+. \end{aligned}\right. \tag{2.2}$$

Definition 2.1 (Mild solution). *Let* $F \in L^1_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d))$ *and* $g \in L^2(\mathbb{R}^d)$. *If the function* $u \in L^1_{loc}(\mathbb{R}^{d+1}_+)$ *has the well-defined representation*

$$u(t) = e^{-t\mathbf{P}_{\mathcal{A}}}g + \int_0^t e^{-(t-s)\mathbf{P}_{\mathcal{A}}}\int_s^\infty e^{-(\tau-s)\mathcal{Q}_{\mathcal{A}}}M_{1/b}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \tag{2.3}$$

then we call u *a mild solution to* (2.1). *Similarly, if the function* $v \in L^1_{loc}(\mathbb{R}^{d+1}_+)$ *has the well-defined representation*

$$\begin{aligned} v(t) = {} & e^{-t\mathbf{P}_{\mathcal{A}}}\Lambda_{\mathcal{A}}^{-1}\big(g + M_b\int_0^\infty e^{-s\mathcal{Q}_{\mathcal{A}}}M_{1/b}F(s)\,\mathrm{d}s\big) \\ & + \int_0^t e^{-(t-s)\mathbf{P}_{\mathcal{A}}}\int_s^\infty e^{-(\tau-s)\mathcal{Q}_{\mathcal{A}}}M_{1/b}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \end{aligned} \tag{2.4}$$

then we call v *a mild solution to* (2.2).

We note that our approach using Theorem 1.1 provides a unified view for (2.1) and (2.2) through mild solutions. As applications to Theorem 1.1, we consider the solvability of inhomogeneous problem in Section 2.1, and in Section 2.2 we show the validity of the Helmholtz decomposition for vector fields in a domain with a graph boundary when the function space of vector fields is chosen as certain anisotropic Lebesgue space.

2.1. Application to inhomogeneous problem with non $\dot{H}^{-1}(\mathbb{R}^{d+1}_+)$ data.

Firstly let us state some results on L^2 solvability of (2.1) and (2.2) in the simplest form. We set $\overline{\mathbb{R}_+} = [0,\infty)$, and for a Banach space X we write $f \in C(\overline{\mathbb{R}_+}; X)$ if and only if $f \in C([0,T); X)$ for all $T > 0$. For the homogeneous problems (i.e., $F = 0$ in (2.1) or (2.2)), Theorem 1.1 implies the following result:

Theorem 2.1 ([31]). *Under the assumptions of Theorem 1.1, there exists a unique weak solution* u *to* (2.1) *with* $F = 0$ *and* $g \in L^2(\mathbb{R}^d)$ *such that* $u \in C(\overline{\mathbb{R}_+}; L^2(\mathbb{R}^d)) \cap \dot{H}^1(\mathbb{R}^d \times (\delta,\infty))$ *for any* $\delta > 0$. *If in addition* g

belongs to the range of Λ_A, then there exists a unique weak solution v to (2.2) with $F = 0$ such that $v \in C(\overline{\mathbb{R}_+}; H^{1/2}(\mathbb{R}^d)) \cap \dot{H}^1(\mathbb{R}^{d+1}_+)$.

Remark 2.1. As we mentioned before, if A is Hermite then $D_{L^2}(\Lambda_A) = H^1(\mathbb{R}^d)$ holds. In this case the weak solution to (2.2) obtained in Theorem 2.1 possesses further regularity such as $C(\overline{\mathbb{R}_+}; H^1(\mathbb{R}^d))$.

Remark 2.2. It is well-known that solvability of the elliptic boundary value problems in $\mathbb{R}^{d+1}_+$ can be extended to that in the domain above a Lipschitz graph. The L^2 solvability of the Laplace equation (i.e., $A = I$) in Lipschitz domains was shown in [11, 24, 44]. In [13] the relation $D_{L^2}(\mathcal{P}_A) = H^1(\mathbb{R}^d)$ is proved in this case. This result was extended by [25, 29, 1] to the case when A is real symmetric, and by [5] to the case when A is Hermite. In view of L^2 solvability of the homogeneous boundary value problems, Theorem 2.1 gives a new contribution under the conditions (iiia) - (iiib) in Theorem 1.1.

When A is not Hermite and nonsmooth, the boundary value problems are not always solvable for L^2 boundary data. If A is a typical block matrix, $\mathbf{r_1} = \mathbf{r_2} = \mathbf{0}$ and $b = 1$, then the homogenous Dirichlet problem is easily solved by using the semigroup theory, while the homogeneous Neumann problem in this case is essentially equivalent with the Kato square root problem solved in [6]; see also [9]. Recently the authors in [4] showed L^2 solvability of the homogeneous Dirichlet and Neumann problems when A is a small L^∞ perturbation of a block matrix; see also [14, 23, 5, 1, 3, 8, 7] for related stability result. In fact, Theorem 2.1 with the conditions (iiia) - (iiib) can be regarded as another stability result for the block matrix case. Note that $\|\nabla_x \cdot \mathbf{r}_j\|_{L^{d,\infty}(\mathbb{R}^d)}$ for $d \geq 2$ or $\|\nabla_x \cdot \mathbf{r}_j\|_{\mathcal{M}(\mathbb{R}^d)}$ for $d = 1$, is in the same order as $\|a_{i,j}\|_{L^\infty(\mathbb{R}^d)}$ in view of scaling. this implies that, the condition (iiia) of Theorem 2.1 is comparable to L^∞ perturbations discussed in [4, 5, 1] in view of scaling. On the other hand, as stated in the introduction, the authors of [30] gave an example of the matrix A such that the homogeneous Dirichlet problem in $\mathbb{R}^2_+$ is not solvable for the boundary data in $L^2(\mathbb{R})$. In their example, A is real but nonsymmetric, and $\nabla_x \cdot \mathbf{r_j}$ ($j = 1, 2$) is a Dirac measure whose mass is not small. This example shows the optimality of our condition (iiia) for the case of real nonsymmetric matrices when $d = 1$. For further results on solvability of the homogeneous problems, see [28] and references therein.

The next result concerns L^2 solvability of the inhomogeneous problems. For simplicity of the presentation, we will assume the boundary data are zero. It is classical that if F belongs to $\dot{H}^{-1}(\mathbb{R}^{d+1}_+)$ then there is a unique solution $u \in \dot{H}^1(\mathbb{R}^{d+1}_+)$ to (2.1) with $g = 0$. The novelty of our result below is that, for some class of A, we can handle with the inhomogeneous term F which does not necessarily belong to $\dot{H}^{-1}(\mathbb{R}^{d+1}_+)$.

Theorem 2.2 ([31]). *Suppose that either*

(ii) *A is Hermite or both*

(iiia') $\nabla_x \cdot \mathbf{r}_1 = 0$ *and* $\nabla_x \cdot \mathbf{r}_2$ *belongs to* $L^{d,\infty}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *if* $d \geq 2$ *(or* $\nabla_x \cdot \mathbf{r}_2$ *belongs to* $\mathcal{M}(\mathbb{R}) + L^\infty(\mathbb{R})$ *if* $d = 1$*) with small* $L^{d,\infty}(\mathbb{R}^d)$ *parts (or small* $\mathcal{M}(\mathbb{R})$ *parts resp.) and*

(iiib') $\mathbf{r}_1$, $\mathbf{r}_2$, *and* b *are real-valued.*

Then for given $F \in L^1(\mathbb{R}_+; L^2(\mathbb{R}^d))$ *there exists a weak solution* u *to* (2.1) *with* $g = 0$ *satisfying*

$$u \in C(\overline{\mathbb{R}_+}; L^2(\mathbb{R}^d)) \quad \text{and} \quad \nabla u \in L^p_{loc}(\overline{\mathbb{R}_+}; L^2(\mathbb{R}^d)) \ \text{ for any } p \in [1, \infty).$$

If in addition $h = M_b \int_0^\infty e^{-s\mathcal{Q}_\mathcal{A}} M_{1/b} F(s)\, \mathrm{d}s$ *belongs to the range of* $\Lambda_\mathcal{A}$*, then there exists a weak solution* v *to* (2.2) *with* $g = 0$ *satisfying*

$$v \in C(\overline{\mathbb{R}_+}; L^2(\mathbb{R}^d)) \quad \text{and} \quad \nabla v \in L^p_{loc}(\overline{\mathbb{R}_+}; L^2(\mathbb{R}^d)) \ \text{ for any } p \in [1, 2).$$

Remark 2.3. Under the assumptions of Theorem 2.2 the Poisson semigroups $\{e^{-t\mathcal{P}_\mathcal{A}}\}_{t\geq 0}$ and $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\geq 0}$ are realized as strongly continuous and analytic semigroups acting on $L^2(\mathbb{R}^d)$ thanks to the results of Theorem 1.1.

Remark 2.4. There is a lot of literature for the inhomogeneous boundary value problems in bounded Lipschitz domains; see, e.g., [12, 26, 15, 34, 35, 36] and references therein. As well as the case for the homogeneous problem, Theorem 2.2 for Hermite matrices yields L^2 solvability of the inhomogeneous problems for matrices of the same type in domains above Lipschitz graphs. For the Laplace equation, L^p solvability of the inhomogenous problems in bounded Lipschitz domains was proved in [12, 26, 15]. Our result also shows the gradient of the Dirichlet Green operator (i.e., the solution map for (2.1) with the zero boundary data: $F \mapsto \nabla u$) maps $L^1(\mathbb{R}_+; L^2(\mathbb{R}^d))$ continuously to $L^p_{loc}(\mathbb{R}_+, L^2(\mathbb{R}^d))$. Results of this type go back to [12] where the author showed that the gradient of the Dirichlet Green operator for $\mathcal{A} = -\Delta$ in the bounded Lipschitz domain is a continuous map from $L^1(\Omega)$ to $L^{n/(n-1),\infty}(\Omega)$. Recently, it was generalized in [34] for the Neumann Green operator by using potential technique; see also [35, 36] for further results.

As is well-known in the spectral theory, it is a subtle problem to determine sufficient conditions for F to solve the problems (2.1) or (2.2). Indeed, due to the lack of the Poincaré inequality, the origin belongs to the continuous spectrum of $\mathcal{A}$ (with the zero boundary condition) in $L^2(\mathbb{R}^{d+1}_+)$. Hence the inhomogeneous problem is not always solvable for $F \in L^2(\mathbb{R}^{d+1}_+)$, even if A is real symmetric and smooth. Therefore some additional conditions related to the spatial decay have to be imposed on F to find the solution. Furthermore, the solution may fail to decay at spatial infinity even if it exists. To show Theorem 2.2 we will make use of the representation formulas (2.3) and (2.4). Then it is clear that the temporal decay of $e^{-t\mathcal{Q}_\mathcal{A}}$ is crucial for solving our problems. In fact, the conditions in Theorem 2.2 guarantee the boundedness of the semigroup $\{e^{-tQ_A}\}_{t\geq 0}$ in $L^2(\mathbb{R}^d)$, and hence, the integrals in (2.3) and (2.4) converge absolutely if $F \in L^1(\mathbb{R}_+; L^2(\mathbb{R}^d))$. By a simple observation of the scaling, it is easy to see that the space $L^1(\mathbb{R}_+; L^2(\mathbb{R}^d))$ includes some functions decaying more slowly at (time) infinity than those in $\dot{H}^{-1}(\mathbb{R}^{d+1}_+)$. In this sense, our result generalizes the class of the inhomogeneous terms for the solvability in terms of the decay at infinity. In should be emphasized here that the factorization in Theorem 1.1 plays an essential role behind the proof of Theorem 2.2, for the representation formulas such as (2.3) and (2.4) are nothing but a result of (1.9). In [31, Section 5] a detailed version of Theorem 2.2 is also stated.

2.2. **Application to Helmholtz decomposition in unbounded domain with graph boundary.** In this section we apply the solution formula (2.4) to the analysis of the Helmholtz decomposition for vector fields in the domain above a Lipschitz graph:

$$\Omega = \{\tilde{x} = (x, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mid x_{d+1} > \eta(x)\}. \tag{2.5}$$

Here η is a given function satisfying $\|\nabla_x \eta\|_{L^\infty(\mathbb{R}^d)} < \infty$.

The Helmholtz decomposition, the decomposition of a given vector field into a solenoidal field and a potential one is the fundamental tool in the mathematical analysis of the incompressible flow. In the energy space $(L^2(\Omega))^{d+1}$ this decomposition is easily derived for any domain Ω from the standard theory of the Hilbert space. On the other hand, if the space $(L^2(\Omega))^{d+1}$ is replaced by other function spaces such as $(L^q(\Omega))^{d+1}$, then the verification of the Helmholtz decomposition requires detailed analysis in general. In the case when Ω is a bounded domain or an exterior domain with smooth boundaries, the validity of the decomposition in $(L^q(\Omega))^{d+1}$, $1 < q < \infty$, is shown by [20] and [37] respectively, and then their results are extended to these domains but with C^1-boundary by [41]. Moreover, for the bounded Lipschitz domains, the validity is proved around $3/2 < q < 3$ in [15], and for any $1 < q < \infty$ by [22] when the domain is convex. However, even if the boundary is smooth enough, the problem becomes subtle when the boundary is noncompact. Although the decomposition is still valid for $1 < q < \infty$ for some special cases, e.g., aperture domains [16, 19], layers [38], cylinders [42], half spaces and their small perturbations [41], it is known that the domain of simple form

$$\Omega = \{\tilde{x} = (x, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mid x_{d+1} > \eta(x)\}, \tag{2.6}$$

with a given function η does not always admit the Helmholtz decomposition in $(L^q(\Omega))^{d+1}$ if $q \neq 2$, even if η is smooth, see [10] and [21, III.1]. Hence it is an important question to ask which function space, other than $(L^2(\Omega))^{d+1}$, admits the Helmholtz decomposition. In [17, 18], the authors considered

$\tilde{L}^q(\Omega)$ defined by

$$\tilde{L}^q(\Omega) = \begin{cases} L^2(\Omega) \cap L^q(\Omega), & 2 \le q < \infty, \\ L^2(\Omega) + L^q(\Omega), & 1 < q < 2, \end{cases}$$

and showed that general domains with uniform C^1 boundaries admit the Helmholtz decomposition in these spaces. In this section we will give an alternative approach for this question in the domain of the form (2.6).

Before stating the result, it would be convenient to formulate our problem more systematically. Let $X(\Omega)$ be a Banach space of functions in Ω satisfying $C_0^\infty(\Omega) \subset X(\Omega) \subset L^1_{loc}(\Omega)$. Set

$$X_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{X(\Omega)}} \quad X_G(\Omega) = \{\nabla f \in (X(\Omega))^{d+1} \mid f \in L^1_{loc}(\Omega)\}. \tag{2.7}$$

Here $C^\infty_{0,\sigma}(\Omega)$ is a set of all smooth, compactly-supported, and divergence-free vector fields in Ω. For simplicity of notations we write $\|\cdot\|_{X(\Omega)}$ for $\|\cdot\|_{(X(\Omega))^{d+1}}$.

Definition 2.2. *We say that the space $(X(\Omega))^{d+1}$ admits the Helmholtz decomposition if each $f \in (X(\Omega))^{d+1}$ has a unique decomposition $f = u + \nabla p$, $u \in X_\sigma(\Omega)$, $\nabla p \in X_G(\Omega)$, satisfying*

$$\|u\|_{X(\Omega)} + \|\nabla p\|_{X(\Omega)} \le C\|f\|_{X(\Omega)}. \tag{2.8}$$

Here C is a positive constant independent of f.

In order to consider the domain Ω of the form (2.6) we introduce the Lipschitzmorphism $\Phi : \Omega \in \tilde{x} \mapsto \Phi(\tilde{x}) \in \mathbb{R}^{d+1}_+$ by

$$\Phi_j(\tilde{x}) = \begin{cases} x_j & \text{if } 1 \le j \le d, \\ x_{d+1} - \eta(x) & \text{if } j = d+1. \end{cases} \tag{2.9}$$

Let $1 < q, r < \infty$ and let $Y^{q,r}(\Omega)$ be the Banach space defined by

$$Y^{q,r}(\Omega) = \{f \in L^1_{loc}(\Omega) \mid \|f\|_{Y^{q,r}(\Omega)} = \|f \circ \Phi^{-1}\|_{L^q(0,\infty;L^r(\mathbb{R}^d))} < \infty\} \tag{2.10}$$

with the norm $\|\cdot\|_{Y^{q,r}(\Omega)}$. Our result reads as follows:

Theorem 2.3 ([33]). *Let Ω be a domain of the form* (2.6) *with uniform Lipschitz boundary. Then the space $\left(Y^{q,2}(\Omega)\right)^{d+1}$ admits the Helmholtz decomposition for all $1 < q < \infty$. Moreover, the constant C in* (2.8) *depends only on d, q, and $\|\nabla_x \eta\|_{L^\infty(\mathbb{R}^d)}$.*

Remark 2.5. The growth of η itself is allowed in our result. The space $(Y^{q,q}(\Omega))^{d+1}$ coincides with $(L^q(\Omega))^{d+1}$. Due to the well-known counterexample of the weak Neumann problem in the exterior of the cone-like domain, one cannot expect the validity of the Helmholtz decomposition in the usual L^q space for domains like (2.6). Roughly speaking, Theorem 2.3 asserts that the Helmholtz decomposition is valid even if the vector fields decay slowly in the x^{d+1} direction.

It is well known that the validity of the Helmholtz decomposition is equivalent with the unique solvability of the Neumann problem

$$\Delta p = \nabla \cdot f \text{ in } \Omega, \qquad n \cdot \nabla p = n \cdot f \text{ on } \partial\Omega. \tag{2.11}$$

Here n stands for the exterior unit normal to $\partial\Omega$. Through the standard homeomorphism (2.9), that is, by setting $w = p \circ \Phi^{-1}$, $F = f \circ \Phi^{-1}$, the problem is reduced to the following Neumann problem in $\mathbb{R}^{d+1}_+$:

$$\begin{cases} \mathcal{A}w &= -\nabla_x \cdot F' - \partial_t(F_{d+1} + M_{\mathbf{r}} \cdot F'), \quad \text{in } \mathbb{R}^{d+1}_+, \\ -\mathbf{e}_{d+1} \cdot A\nabla w &= -(F_{d+1} + M_{\mathbf{r}} \cdot F'), \quad \text{on } \partial\mathbb{R}^{d+1}_+. \end{cases} \tag{2.12}$$

Here the matrix A in this case is real symmetric and positive definite with $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r} = -\nabla_x \eta$, $b = 1 + |\nabla_x \eta|^2$, and $A' = (a_{i,j})_{1 \le i,j \le d} = I'$ (the identity matrix). Let $1 < q, r < \infty$ and set

$$\begin{aligned} Z^{q,r}(\mathbb{R}^{d+1}_+) : &= \dot{W}^{1,q}(\mathbb{R}_+; L^r(\mathbb{R}^d)) \cap L^q(\mathbb{R}_+; \dot{W}^{1,r}(\mathbb{R}^d)) \\ &= \{\phi \in L^1_{loc}(\mathbb{R}^{d+1}_+) \mid \partial_i \phi \in L^q(\mathbb{R}_+; L^r(\mathbb{R}^d)) \ 1 \le i \le d+1\}. \end{aligned} \tag{2.13}$$

Let $F = (F', F_{d+1}) \in \big(L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))\big)^{d+1}$. Then the weak formulation of (2.12) is to look for $w \in Z^{q,r}(\mathbb{R}^{d+1}_+)$ such that

(2.14)
$$\langle A\nabla w, \nabla\phi \rangle_{L^2(\mathbb{R}^{d+1}_+)} = \langle F', \nabla_x \phi + M_{\mathbf{r}} \partial_t \phi \rangle_{L^2(\mathbb{R}^{d+1}_+)} + \langle F_{d+1}, \partial_t \phi \rangle_{L^2(\mathbb{R}^{d+1}_+)}$$

for all $\phi \in Z^{q',r'}(\mathbb{R}^{d+1}_+)$ with $1/q + 1/q' = 1$, $1/r + 1/r' = 1$.

In the following paragraphs we abbreviate $\mathcal{P}_{\mathcal{A}}$ ($\mathcal{Q}_{\mathcal{A}}$, $\Lambda_{\mathcal{A}}$) to $\mathcal{P}$ ($\mathcal{Q}$ and Λ as well) for simplicity of the notation. The most important step in the analysis of (2.12) is to derive the estimate corresponding with (2.8), which is closely related to the spectral properties of $\mathcal{P}$ and Λ. We focus on the a priori estimate for the gradient of w. To this end we assume that $F \in (C_0^\infty(\mathbb{R}^{d+1}_+))^{d+1}$. Set

$$G = -(F_{d+1} + M_{\mathbf{r}} \cdot F').$$

Due to the solution formula for the Neumann problem (2.4), for the solution w to (2.12) we have the representation

$$
\begin{aligned}
(2.15)\quad w(t) &= e^{-t\mathcal{P}}\Lambda^{-1}\Big(\gamma G + M_b \int_0^\infty e^{-s\mathcal{Q}} M_{1/b}(-\nabla_x\cdot F' + \partial_s G)(s)\,\mathrm{d}s\Big) \\
&\quad + \int_0^t e^{-(t-s)\mathcal{P}} \int_s^\infty e^{-(\tau-s)\mathcal{Q}} M_{1/b}(-\nabla_x\cdot F' + \partial_s G)(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= e^{-t\mathcal{P}}\Lambda^{-1} M_b \mathcal{Q} \int_0^\infty e^{-s\mathcal{Q}}\big(M_{1/b}G - \mathcal{Q}^{-1}M_{1/b}\nabla_x\cdot F'\big)(s)\,\mathrm{d}s \\
&\quad + \int_0^t e^{-(t-s)\mathcal{P}}\Big(- M_{1/b}G(s) \\
&\quad + \mathcal{Q}\int_s^\infty e^{-(\tau-s)\mathcal{Q}}\big(M_{1/b}G - \mathcal{Q}^{-1}M_{1/b}\nabla_x\cdot F'\big)(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.
\end{aligned}
$$

Here we have used the integration by parts in the t variable. Set

$$h(t) = -M_{1/b}G(t) + \mathcal{Q}\int_t^\infty e^{-(s-t)\mathcal{Q}}\big(M_{1/b}G - \mathcal{Q}^{-1}M_{1/b}\nabla_x\cdot F'\big)(s)\,\mathrm{d}s.$$

Then, by using the fact $\gamma G = 0$ due to $F \in (C_0^\infty(\mathbb{R}_+^{d+1}))^{d+1}$, the solution w is written in the form $w = w_1 + w_2$, where each w_i is given by

$$(2.16)\qquad w_1(t) = \int_0^t e^{-(t-s)\mathcal{P}} h(s)\,\mathrm{d}s, \qquad w_2(t) = e^{-t\mathcal{P}}\Lambda^{-1}M_b\gamma h.$$

To prove Theorem 2.3 we need to establish the estimate

$$(2.17)\qquad \|\nabla w_i\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))} \le C\|F\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))}, \quad i = 1, 2,$$

where C depends only on d, q, and $\|\nabla_x \eta\|_{L^\infty(\mathbb{R}^d)}$. As is observed in [33], (2.17) follows from the next three properties of $\mathcal{P}$ and Λ:

(I) Boundednes of semigroups: $\{e^{-t\mathcal{P}}\}_{t\ge 0}$ and $\{e^{-t\Lambda}\}_{t\ge0}$ in $L^2(\mathbb{R}^d)$ are strongly continuous and bounded, i.e.,

$$(2.18)\quad \|e^{-t\mathcal{P}}\varphi\|_{L^2(\mathbb{R}^d)} + \|e^{-t\Lambda}\varphi\|_{L^2(\mathbb{R}^d)} \le C\|\varphi\|_{L^2(\mathbb{R}^d)}, \quad t > 0,\ \varphi \in L^2(\mathbb{R}^d).$$

(II) Coercive estimates: $D_{L^2}(\mathcal{P}) = D_{L^2}(\Lambda) = H^1(\mathbb{R}^d)$ and

$$(2.19)\qquad \|\nabla_x\varphi\|_{L^2(\mathbb{R}^d)} \le C\|\mathcal{P}\varphi\|_{L^2(\mathbb{R}^d)}, \quad \varphi\in H^1(\mathbb{R}^d),$$

$$(2.20)\qquad \|\nabla_x\varphi\|_{L^2(\mathbb{R}^d)} \le C\|\Lambda\varphi\|_{L^2(\mathbb{R}^d)}, \quad \varphi\in H^1(\mathbb{R}^d).$$

(III) Maximal regularity: $\Psi_{\mathcal{P}}[\phi](t) = \int_0^t e^{-(t-s)\mathcal{P}}\phi(s)\,\mathrm{d}s$ satisfies

$$(2.21)\qquad \|\mathcal{P}\Psi_{\mathcal{P}}[\phi]\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))} \le C\|\phi\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))}, \quad \phi\in L^2(\mathbb{R}_+^{d+1}).$$

We note that (I) and (III) imply the analyticity of $\{e^{-t\mathcal{P}}\}_{t\ge0}$ in $L^2(\mathbb{R}^d)$ and the estimate

$$(2.22)\quad \|\mathcal{P}\Psi_{\mathcal{P}}[\phi]\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))} \le C\|\phi\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))}, \quad \phi\in L^q(\mathbb{R}_+;L^2(\mathbb{R}^d)),$$

holds for all $1 < q < \infty$. Then, by the duality argument, we also have

$$\|\mathcal{Q}\int_t^\infty e^{-(s-t)\mathcal{Q}}\phi(s)\,\mathrm{d}s\|_{L^q_t(\mathbb{R}_+;L^2(\mathbb{R}^d_x))} \le C\|\phi\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))} \tag{2.23}$$

for $\phi \in L^q(\mathbb{R}_+; L^2(\mathbb{R}^d))$. The estimate (2.17) is proved by using (2.18)-(2.23). For example, we have from (2.22),

$$\|\nabla w_1\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))} \le C\|h\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))},$$

and by using (2.23) the norm of h is estimated as

$$\|h\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))} \le C(\|F\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))} + \|\mathcal{Q}^{-1}M_{1/b}\nabla_x\cdot F'\|_{L^q(\mathbb{R}_+;L^2(\mathbb{R}^d))}).$$

Hence (2.19) and the duality argument imply (2.17). The proof for w_2 is similar, though we need to use the Marcinkiewicz interpolation theorem in this case. For details, see [33].

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