数理解析研究所講究録
第 1914 巻 2014 年 104-113



# Strichartz Estimate for Schrödinger Equation of Fourth Order with Periodic Boundary Condition

Yoshio Tsutsumi (堤 誉志雄)

Department of Mathematics, Kyoto University,
Kyoto 606-8502, JAPAN

**Abstract**

We show the $L^4$ space-time integrability of solution for the Schrödinger equation of fourth order with periodic boundary condition, that is, on the one dimensional torus.

*Keywords*: Schrödinger equation of fourth order, Strichartz estimate on one dimensional torus

## 1 Introduction and Main Theorem

We consider the following inhomogeneous linear Schrödinger equation of fourth order on the one dimensional torus $\mathbf{T}=\mathbf{R}/2\pi\mathbf{Z}$.

$$i\partial_t u+\partial_x^2 u-\partial_x^4 u=f, \qquad t\in\mathbf{R}, \quad x\in\mathbf{T}, \tag{1}$$

$$u(0,x)=u_0(x), \qquad x\in\mathbf{T}. \tag{2}$$

The Schrödinger equation (1) of fourth order appears as mathematical models in various fields, for example, in plasma physics when the quantum effect is taken into account (see [5]) or in fluid mechanics when an isolated vortex filament is embedded in an inviscid incompressible fluid filling an infinite region (see [4]).

In this note, we prove the space-time integrability of solution for (1)-(2), which is called the Strichartz estimate. The Strichartz estimate of solution to (1)-(2) is expected to be useful for the study of nonlinear evolution equations of the fourth order Schrödinger type such as the qauntum Zkakharov equations.

**Theorem 1.1.** *Let $T > 0$ and let $1/2 > b > 5/16$. Then, we have*

$$\|u\|_{L^4((-T,T)\times\mathbf{T})} \leq CT^{1/2}\mathcal{T}^{-b}\big[\|u_0\|_{L^2(\mathbf{T})} + T^{1/2}\mathcal{T}^{-b}\|f\|_{L^{4/3}((-T,T)\times\mathbf{T})}\big], \tag{3}$$

*where $\mathcal{T} = \min\{T, 1\}$ and $C$ is a positive constant dependent only on $b$.*

Theorem 1.1 is more or less known (for the Schrödinger equation of second order and the linear KdV equation, see [1] and for the linear Boussinesq type equation, see [3]), but there seems to be no literature which contains the statement and the proof of Theorem 1.1 explicitly. Moreover, the problem in the case of $\mathbf{T}$ has not been studied as well as in the case of $\mathbf{R}$ (for the results about the $\mathbf{R}$ case, see, e.g., Segata [8] and Jian, Lin and Shao [6]). So we present the proof of Theorem 1.1 in this note.

**Remark 1.2.** It is presumed that Theorem 1.1 may hold with the $L^4$ norm replaced by the $L^p$ norm for some $p > 4$ on the left hand side of (3) for the same reason as it is conjectured for the Schrödinger equation of second order and the linear KdV equation (see [1]). The Strichartz estimate in the case of $\mathbf{T}$ is more complicated than that in the case of $\mathbf{R}$. For example, a sharp necessary condition for the Strichartz estimate in the $\mathbf{R}$ case follows directly from the scaling, but it is not the case with the Strichartz estimate on $\mathbf{T}$. The specific property of each equation only reflects on the lower bound of the index $b$ for the $L^4$ type Strichartz estimate (see, e.g., the proof of Proposition 2.2 in Section 2).

We now list notations which are used throught this note. For any $a \in \mathbf{C}$, we put $\langle a\rangle = 1+|a|$. Let $U(t) = e^{it(\partial_x^2-\partial_x^4)}$. Let $\tilde{f}$ denote the Fourier transform of $f$ in both the time and spatial variables. For $T > 0$, we put $\mathcal{T} = \min\{T, 1\}$. For $b$, $s \in \mathbf{R}$, we define the Fourier restriction norms $\|\cdot\|_{Y^{b,s}}$ and$\|\cdot\|_{\bar{Y}^{b,s}}$ as follows.

$$\|f\|_{Y^{b,s}} = \left\{\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\langle k\rangle^{2s}\langle \tau - k^2 - k^4\rangle^{2b}|\tilde{f}(\tau,k)|^2\,d\tau\right\}^{1/2},$$

$$\|f\|_{\bar{Y}^{b,s}} = \left\{\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\langle k\rangle^{2s}\langle \tau + k^2 + k^4\rangle^{2b}|\tilde{f}(\tau,k)|^2\,d\tau\right\}^{1/2}.$$

We also define the spaces $Y^{b,s}$ and $\bar{Y}^{b,s}$ by the completions of $C_0^\infty(\mathbf{R}\times\mathbf{T})$ in the norms $\|\cdot\|_{Y^{b,s}}$ and $\|\cdot\|_{\bar{Y}^{b,s}}$, respectively.

## 2 Proof of Theorem 1

In this section, we describe the proof of Theorem 1.1. We begin with the following lemma about the estimate of the integral of the convolution type.

**Lemma 2.1.** *Let $b > 1/4$ and $0 < \varepsilon < 4b-1$. Then, for any $a \in \mathbf{R}$, we have*

$$\int_{-\infty}^{\infty}\frac{1}{\langle a-x\rangle^{2b}\langle x\rangle^{2b}}\,dx \le \frac{C}{\langle a\rangle^{4b-1-\varepsilon}},$$

*where $C$ is a positive constant independent of $a$.*

**Proof.** We denote the integral on the left hand side of the inequality by $I$. We split the integral into two parts as follows.

$$I = \int_{|x|\ge|a|/2} + \int_{|x|\le|a|/2} =: I_1 + I_2.$$

When $|x| \ge |a|/2$, we have

$$\begin{aligned} I_1 &\le \tfrac{C}{\langle 1+|a|/2\rangle^{4b-1-\varepsilon}}\int_{-\infty}^{\infty}\tfrac{dt}{\langle x-a\rangle^{2b}\langle x\rangle^{-2b+1+\varepsilon}} \\ &\le \tfrac{C}{\langle a\rangle^{4b-1-\varepsilon}}. \end{aligned}$$

Since $|x-a| \ge |a| - |x| \ge |a|/2$ for $|x| \le |a|/2$, we have

$$\begin{aligned} I_2 &\le \tfrac{C}{\langle 1+|a|/2\rangle^{4b-1-\varepsilon}}\int_{-\infty}^{\infty}\tfrac{dt}{\langle x-a\rangle^{-2b+1+\varepsilon}\langle x\rangle^{2b}} \\ &\le \tfrac{C}{\langle a\rangle^{4b-1-\varepsilon}}. \end{aligned}$$

Therefore, we obtain the desired inequality. □

We next prove the $L^4$ space-time estimate, which is a variant of the so-called Strichartz estimate for the Schrödinger equation of fourth order.

**Proposition 2.2.** *Let $b > 5/16$. Then, we have*

$$\|f\|_{L^4(\mathbf{R}\times\mathbf{T})} \le C\|f\|_{Y^{b,0}},$$

*where $C$ is a positive constant dependent only on $b$.*

**Proof.** We follow the argument by Kenig, Ponce and Vega [7, the proof of Lemma 5.2] (see also [10, the proof of Lemma 2.1]).

By the Parseval identity, we have

$$\begin{aligned} &\|\widetilde{f\times f}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \qquad (4)\\ &\le C\sum_{k=-\infty}^{\infty}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\int_{\mathbf{R}}|\tilde{f}(\tau-\tau_1,k_1)||\tilde{f}(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau\\ &= C\sum_{k\neq 0}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\int_{\mathbf{R}}|\tilde{f}(\tau-\tau_1,k_1)||\tilde{f}(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau\\ &\quad +C\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=0}\int_{\mathbf{R}}|\tilde{f}(\tau-\tau_1,k_1)||\tilde{f}(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau\\ &=: I_1+I_2. \end{aligned}$$

By the Schwarz inequality and the Minkowski inequality, we see that when $b > 1/4$,

$$\begin{aligned} I_2 &\le C\int_{\mathbf{R}}\Big[\sum_{k_1=-\infty}^{\infty}\Big\{\Big(\int_{\mathbf{R}}\langle\tau-\tau_1-k_1^2-k_1^4\rangle^{-2b}\\ &\quad\times\langle\tau_1-k_1^2-k_1^4\rangle^{-2b}\,d\tau_1\Big)^{1/2}\\ &\quad\times\Big(\int_{\mathbf{R}}\langle\tau-\tau_1-k_1^2-k_1^4\rangle^{2b}|\tilde{f}(\tau-\tau_1,k_1)|^2\\ &\quad\times\langle\tau_1-k_1^2-k_1^4\rangle^{2b}|\tilde{f}(\tau_1,-k_1)|^2\,d\tau_1\Big)^{1/2}\Big\}\Big]^2\,d\tau\\ &\le C\Big[\sum_{k_1=-\infty}^{\infty}\Big\{\int_{\mathbf{R}}\int_{\mathbf{R}}\langle\tau-\tau_1-k_1^2-k_1^4\rangle^{2b}|\tilde{f}(\tau-\tau_1,k_1)|^2\\ &\quad\times\langle\tau_1-k_1^2-k_1^4\rangle^{2b}|\tilde{f}(\tau_1,-k_1)|^2\,d\tau_1 d\tau\Big\}^{1/2}\Big]^2\\ &\le C\|f\|^4_{Y^{b,0}}. \end{aligned}$$

Next we suppose that

$$\tilde{g}(\tau,k)=\tilde{h}(\tau,k)=0 \quad (\tau\in\mathbf{R},\ k<0).$$

Then, for the estimate of $I_1$, it suffices to show that

$$\sum_{k\neq 0}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\int_{\mathbf{R}}|\tilde{g}(\tau-\tau_1,k_1)||\tilde{h}(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau \le C\|g\|^2_{Y^{b,0}}\|h\|^2_{Y^{b,0}}, \tag{5}$$

$$\sum_{k\neq 0}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\int_{\mathbf{R}}|\tilde{g}(\tau-\tau_1,k_1)||\tilde{h}(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau \le C\|g\|^2_{\bar{Y}^{b,0}}\|h\|^2_{\bar{Y}^{b,0}} \tag{6}$$

for $b > 5/16$. In fact, if we write $f = f_1 + f_2$ with $\tilde{f}_1(\tau,k) = \tilde{f}(\tau,k)$ $(k \ge 0)$ and $\tilde{f}_2(\tau,k) = \tilde{f}(\tau,k)$ $(k < 0)$, the $L^2(\mathbf{T})$ norms of $(f_1)^2$, $f_1f_2$ and $(f_2)^2$ can be evaluated by virtue of the above estimate (5). Because we have by the Parseval identity and the fact that $\tilde{\bar{f}}(\tau,k) = \overline{\tilde{f}}(-\tau,-k)$,

$$\|(f_2)^2\|_{L^2(\mathbf{R}\times\mathbf{T})} = \|(\bar{f}_2)^2\|_{L^2(\mathbf{R}\times\mathbf{T})} = \|(\tilde{f}_2)^- * (\tilde{f}_2)^-\|_{L^2(\mathbf{R}\times\mathbf{T})}, \tag{7}$$

$$\|f_1f_2\|_{L^2(\mathbf{R}\times\mathbf{T})} = \|f_1\bar{f}_2\|_{L^2(\mathbf{R}\times\mathbf{T})} \le \big\||\tilde{f}_1| * |(\tilde{f}_2)^-|\big\|_{L^2(\mathbf{R}\times\mathbf{T})}, \tag{8}$$

where $(\tilde{f}_2)^-(\tau,k) = \tilde{f}_2(-\tau,-k)$ and $\tilde{f} * \tilde{g}$ denotes the convolution in both $\tau$ and $k$ of $\tilde{f}$ and $\tilde{g}$. Here, we note that if $f \in Y^{b,s}$, then $\mathcal{F}^{-1}(\tilde{f})^-$, $\mathcal{F}^{-1}|(\tilde{f})^-| \in \bar{Y}^{b,s}$, where , $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of $f$. Therefore, the right hand side of (7) can be estimated by (5) and the right hand side of (8) can be estimated by (5) and (6).

We only show the estimate (5), since (6) can be proved in the same wasy as (5). We denote the left hand side of (5) by J and we have by the Schwarz inequality

$$\begin{aligned}J \le C\sum_{\substack{k\in\mathbf{Z}\\k\neq 0}}\int_{\mathbf{R}}\Big(&\sum_{\substack{k_1+k_2=k\\k_1,k_2\ge 0}}\int_{\mathbf{R}}\langle\tau-\tau_1-k_1^2-k_1^4\rangle^{-2b}\\ &\times\langle\tau_1-k_2^2-k_2^4\rangle^{-2b}\,d\tau_1\Big)\\ \times\Big(&\sum_{k_1+k_2=k}\int_{\mathbf{R}}\langle\tau-\tau_1-k_1^2-k_1^4\rangle^{2b}|\tilde{g}(\tau-\tau_1,k_1)|^2\\ &\times\langle\tau_1-k_2^2-k_2^4\rangle^{2b}|\tilde{h}(\tau_1,k_2)|^2\,d\tau_1\Big)\,d\tau\\ \le CM&\|g\|^2_{Y^{b,0}}\|h\|^2_{Y^{b,0}},\end{aligned}$$

where

$$\begin{aligned}M = \sup_{(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{0\})}\Big[&\sum_{\substack{k_1+k_2=k\\k_1,k_2\ge 0}}\int_{\mathbf{R}}\langle\tau-\tau_1-k_1^2-k_1^4\rangle^{-2b}\\ &\times\langle\tau_1-k_2^2-k_2^4\rangle^{-2b}\,d\tau_1\Big].\end{aligned}$$

Consequently, for the proof of (5), it suffices to show that $M < \infty$. A simple computation and Lemma 2.1 yield

$$\begin{aligned} M \le C & \qquad (9) \\ \times \sup_{(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{0\})} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 0}} \big\langle \tau - (k-k_1)^2 - (k-k_1)^4 & \\ + k_1^2 + k_1^4 \big\rangle^{-4b+1+\varepsilon} & \end{aligned}$$

for any $\varepsilon$ with $0 < \varepsilon < 4(b - 1/4)$

For each $(\tau, k) \in \mathbf{R} \times (\mathbf{Z} \setminus \{0\})$, we consider the following algebraic equation with respect to $k_1$, which corresponds to the formula inside the brackets on the right hand side of (9).

$$k\big(4k_1^3 - 6kk_1^2 + 2(k^2+1)k_1 - k(1+k^2)\big) + \tau = 0. \qquad (10)$$

We denote three roots of the algebraic equation (10) with respect to $k_1$ by $\alpha$, $\beta$ and $\gamma$, respectively. Since (10) is a cubic equation, one of the three roots is necessarily real, which is denoted by $\alpha$. If the two other roots are real, we write $\beta$ and $\gamma$ for those real roots. If the two other roots are complex, that is, if $\beta = \bar{\gamma}$ and $\Im\beta \neq 0$, then we simply use the same notation $\beta$ and $\gamma$ for the real part of $\beta$ and $\gamma$. In either case, there exist at most 12 $k_1$'s such that

$$|k_1 - \alpha| < 2, \ |k_1 - \beta| < 2 \text{ or } |k_1 - \gamma| < 2,$$

and we can choose $\eta > 0$ so that for the other $k_1$'s,

$$\begin{aligned} &\big|k_1^3 - \tfrac{3}{2}kk_1^2 + \tfrac{1}{2}(k^2+1)k_1 - \tfrac{1}{4}k(1+k^2) - \tfrac{\tau}{4k}\big| \\ &\quad \ge \big|(k_1-\alpha)(k_1-\beta)(k_1-\gamma)\big| \\ &\quad \ge \eta\langle k_1-\alpha\rangle\langle k_1-\beta\rangle\langle k_1-\gamma\rangle. \end{aligned}$$

On the other hand, the condition $k_1 \ge 0$ and $k - k_1 \ge 0$ implies that $k \ge k_1 \ge 0$. Furthermore, we can choose $\varepsilon > 0$ so small that $4(4b - 1 - \varepsilon) > 1$.

Therefore, the right hand side of (9) is bounded by the following:

$$\begin{aligned}
&C\sup_{(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{0\})}\sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 0\\ k\neq 0}}\frac{1}{\langle k(k_1-\alpha)(k_1-\beta)(k_1-\gamma)\rangle^{4b-1-\varepsilon}}\\
&\leq C\sum_{k_1\in\mathbf{Z}}\frac{1}{\langle(|k_1|+1)(k_1-\alpha)(k_1-\beta)(k_1-\gamma)\rangle^{4b-1-\varepsilon}}\\
&\leq C\Big(12+\sum_{\substack{|k_1-\alpha|\geq 2\\ |k_1-\beta|\geq 2\\ |k_1-\gamma|\geq 2}}\frac{1}{\langle k_1\rangle^{4b-1-\varepsilon}\langle k_1-\alpha\rangle^{4b-1-\varepsilon}\langle k_1-\beta\rangle^{4b-1-\varepsilon}\langle k_1-\gamma\rangle^{4b-1-\varepsilon}}\Big)\\
&\leq C\Big\{12+\Big(\sum_{k_1\in\mathbf{Z}}\frac{1}{\langle k_1\rangle^{4(4b-1-\varepsilon)}}\Big)^{1/4}\Big(\sum_{|k_1-\alpha|\geq 2}\frac{1}{\langle k_1-\alpha\rangle^{4(4b-1-\varepsilon)}}\Big)^{1/4}\\
&\times\Big(\sum_{|k_1-\beta|\geq 2}\frac{1}{\langle k_1-\beta\rangle^{4(4b-1-\varepsilon)}}\Big)^{1/4}\Big(\sum_{|k_1-\gamma|\geq 2}\frac{1}{\langle k_1-\gamma\rangle^{4(4b-1-\varepsilon)}}\Big)^{1/4}\Big\}<\infty,
\end{aligned}$$

since $4(4b-1-\varepsilon)>1$. This inequality shows that $M<\infty$ and so the proof is complete. □

**Remark 2.3.** (i) We use Lemma 2.1 to show (9) in the above proof of Proposition 2.2. Therefore, we need to assume that $b>1/4$, which corresponds to the Sobolev embedding in the time variable: $H^b(\mathbf{R})\subset L^4(\mathbf{R})$ $(b\geq 1/4)$.

(ii) For $H>0$, we consider the Fourier restriction norm $\|\cdot\|_{Z^{s,b}}$ with the Fourier restriction wight $\langle\tau-k^2-k^4\rangle$ replaced by $\langle\tau-k^2-Hk^4\rangle$ in the definition of the norm $\|\cdot\|_{Y^{b,s}}$. This Fourier restriction norm $\|\cdot\|_{Z^{b,s}}$ is related to the following equation (see [5]).

$$i\partial_t u+\partial_x^2 u-H\partial_x^4 u=f,\qquad t\in\mathbf{R},\quad x\in\mathbf{T}. \tag{11}$$

Proposition 2.4 also holds with $\|\cdot\|_{Y^{b,s}}$ replaced by $\|\cdot\|_{Z^{b,s}}$. Because in that case, the algebraic equation(10) is changed to

$$k\big(4Hk_1^3-6Hk_1^2+2(2Hk^2+1)k_1-k(1+Hk^2)\big)+\tau=0,$$

which causes no trouble to the proof of Proposition 2.2.

The following corollary is an immediate consequence of Proposition 2.2.

**Corollary 2.4.** *Let $T>0$ and let $1/2>b>5/16$. Then, we have*

$$\|U(\cdot)u_0\|_{L^4((-T,T)\times\mathbf{T})}\leq CT^{1/2}\mathcal{T}^{-b}\|u_0\|_{L^2(\mathbf{T})},$$

*where $C$ is a positive constant dependent only on $b$.*

**Proof.** Let $\varphi$ be a time cut-off function in $C_0^\infty(\mathbf{R})$ such that $\varphi(t)=1$ for $|t|\le 1$ and $\varphi(t)=0$ for $|t|\ge 2$. We put $\varphi_T(t)=\varphi(t/T)$ for $T>0$. We note that $\varphi_T(t)U(t)u_0\in Y^{b,0}$ for any $b\in\mathbf{R}$, since a simple computation yields

$$\widetilde{\varphi_T U(\cdot)}u_0 = T\hat{\varphi}(T(\tau-k^2-k^4))\hat{u}_0(k),$$

where $\hat{}$ denotes either the Fourier transform in the time variable or the Fourier coefficient in the spatial variable. Furthermore, for $b>0$,

$$\langle\tau\rangle^{2b}=\left(1+T^{-1}|T\tau|\right)^{2b}\le \mathcal{T}^{-2b}\langle T\tau\rangle^{2b}.$$

Therefore, for $1/2>b>0$, we have by the change of variables

$$\begin{aligned}
&\|\varphi_T U(\cdot)u_0\|_{Y^{b,0}}^2\\
&=\textstyle\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\tau-k^2-k^4\rangle^{2b}|T\hat{\varphi}(T(\tau-k^2-k^4))\hat{u}_0(k)|^2\,d\tau\\
&\le\left(\textstyle\sum_{k=-\infty}^{\infty}|\hat{u}_0(k)|^2\right)\left(\int_{-\infty}^{\infty}T\mathcal{T}^{-2b}\langle\tau\rangle^{2b}|\hat{\varphi}(\tau)|^2\,d\tau\right)\\
&\le CT\mathcal{T}^{-2b}\|u_0\|_{L^2(\mathbf{R})}^2.
\end{aligned}$$

Therefore, Proposition 2.2 implies Corollary 2.4. □

We are now in a position to show Theorem 1.1.

**Proof of Theorem 1.1.** Lemma 1.1 without external forceing $f$ is reduced to Corollary 2.4. When $u_0=0$, it is sufficient to prove that

$$\left\|\int_0^t U(t-\tau)f(\tau)\,d\tau\right\|_{L^4((0,T)\times\mathbf{T})}\le CT\mathcal{T}^{-2b}\|f\|_{L^{4/3}((0,T)\times\mathbf{T})},\tag{12}$$

where $C$ is a positive constant dependent only on $T$. Because we can easily prove the estimate (12) on $(-T,0)$ in the same way. From the Christ-Kiselev lemma (see [2] and [9, Lemma 3.1 on page 2179]), it follows that the proof of (12) is reduced to that of the following inequality.

$$\left\|\int_0^T U(t-\tau)f(\tau)\,d\tau\right\|_{L^4((0,T)\times\mathbf{T})}\le CT\mathcal{T}^{-2b}\|f\|_{L^{4/3}((0,T)\times\mathbf{T})},\tag{13}$$

where $C$ is a positive constant dependent only on $T$. Then, Corollay 2.4 yields that

$$\begin{aligned}
&\left\|\textstyle\int_0^T U(t-\tau)f(\tau)\,d\tau\right\|_{L^4(0,T)\times\mathbf{T})}\\
&=\left\|U(t)\textstyle\int_0^T U(-\tau)f(\tau)\,d\tau\right\|_{L^4(0,T)\times\mathbf{T})}\\
&\le CT^{1/2}\mathcal{T}^{-b}\left\|\textstyle\int_0^T U(-\tau)f(\tau)\,d\tau\right\|_{L^2(\mathbf{T})}.
\end{aligned}\tag{14}$$

Furthermore, we have by the Fubini theorem, Hölder's inequality and Corollary 2.4

$$\begin{aligned} \left|\left(\int_0^T U(-\tau)f(\tau)\ d\tau, v\right)\right| &= \left|\int_0^T (f(\tau), U(\tau)v)\ d\tau\right| \\ &\leq \|f\|_{L^{3/4}((0,T)\times\mathbf{T})}\|U(\cdot)v\|_{L^4((0,T)\times\mathbf{T})} \\ &\leq CT^{1/2}\mathcal{T}^{-b}\|f\|_{L^{4/3}((0,T)\times\mathbf{T})}\|v\|_{L^2(\mathbf{T})}, \quad v \in L^2(\mathbf{T}), \end{aligned} \tag{15}$$

where $(\cdot,\cdot)$ denots the $L^2(\mathbf{T})$ saclar product and $C$ is a positive constant dependent only on $b$. Accordingly, inequalities (14), (15) and the duality argument imply (13), which completes the proof of Theorem 1.1. □

**Remark 2.5.** (i) When we use the Christ-Kiselev lemma to derive (12) from (13) in the above proof of Theorem 1.1, we can see explicitly how the right hand side of (12) depends on $T$ and $\mathcal{T}$ (see, e.g., [9, Lemma 3.1 on page 2179]).

(ii) We consider the equation (11) with parameter $H > 0$ instead of (1). In that case, Theorem 1.1 also holds for any $H > 0$, because the introduction of parameter $H$ gives rise to no change in the above proof as long as $H$ is positive (see Remark 2.3 (ii)).

# References


[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schödinger equations, II. The KdV-equation, *Geom. Funct. Anal.*, **3** (1993), 107–156, 209–262.

[2] M. Christ and A. Kiselev, Maximal functions associated to filtrations, *J. Funct. Anal.* , **179** (2001), 409–425.

[3] Y.-M. Fang and M. Grillakis, Existence and uniqueness for Boussinesq type equations on a circle, *Comm. Part. Diff. Eqs.* , **21** (1996), 1253-1277.

[4] Y. Fukumoto and H.K. Moffat, Motion and expansion of a viscous vortex ring. Part I. A higher-order asymptotic formula for the velocity, *J. Fluid Mech.*, **417** (2000), 1–45.

[5] L. G. Garcia, F. Haas, L. P. L. de Oliveira and J. Goedert, Modified Zakharov equations for plasmas with a quntum correction, *Phys. Plasumas*, **12**, 012302 (2005).

[6] J.-C. Jiang, C.-K. Lin and S. Shao, On one dimensional quantaum Zakharov system, preprint, 2013.

[7] C. E. Kenig, G. Ponce and L. Vega, A bilinear estimate with applications to the KdV equation, *J. Amer. Math. Soc.*, **9** (1996), 573–603.

[8] J. Segata, Well-posedness and existence of standing waves for the fourth order nonlinear Schrödinger type equation, *Discr. Cont. Dyn. Syst.*, **27** (2010), 1093–1105.

[9] H. F. Smith and C. D. Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian, *Comm. Part. Diff. Eqs.*, **25** (2000), 2171–2183.

[10] H. Takaoka and Y. Tsutsumi, Well-posedness of the Cauchy problem for the modified KdV equation with periodic boundary condition, *Int. Math. Res. Not.*, **2004**, no. 56, 3009–3040.