数理解析研究所講究録
第 1914 巻 2014 年 121-138

RESONANCES GENERATED BY A HOMOCLINIC CURVE

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ABSTRACT. We study here semiclassical resonances for the Schrödinger operator on $L^2(\mathbb{R}^n)$, when the associated trapped set consists of a hyperbolic fixed point and an associated homoclinic trajectory. Thanks to a new approach for this kind of problem, well-suited for the geometry we consider, we obtain a quantization condition for resonances, and describe precisely their location. Our proof also provides polynomial semiclassical resolvent estimates.

1. INTRODUCTION

We are interested in resonances for Schrödinger operators on $\mathbb{R}^n$

$$P=-h^2\Delta+V(x), \tag{1.1}$$

with a smooth potential V. We consider the semiclassical regime $h\to 0$, and Bohr's correspondance principle asserts that the underlying classical system should play an important role. Indeed, it is now well known that the main role is played by the set of bounded classical trajectories, which is usually referred to as the trapped set. We recall in Section 1.3 below some of the known results for existence and location of resonances related to the nature of trapped set.

In some sense, we study here the simplest configuration for which there is no result so far, namely that where the trapped set consists exactly of a hyperbolic fixed point, and an associated homoclinic trajectory γ. The 1-dimensional case has been studied in [9], and we are interested here in the full n-dimensional setting.

In this short introduction, we would like to emphasize that we have had to develop a new approach to obtain the asymptotics of the resonances, due to the geometry of the problem. Indeed, our results rely heavily on the study of what we call a microlocal Cauchy problem, of the form

$$\begin{cases}(P-E)u=v & \text{microlocally in } \Omega,\\ u=u_- & \text{microlocally near } S_-,\end{cases} \tag{1.2}$$

where Ω is some neighborhood of the trapped set, and S_- is a suitable hypersurface in the incoming region of Ω. As a matter of fact, we prove our main resolvent estimate (and therefore absence of resonances) in some region of the complex plane of energies (see Section 3.2) by establishing a uniqueness result for the above problem. On the other hand, we prove the existence and give the precise location of resonances using the existence of solution to the inhomogeneous microlocal Cauchy problem (see Section 3.3). One may notice also that we obtain that way a Bohr-Sommerfeld like quantization condition for the resonances, which implies in particular that they accumulate on precisely defined curves as $h\to 0$.

For the study of (1.2), one of the main tools is the connection formula for microlocal solutions near a hyperbolic fixed point that we have established in [2]. This result permits us to describe the solution on the outgoing stable manifold associated with the fixed point, in terms of its value on the incoming stable manifold.

This note is a brief review of part of the forthcoming paper [3]. In that paper, we also obtain asymptotics of the resonances for more general trapped set, for example consisting of several singular points, homoclinic or heteroclinic orbits.

The first and third authors acknowledge partial support from the ANR project Nosevol. The second author is supported by a JSPS Grants-in-Aid for Scientific Research.

1.1. **Definition of resonance.** For the potential $V(x)$, we assume

(A1) $V(x)$ is a real-valued smooth function on $\mathbb{R}^n$ and analytic in a sector

$$\mathcal{S} = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| \leq (\tan\theta_0)|\operatorname{Re} x| \text{ and } |\operatorname{Re} x| > C\},$$

for some positive constants $0 < \theta_0 < \frac{\pi}{2}$ and C. Moreover

$$V(x) \longrightarrow 0 \text{ as } |x| \to \infty \text{ in } \mathcal{S}.$$

The opertor P is self-adjoint on $L^2(\mathbb{R}^n)$ and its essential spectrum $\sigma_{\mathrm{ess}}(P)$, is equal to $\mathbb{R}_+$.

First, we define the resonances of P by complex dilation (see [1]), assuming a stronger condition $(\mathrm{A1})_{\mathrm{gl}}$ with

$$\mathcal{S}_{\mathrm{gl}} = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| \leq (\tan\theta_0)\langle \operatorname{Re} x\rangle\},$$

including the whole real space, instead of $\mathcal{S}$ in (A1). For a function analytic in $\mathcal{S}_{\mathrm{gl}}$, let

$$(U_\theta f)(x) := f(e^{i\theta}x)$$

for real small θ, and $P_\theta := U_\theta P U_\theta^{-1} = e^{-2i\theta}\left(-h^2\Delta + e^{2i\theta}V(e^{i\theta}x)\right)$. This new operator P_θ is no longer self-adjoint and $\sigma_{ess}(P_\theta) = e^{-2i\theta}\mathbb{R}_+$. There eventually exist eigenvalues (with finite multiplicities) in the sector $\mathcal{E}_\theta := \{z \in \mathbb{C}\setminus\{0\};\ \arg z \in (-2\theta, 0)\}$. We call these eigenvalues *resonances* and the corresponding eigenfunctions *resonant states*, and we write

$$\operatorname{Res}(P) := \sigma_{\mathrm{disc}}(P_\theta) \cap \mathcal{E}_\theta.$$

Remark that the set $\operatorname{Res}(P)$ is independent of θ in the sense that

$$\operatorname{Res}_{\theta'}(P) \cap \mathcal{E}_\theta = \operatorname{Res}_\theta(P) \quad \text{for } 0 < \theta < \theta'.$$

When we assume only (A1), we replace $e^{i\theta}x$ in the definition of U_θ by $x + i\theta F(x)$, with $F(x) = 0$ in a compact set and $F(x) = x$ for large $|x|$, it corresponds to a complex distortion introduced by Hunziker, see [16].

Equivalently, we can define the resonances of P by showing that the resolvent $(P-E)^{-1}$: $L^2_{\mathrm{comp}}(\mathbb{R}^n) \to L^2_{\mathrm{loc}}(\mathbb{R}^n)$ has a meromorphic extension from the upper half plane to $\mathcal{E}_\theta$ across $(0, \infty)$. We have

$$\chi(P-E)^{-1}\chi = \chi(P_\theta - E)^{-1}\chi \tag{1.3}$$

for any cut-off function χ whose support is in $\{x \in \mathbb{R}^n;\ F(x) = 0\}$. The poles are the resonances and the multiplicity of a resonance is the rank of the projector

$$-\frac{1}{2\pi i}\oint (P-E)^{-1} dE.$$

In particular when $n=1$ and $V(x) \in C^\infty(\mathbb{R})$ is compactly supported, E is a resonance if and only if there exist a non-trivial u and $(\alpha, \beta) \neq (0,0)$ such that

$$\begin{cases} Pu = Eu, \\ u = \begin{cases} \alpha e^{i\sqrt{E}x/h} & x \gg 1 \\ \beta e^{-i\sqrt{E}x/h} & x \ll -1. \end{cases} \end{cases}$$

The function $\Psi(t,x) := e^{-iEt/h}u(x)$ is a time decaying solution of

$$\frac{h}{i}\frac{\partial}{\partial t}\Psi + P\Psi = 0,$$

which behaves like an outgoing wave near the spatial infinity. The decay rate is formally given by $|e^{-iEt/h}| = e^{-|\operatorname{Im} E|t/h}$, hence resonances close to the real axis correspond to resonant states with long life time.

Resonances are defined also as poles of scattering matrix. In fact, in this one-dimensional case, the scattering matrix is a 2×2 matrix expressing the outgoing waves in terms of the incoming ones. In [12], Helffer and Martinez have established the equivalence between the different notions of resonances.

1.2. **Trapped set and resonances.** Let

$$p(x,\xi) = \xi^2 + V(x), \tag{1.4}$$

be the classical Hamiltonian corresponding to the Schrödinger operator (1.1) and

$$H_p := \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi = 2\xi \cdot \partial_x - \nabla_x V \cdot \partial_\xi,$$

the corresponding Hamiltonian vector field. We denote by $\exp(tH_p)(x_0,\xi_0)$ the integral curve of the Hamiltonian vector field and we call it a Hamiltonian curve starting from the point (x_0, ξ_0). Note that the symbol $p(x,\xi)$ is invariant along any Hamiltonian curve.

Now we fix $E_0 > 0$. Let $K(E_0)$ be the set of *trapped trajectories* on the energy surface $p^{-1}(E_0)$:

$$K(E_0) = \{(x,\xi) \in p^{-1}(E_0);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\}.$$

This set is compact (see [10]). There is a close relationship between the semiclassical distribution of resonances near a real energy E_0 and the geometry of $K(E_0)$ of the corresponding classical dynamics.

We will use throughout this report the following notations.

$$\begin{aligned} D(r) &:= \{E \in \mathbb{C};\ |E| < r\}, \\ \operatorname{Box}(c_1, c_2) &:= \{E \in \mathbb{C};\ |\operatorname{Re} E - E_0| < c_1,\ -c_2 < \operatorname{Im} E < 0\}. \end{aligned} \tag{1.5}$$

In the non-trapping case, the following results have been obtained by Briet, Combes, Duclos [5], and Helffer, Sjöstrand [14] in the analytic case, and by Martinez in [17] in the C^∞ case. The Gevrey case has been treated by Rouleux [18].

Theorem 1.1 ([5], [14]). *Assume* $(\mathrm{A1})_{\mathrm{gl}}$ *and that* $K(E_0) = \emptyset$. *Then there exists* $\varepsilon > 0$ *such that* $\operatorname{Res}(P) \cap \operatorname{Box}(\varepsilon, \varepsilon) = \emptyset$ *for sufficiently small* h.

Theorem 1.2 ([17]). *Assume* (A1) *and that* $K(E_0) = \emptyset$. *Then there exists* $\varepsilon > 0$ *such that, for any* $C > 0$, $\operatorname{Res}(P) \cap \operatorname{Box}(\varepsilon, Ch|\log h|) = \emptyset$ *for sufficiently small* h.

1.3. **Some known results about the asymptotic distribution of resonances.** When $K(E_0) \neq \emptyset$, on the contrary, what is the asymptotic distribution of resonances as $h \to 0$ in a complex neighborhood of E_0? Here we recall the most closely related known results.

The first one is about resonances created by a simple closed trajectory and is due to Gérard and Sjöstrand [10].

Theorem 1.3 ([10]). *Assume $(A1)_{\rm gl}$ and $K(E_0)$ consists of a simple closed trajectory whose Poincaré map is hyperbolic (i.e. none of the eigenvalues is of modulus 1). Then, in $\operatorname{Box}(\varepsilon, Ch)$, there is a bijection b_h between $\operatorname{Res}(P)$ (counted with their multiplicity) and the set of roots of the equation*

$$A(E) = 2k\pi h + ih \log \rho(E) - ih \sum_{j=1}^{n-1} \alpha_j \log \theta_j(E), \quad k \in \mathbb{Z}, \ \alpha_j \in \mathbb{N},$$

such that $b_h(E) = E + o(h)$. Here $A(E)$ is the action, $\theta_1, \dots, \theta_{n-1}$ are the eigenvalues of the Poincaré map with modulus > 1 and $\rho(E)$ is some analytic function of E satisfying

$$|\rho(E)| = |\theta_1(E) \cdots \theta_{n-1}(E)|^{-\frac{1}{2}}.$$

The second one is the case where $K(E_0)$ reduces to a point $K(E_0) = \{(x_0, \xi_0)\}$, necessarily with $\xi_0 = 0$. We may assume that $x_0 = 0$. In this case, Sjöstrand [20] has obtained the following result.

Theorem 1.4 ([20]). *Assume $(A1)_{\rm gl}$ and $K(E_0)$ consists of a non-degenerate saddle point $(0,0)$. We may assume , after a linear change of variables, that the Taylor expansion of $V(x)$ is*

$$V(x) = E_0 - \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + \sum_{j=d+1}^{n} \frac{\mu_j^2}{4} x_j^2 + \mathcal{O}(|x|^3) \quad \text{as } x \to 0 \tag{1.6}$$

with $E_0 > 0$, $\lambda_j, \mu_j > 0$. Define the pseudo-resonances $E_{\alpha,\beta}$ by

$$E_{\alpha,\beta} = E_0 + h \sum_{j=d+1}^{n} \mu_j (\beta_j + \frac{1}{2}) - ih \sum_{j=1}^{d} \lambda_j (\alpha_j + \frac{1}{2})$$

for each $(\alpha, \beta) = (\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_n) \in \mathbb{N}^n$. Then for any $C > 0$ such that no pseudo-resonance is on the boundary of $\operatorname{Box}(Ch, Ch)$, there exists a bijection b_h in $\operatorname{Box}(Ch, Ch)$ between $\operatorname{Res}(P)$ and the set of pseudo-resonances such that $b_h(E) = E + \mathcal{O}(h^{3/2})$.

Remark 1.5. *The case $d = n$ was also studied by Briet, Combes and Duclos [6] under a virial condition.*

Remark 1.6. *The case $d = 0$ has been studied in more detail in the so called "well in an island" setting. Especially the imaginary part of resonances is exponentially small and the decay rate is given by the Agmon distance from the well to the sea (see Helffer, Sjöstrand [14] and Lahmar-Benbernou, Martinez with the second author [8]).*

Finally we recall the result in one dimension about the resonances created by a homoclinic trajectory.

Theorem 1.7 ([9]). *Assume $n=1$, (1.6) with $d=1$ and* $(\mathrm{A1})_{\mathrm{gl}}$. *Moreover, we suppose that $K(E_0)=\{(0,0)\}\cup\gamma$, where γ is a homoclinic trajectory associated with $(0,0)$. Then, for any $C>0$, the resonances in $\mathrm{Box}(C\frac{h}{|\log h|},C\frac{h}{|\log h|})$ satisfy*

$$E_k=E_0-\lambda_1\frac{A-(2k+1)\pi h}{|\log h|}-i\frac{\log 2}{2}\lambda_1\frac{h}{|\log h|}+\mathcal{O}\Big(\frac{h}{|\log h|^2}\Big), \tag{1.7}$$

where $A=\int_\gamma \xi\cdot dx$ is the action along the homoclinic curve γ and $k\in\mathbb{Z}$. In particular,

$$\operatorname{Im}E_k=-\frac{\log 2}{2}\lambda_1\frac{h}{|\log h|}+\mathcal{O}\Big(\frac{h}{|\log h|^2}\Big).$$

Our problem is to generalize the last theorem to higher dimensions. Theorem 1.7 together with Theorem 1.4 suggest a lattice structure of resonances, where the first (closest to the real axis) horizontal line has imaginary part of order $-Lh$ with $L=\frac{1}{2}\sum_{j=2}^n\lambda_j$ and the distance between two neighboring resonances is of order $2\pi\lambda_1\frac{h}{|\log h|}$, where the homoclinic trajectory is assumed to be in the x_1-direction.

In the next section, we recall the microlocal study of solutions near a hyperbolic fixed point in [2], more precisely we state without proof the uniqueness and construction theorems of microlocal Cauchy problem (Theorems 2.5 and 2.8).

The main results will be given in the last section. We assume that the trapped set consists of a hyperbolic fixed point and an associated single homoclinic trajectory, which is a transversal intersection of the outgoing and incoming stable manifolds . We first give the quantization condition of resonances which defines pseudo-resonances (Definition 3.3), and then prove the polynomial estimate of the resolvent away from the pseudo-resonances (Theorem 3.6) and the existence of resonances close to each pseudo-resonances (Theorem 3.12).

2. Microlocal study of solutions near a hyperbolic fixed point

In this section, we study the operator (1.1) locally in a neighborhood of the origin, where we assume that the potential is smooth and satisfies:

(A2) The origin $x=0$ is a non-degenerate local maximum of $V(x)$, i.e., after a linear change of variables, we have

$$V(x)=E_0-\sum_{j=1}^n\frac{\lambda_j^2}{4}x_j^2+\mathcal{O}(|x|^3)\quad\text{as } x\to 0,$$

with $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n$.

2.1. **Stable/unstable manifolds.** Under (A2), the origin $(x,\xi)=(0,0)$ is a fixed point of the Hamiltonian vector field H_p. In the above coordinates, the corresponding fundamental matrix

$$F_p:=D_{(0,0)}H_p=\begin{pmatrix}\frac{\partial^2 p}{\partial x\partial\xi} & \frac{\partial^2 p}{\partial\xi^2}\\ -\frac{\partial^2 p}{\partial x^2} & -\frac{\partial^2 p}{\partial\xi\partial x}\end{pmatrix}\Bigg|_{(0,0)}=\begin{pmatrix}0 & 2\,\mathrm{Id}\\ \frac{1}{2}\mathrm{diag}\,(\lambda_j)^2 & 0\end{pmatrix},$$

has n positive eigenvalues $\{\lambda_j\}_{j=1}^n$ and n negative eigenvalues $\{-\lambda_j\}_{j=1}^n$. The eigenspaces $\Lambda_\pm^0$ corresponding to these positive and negative eigenvalues are respectively outgoing and incoming stable manifolds for the quadratic part $p_0 := \xi^2 - \sum \lambda_j^2 x_j^2/4$ of p:

$$\begin{aligned}\Lambda_\pm^0 =&\big\{(x,\xi)\in\mathbb{R}^{2n};\ \exp(tH_{p_0})(x,\xi)\to(0,0) \text{ as } t\to\mp\infty\big\}\\ =&\big\{(x,\xi)\in\mathbb{R}^{2n};\ \xi_j=\pm\frac{\lambda_j}{2}x_j,\ j=1,\dots,n\big\}.\end{aligned}$$

By the stable/unstable manifold theorem, we also have outgoing and incoming stable manifolds for p:

$$\Lambda_\pm=\big\{(x,\xi)\in\mathbb{R}^{2n};\ \exp(tH_p)(x,\xi)\to(0,0) \text{ as } t\to\mp\infty\big\},$$

which are tangent to $\Lambda_\pm^0$ at $(0,0)$. The manifolds $\Lambda_\pm$ are Lagrangian manifolds and can be written near $(0,0)$ as

$$\Lambda_\pm=\Big\{(x,\xi)\in\mathbb{R}^{2n};\ \xi=\frac{\partial\phi_\pm}{\partial x}(x)\Big\},$$

where the generating functions $\phi_\pm(x)$ behave like

$$\phi_\pm(x)=\pm\sum_{j=1}^{n}\frac{\lambda_j}{4}x_j^2+\mathcal{O}\big(|x|^3\big)\quad \text{as } x\to 0. \tag{2.1}$$

2.2. **Microlocal Cauchy problem near a hyperbolic fixed point.** First we recall some microlocal terminology.

Definition 2.1. *Let $\chi(x,\xi)$ be a function in $S(1)$, i.e. smooth and bounded with all its derivatives. The pseudodifferential operator $\chi^w(x,hD)$ with symbol χ is defined, for all $u\in\mathcal{S}(\mathbb{R}^n)$, by*

$$\big(\chi^w(x,hD)u\big)(x)=\frac{1}{(2\pi h)^n}\iint e^{i(x-y)\cdot\xi/h}\chi\Big(\frac{x+y}{2},\xi\Big)u(y)\,dy\,d\xi.$$

Definition 2.2. *Let $u(x,h)\in L^2(\mathbb{R}^n)$ with $\|u\|=\mathcal{O}(h^{-c})$ for some $c\in\mathbb{R}$. We say that $u=0$ microlocally at (x_0,ξ_0) if there exists $\chi(x,\xi)\in S(1)$ with $\chi(x_0,\xi_0)=1$ such that*

$$\big\|\chi^w(x,hD)u\big\|_{L^2(\mathbb{R}^n)}=\mathcal{O}(h^\infty)\qquad \text{as } h\to 0. \tag{2.2}$$

The complement of the set of such points is called frequency set and denoted by FS(u), *see Guillemin and Sternberg* [11].

The frequency set of a function u is a closed set.

Example 2.3. *Let $u(x,h)=a(x,h)e^{i\phi(x)/h}$, where $\phi(x)$ is a real-valued C^∞ function and $a(x,h)\in S(1)$. Then* $\mathrm{FS}(u)\subset\Big\{(x,\xi)\in\mathbb{R}^{2n};\ \xi=\frac{\partial\phi}{\partial x}(x)\Big\}$.

Eventually, we state the theorem of propagation of singularities which was first proved by Hörmander [15] in the classical setting.

Theorem 2.4 (Propagation of singularities). *Let u be a solution to $Pu=Eu$ with $\|u\|\le 1$ and $E=E_0+\mathcal{O}(h)$. Then $\mathrm{FS}(u)\subset p^{-1}(E_0)$. Moreover, for all $(x_0,\xi_0)\in p^{-1}(E_0)$,*

$$(x_0,\xi_0)\in\mathrm{FS}(u)\Longleftrightarrow \forall t\in I,\quad \exp(tH_p)(x_0,\xi_0)\in\mathrm{FS}(u),$$

where $0\in I$ is the maximal interval of existence of $\exp(tH_p)(x_0,\xi_0)$.

Let us come back to our problem. Suppose $V(x)$ is smooth and satisfies (A2) with $E_0 = 0$. For $\varepsilon > 0$ small, let $S^\varepsilon_\pm = \{(x,\xi) \in \Lambda_\pm;\ |x| = \varepsilon\}$.

Let $u_- \in L^2$ with $\|u_-\| \leq 1$ satisfy $(P-E)u_- = 0$ microlocally near S^ε_- (see Definition 2.2). We consider the microlocal Cauchy problem:

$$\begin{cases} (P-E)u = 0 & \text{microlocally near } (0,0), \\ u = u_- & \text{microlocally near } S^\varepsilon_-. \end{cases} \tag{2.3}$$

Remark that the initial surface S^ε_- is transversal to the Hamiltonian flow for sufficiently small ε. Since we want to study quantities associated to the resonances which are non real in general, the spectral parameter z may be complex but in a disc of center 0 with radius bounded with respect to h.

Theorem 2.5 ([2]). *There exists a discrete set $\Gamma(h) \subset \mathbb{C}_-$ satisfying*

(i) for any $C > 0$, $\#(\Gamma(h) \cap D(Ch)) = \mathcal{O}(1)$ as $h \to 0$,

(ii) $\Gamma(h) \subset \{\operatorname{Im} E < -\delta h\}$ for some $\delta > 0$,

such that if

$$\operatorname{dist}(E, \Gamma(h)) > \nu h \tag{2.4}$$

for some $\nu > 0$ independent of h, then, if $u_- = 0$ and u is a corresponding solution of (2.3) with $\|u\| \leq 1$, then $u = 0$ microlocally in a neighborhood of $(0,0)$.

Remark 2.6. *In the analytic category (i.e. the symbol p is analytic near the origin and the notion of C^∞-microsupport is replaced by the analytic microsupport (see Sjöstrand [19]), we have the same theorem with more precision on the set $\Gamma(h)$. In fact, one has*

$$\Gamma(h) = \left\{ -ih\sum_{j=1}^{n} \lambda_j \left(\alpha_j + \frac{1}{2}\right);\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \right\},$$

and, modulo $o(h)$, $E_0 + \Gamma(h)$ is the set of resonances generated by the barrier top (see Theorem 1.4).

Now we state the construction theorem of [2], i.e. we give the asymptotic formula of the solution u to the microlocal Cauchy problem (2.3) microlocally near a point $\rho_+ := (x_+, \xi_+)$ on $\Lambda_+ \setminus \{(0,0)\}$ for a Cauchy datum u_- which we may suppose, thanks to the linearity, to be supported microlocally in a small vicinity of a point $\rho_- := (x_-, \xi_-)$ in S^ε_-.

From Helffer and Sjöstrand [13], the Hamiltonian flows $(x_\pm(t), \xi_\pm(t)) = \exp(tH_p)(\rho_\pm)$ are known to behave as follows.

Lemma 2.7 ([13]). *We have*

$$x_\pm(t) = g_\pm(x_\pm)e^{\pm\lambda_1 t} + o(e^{\pm\lambda_1 t}) \quad \text{as } t \to \mp\infty,$$

where $g_\pm(x_\pm)$ is the x-space projection of an eigenvectors of F_p corresponding to $\pm\lambda_1$.

We assume

$$g_-(x_-) \cdot g_+(x_+) \neq 0. \tag{2.5}$$

It follows, in particular, $g_-(x_-) \neq 0$, and we may assume without loss of generality that the curve $x_-(t)$ is tangent to the positive x_1-axis at the origin.

Theorem 2.8 ([2]). *Assume (2.4) and (2.5). Then the microlocal Cauchy problem (2.3) has a unique solution $u(x,h)$. Microlocally near ρ_+, it has the form*

$$u(x,h) = a_+(x,h)e^{i\phi_+(x)/h}$$

with

$$a_+(x,h) = \frac{h^{\zeta(z)+\frac{1}{2}}}{(2\pi h)^{n/2}} \int_{\mathbb{R}^{n-1}} e^{-i\phi_-(\varepsilon,y')/h} d(x,y';h) u_-(\varepsilon,y')\,dy', \tag{2.6}$$

where

$$E = hz, \quad \zeta(z) := \frac{1}{\lambda_1}(L - iz) \quad \text{and} \quad L := \frac{1}{2}\sum_{j=2}^{n} \lambda_j. \tag{2.7}$$

The principal term of the symbol $d(x,y';h) \in S(1)$ is computed explicitly (see [2]).

If, in particular, u_- is of WKB form

$$u_-(x,h) = a_-(x,h)e^{i\psi(x)/h}, \quad a_- \in S(1),$$

where Λ_- and $\Lambda_\psi := \{(x,\xi);\ \xi = \partial_x\psi(x)\}$ intersect transversally along $\gamma = \exp tH_p(\rho_-)$, then, microlocally near ρ_+, the symbol $a_+(x,h)$ is given by

$$a_+(x_+,h) = J(x_+,x_-;\zeta,h)a_-(x_-,h) + S(h^{\operatorname{Re}\zeta+1-\mu}), \tag{2.8}$$

for all $\mu > 0$, and

$$J(x_+,x_-;\zeta,h) = \sqrt{\frac{\lambda_1}{2\pi}} e^{-\frac{\pi}{4}i} I_{\rho_-\to 0} I_{0\to\rho_+} h^{\zeta(z)} \frac{|g_-(x_-)|\Gamma(\zeta(z)+\frac{1}{2})}{(i\lambda_1 g_+(x_+)\cdot g_-(x_-))^{\zeta(z)+\frac{1}{2}}}. \tag{2.9}$$

Here $I_{0\to\rho_+}$ and $I_{\rho_-\to 0}$ are defined as follows. Let $(x_+(t,y'),\xi_+(t,y'))$ and $(x_-(t,y'),\xi_-(t,y'))$ be parametrizations of Λ_+ and Λ_ψ respectively near γ by Hamiltonian flows such that

$$(x_\pm(0,0),\xi_\pm(0,0)) = (x_\pm,\xi_\pm), \quad \exp(tH_p)(x_\pm(0,y'),\xi_\pm(0,y')) = (x_\pm(t,y'),\xi_\pm(t,y')).$$

Set

$$D_\pm(t,y') := \left|\det\frac{\partial x_\pm(t,y')}{\partial(t,y')}\right|, \quad D_\pm(t) = D_\pm(t,0). \tag{2.10}$$

Then the following limits exist and do not vanish

$$I_{0\to\rho_+} := \lim_{t\to-\infty}\sqrt{\frac{D_+(t)}{D_+(0)}}\, e^{-(\frac{\lambda_1}{2}+L)t}, \quad I_{\rho_-\to 0} := \lim_{t\to+\infty}\sqrt{\frac{D_-(0)}{D_-(t)}}\, e^{(-\frac{\lambda_1}{2}+L)t}. \tag{2.11}$$

Remark 2.9. *In (2.9), the term $(i\lambda_1 g_+(x_+)\cdot g_-(x_-))^{\zeta+\frac{1}{2}}$ should be interpreted as*

$$\exp\Big(\frac{\pi}{2}i\Big(\zeta+\frac{1}{2}\Big)\operatorname{sgn}(g_+\cdot g_-)\Big)|\lambda_1 g_+\cdot g_-|^{\zeta+\frac{1}{2}}.$$

Remark 2.10. *The quantities $I_{0\to\rho_+}$ and $I_{\rho_-\to 0}$ come respectively from the transport equation for the principal symbol*

$$2\partial\phi_+\cdot\partial a_+ + (\Delta\phi_+ - iz)a_+ = 0,$$
$$2\partial\psi\cdot\partial a_- + (\Delta\psi - iz)a_- = 0,$$

and they can also be given by

$$I_{0\to\rho_+} = \exp\Big(- \int_{-\infty}^{0} \Big(\Delta\phi_+(x_+(t)) - \sum_{j=1}^{n} \frac{\lambda_j}{2}\Big) dt\Big), \tag{2.12}$$

$$I_{\rho_-\to 0} = \exp\Big(- \int_{0}^{\infty} \Big(\Delta\psi(x_-(t)) - \sum_{j=1}^{n} \frac{\lambda_j}{2} + \lambda_1\Big) dt\Big). \tag{2.13}$$

The existence of the limit in (2.11) is a consequence of the behavior of the generating functions (2.1) and

$$\psi(x) = \frac{\lambda_1}{4}x_1^2 - \sum_{j=2}^{n} \frac{\lambda_j}{4}x_j^2 + \mathcal{O}(|x|^3) \quad \text{as } x \to 0,$$

and Lemma 2.7.

Brief derivation of J:

Here, we explain very briefly the idea of the latter part of Theorem 2.8 where the initial datum is of WKB form.

The main idea goes back to Helffer and Sjöstrand (see [13]). It consists in expressing the solution u microlocally near the fixed point $(0,0)$ as a superposition of WKB solutions to the time-dependent Schrödinger equation:

$$u(x,h) = \frac{1}{\sqrt{2\pi h}} \int_0^\infty e^{i\varphi(t,x)/h} a(t,x;h)\, dt. \tag{2.14}$$

The phase $\varphi(t,x)$ and the principal term $a_0(t,x)$ of the symbol $a(t,x;h) \sim \sum_{\ell=0}^{\infty} a_\ell(t,x)h^\ell$ satisfy respectively the eikonal and transport equations.

$$\partial_t\varphi + p(x, \partial_x\varphi) = 0,$$

$$\partial_t a_0 + \partial_x\varphi \cdot \partial_x a_0 + (\Delta\varphi - iz)a_0 = 0.$$

They should be solved in such a way that u has the prescribed asymptotic expansion $a_- e^{i\psi/h}$ near ρ_-. Then, they have the following asymptotic expansions as $t \to +\infty$ with all its derivatives (called *expandible* in [13]):

$$\varphi(t,x) \sim \phi_+(x) + \sum_{k=1}^{\infty} \phi_k(t,x)e^{-\mu_k t}, \quad a_0(t,x) \sim e^{-(\frac{1}{2}\sum\lambda_j - iz)t} \sum_{k=0}^{\infty} a_{0,k}(t,x)e^{-\mu_k t},$$

where $0 = \mu_0 < \mu_1 < \cdots < \mu_k < \cdots$ is a strictly increasing sequence of linear combinations of $\{\lambda_j\}_{j=1}^n$ over $\mathbb{N}$, and $\phi_k(t,x)$ and $a_{0,k}(t,x)$ are polynomials in t. In particular, ϕ_1 and $a_{0,0}$ are independent of t and

$$\phi_1(x) = -\lambda_1 g_-(x_-) \cdot x + \mathcal{O}(|x|^2). \tag{2.15}$$

Let us now roughly compute the contribution from $t = \infty$ of the integral (2.14). Recalling that $\frac{1}{2}\sum\lambda_j - iz = \lambda_1(\zeta + \frac{1}{2})$, one obtains

$$a_+(x,h) \sim \frac{a_{0,0}(x)}{\sqrt{2\pi h}} \int^\infty e^{\frac{i}{h}\phi_1(x)e^{-\lambda_1 t}} e^{-(\zeta+\frac{1}{2})t} dt,$$

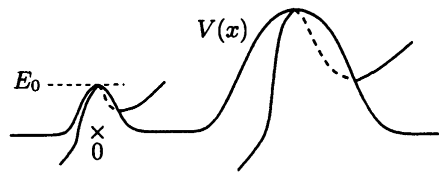

FIGURE 1. An example of potential considered in Section 3.

and, after the change of variable $s = e^{-\lambda_1 t}$, the integral behaves like

$$\frac{1}{\lambda_1}\int_0 e^{\frac{i}{h}\phi_1 s} s^{\zeta-\frac{1}{2}} ds \sim \frac{1}{\lambda_1}\Big(\frac{ih}{\phi_1}\Big)^{\zeta+\frac{1}{2}}\Gamma(\zeta+\frac{1}{2}) \quad \text{as } h \to 0.$$

The right hand side with (2.15) gives a part of J in (2.9) which will play an important role in the asymptotic distribution of resonances. Here we also see that the assumption (2.5) is essential.

3. Quantization of resonances

We come back to the global problem of resonances. Let $\widetilde{\Lambda}_\pm$ be the global evolution by the Hamiltonian flow of $\Lambda_\pm$ which was defined locally near the fixed point $(0,0)$. We assume

(A3) The trapped set is

$$K(E_0) = \{(0,0)\} \cup \gamma,$$

where $\gamma \subset \widetilde{\Lambda}_+ \cap \widetilde{\Lambda}_- = \{(0,0)\} \cup \gamma$ is a single homoclinic trajectory associated with the fixed point $(0,0)$. Moreover we suppose that the intersection is transverse, i.e. $T_\rho\widetilde{\Lambda}_+ \cap T_\rho\widetilde{\Lambda}_- = T_\rho\gamma$ at any point $\rho \in \gamma$.

Fix $\rho_- = (x_-, \xi_-) \in \gamma \cap \Lambda_-$ sufficiently close to $(0,0)$ and set $g_\pm(x_-) =: g_\pm$ i.e.

$$\Pi_x \exp(tH_p)(\rho_-) \sim g_\pm e^{\pm\lambda_1 t} \quad \text{as } t \to \mp\infty.$$

Moreover, we assume

(A4) $g_+ \cdot g_- \neq 0$.

Figure 1 provides a simple example of potential satisfying the assumptions (A1)–(A4)

Remark 3.1. *Since there is only one homoclinic trajectory, and since the Hamiltonian* $p(x,\xi) = \xi^2 + V(x)$ *is even with respect to* ξ, g_+ *is collinear to* g_- *and* (A4) *reduces to* $g_- \neq 0$.

3.1. **Quantization along a homoclinic trajectory.** First, we setup notations for some classical quantities. We recall that $\zeta := \zeta(z) = \frac{1}{\lambda_1}(L - iz)$, $L = \frac{1}{2}\sum_{j=2}^n \lambda_j$ and $E = E_0 + hz$. Let also

$$A := \int_\gamma \xi \cdot dx \tag{3.1}$$

be the classical action along γ, and

$$m(\zeta) := \frac{\Gamma(\zeta+\frac{1}{2})}{(i\lambda_1 g_+ \cdot g_-)^{\zeta+\frac{1}{2}}}\sqrt{\frac{\lambda_1}{2\pi}}e^{-\frac{\pi}{2}(\nu+\frac{1}{2})i}|g_-|\mathcal{I}_{0\to\rho_-}\mathcal{I}_{\rho_-\to 0}, \tag{3.2}$$

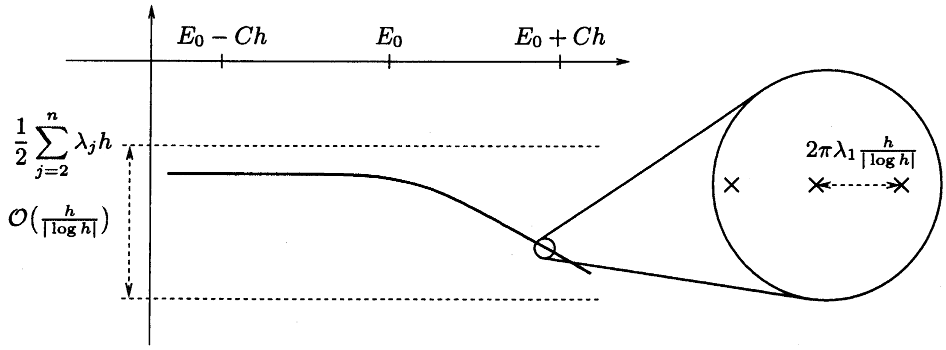

FIGURE 2. The asymptotic description of the pseudo-resonances. At the macroscopic scale h, they accumulate on a curve whose behaviour is governed by the trapping situations at energies lower and higher than E_0. At the microscopic scale $\frac{h}{|\log h|}$, they are regularly distributed on a horizontal line.

be the coefficient which measures the amplification after a complete turn around γ. Here ν is the Maslov index of $\widetilde{\Lambda}_+$ along γ, and the quantities $\mathcal{I}_{0\to\rho_-}$ and $\mathcal{I}_{\rho_-\to 0}$ are defined by

$$\mathcal{I}_{0\to\rho_-} = \lim_{t\to-\infty} \frac{\sqrt{D(t)}}{\sqrt{D(0)}} e^{-(\frac{\lambda_1}{2}+L)t}, \qquad \mathcal{I}_{\rho_-\to 0} = \lim_{t\to+\infty} \frac{\sqrt{D(0)}}{\sqrt{D(t)}} e^{(-\frac{\lambda_1}{2}+L)t},$$

where

$$D(t) := D(t,0), \qquad D(t,y') := \left| \det \frac{\partial x(t,y')}{\partial(t,y')} \right|, \tag{3.3}$$

and $(x,\xi) = (x(t,y'),\xi(t,y'))$ is a parametrization of $\widetilde{\Lambda}_+$ near γ such that

$$(x(0,0),\xi(0,0)) = (x_-,\xi_-), \qquad \exp(tH_p)(x(0,y'),\xi(0,y')) = (x(t,y'),\xi(t,y')).$$

Remark 3.2. *The function $m(\zeta)$ is independent of the choice of the point $\rho_- \in \gamma$. In fact, if $\rho_-' = \exp(\tau H_p)(\rho_-) \in \gamma$, then since $g_+(\rho_-') = e^{\lambda_1\tau} g_+(\rho_-)$, $g_-(\rho_-') = e^{-\lambda_1\tau} g_-(\rho_-)$,*

$$\mathcal{I}_{0\to\rho_-'} = e^{(\frac{\lambda_1}{2}+L)\tau}\mathcal{I}_{0\to\rho_-}, \qquad \mathcal{I}_{\rho_-'\to 0} = e^{-(-\frac{\lambda_1}{2}+L)\tau}\mathcal{I}_{0\to\rho_-},$$

one can check that $|g_-|\mathcal{I}_{0\to\rho_-}\mathcal{I}_{\rho_-\to 0}$ and $g_+ \cdot g_-$ are both independent of ρ_-.

Definition 3.3 (Quantization condition). *The set of pseudo-resonances is*

$$\widetilde{\operatorname{Res}}(P) := \{E \in \mathbb{C};\ h^{\zeta} m(\zeta) e^{iA/h} = 1\},$$

where $\zeta = \zeta(\frac{E-E_0}{h})$.

The following proposition describes the semiclassical distribution of the pseudo-resonances. The setting is illustrated in Figure 2.

Proposition 3.4. *Let $\sigma \in \mathbb{R}$. The pseudo-resonances E, such that $\operatorname{Re} E = E_0 + \sigma h + o(h)$, are given by*

$$E = E_0 + \lambda_1\Big(2k\pi - \frac{A}{h}\Big)\frac{h}{|\log h|} - i\Big(Lh - \lambda_1 \log m\Big(-i\frac{\sigma}{\lambda_1}\Big)\frac{h}{|\log h|}\Big) + o\Big(\frac{h}{|\log h|}\Big), \tag{3.4}$$

for some $k \in \mathbb{Z}$.

We will prove (see Theorem 3.12 below) that the asymptotic of the resonances is given by the pseudo-resonances.

Remark 3.5. *(i) As it can be seen in Figure 2, Proposition 3.4 implies that the pseudo-resonances accumulate on the curve*

$$\operatorname{Im} E = -Lh + \lambda_1 \log \Big| m\Big(-i \frac{\operatorname{Re} E - E_0}{\lambda_1 h}\Big)\Big| \frac{h}{|\log h|}. \tag{3.5}$$

At the same time, two consecutive pseudo-resonances on this curve differ from each over by the almost real and constant quantity

$$2\pi\lambda_1 \frac{h}{|\log h|} + o\Big(\frac{h}{|\log h|}\Big).$$

(ii) Since the asymptotic of the pseudo-resonances given in Proposition 3.4 is valid in $[E_0 - Ch, E_0 + Ch]$ *for any fixed positive constant* C*, it is natural to study the behaviour of the curve* (3.5) *when* $(\operatorname{Re} E - E_0)/h$ *goes to* $\pm\infty$*. Using* $g_+ \cdot g_- > 0$ *and the formula*

$$\Big|\Gamma\Big(\frac{1}{2} - i\sigma\Big)\Big| = \sqrt{\frac{\pi}{\cosh(\pi\sigma)}},$$

we deduce from (3.2) *the asymptotic*

$$|m(-i\sigma)| \sim \frac{|g_-|}{\sqrt{g_+ \cdot g_-}} \mathcal{I}_{0\to\rho_-} \mathcal{I}_{\rho_-\to 0} \begin{cases} e^{-\pi\sigma} & \text{as } \sigma \to +\infty, \\ 1 & \text{as } \sigma \to -\infty. \end{cases}$$

Thus, the concentration curve (3.5) *satisfies*

$$\operatorname{Im} E = -Lh + \frac{h}{|\log h|} \begin{cases} K - \pi \dfrac{\operatorname{Re} E - E_0}{h} + o(1) & \text{as } \dfrac{\operatorname{Re} E - E_0}{h} \to +\infty, \\ K + o(1) & \text{as } \dfrac{\operatorname{Re} E - E_0}{h} \to -\infty, \end{cases} \tag{3.6}$$

where

$$K := \lambda_1 \log \Big(\frac{|g_-|}{\sqrt{g_+ \cdot g_-}} \mathcal{I}_{0\to\rho_-} \mathcal{I}_{\rho_-\to 0} \Big).$$

This estimate illustrates the transition from the geometric situation below E_0 *(which is generally trapping) to the geometric situation above* E_0 *(which is generally non-trapping). Such a behaviour is consistent with the intuition that the homoclinic regimes are generally instable.*

(iii) In the 1-dimensional case, (3.4) *coincides with the asymptotics given in (1.7) for the resonances close to* E_0*. Indeed, when* $n = 1$ *and* $\sigma = o(1)$ *as* $h \to 0$*, one has* $L = 0$*,* $\zeta = o(1)$ *and it turns out that* $m(0) = -2^{-1/2}$*.*

3.2. **Polynomial resolvent estimate away from the pseudo-resonances.** For positive C and ε, we set

$$B_1(C, \varepsilon) := \operatorname{Box}(Ch, (L + \lambda_1 - \varepsilon)h) \setminus (\Gamma(h) + D(\varepsilon h)).$$

Recall that $\Gamma(h)$ is the exceptional set for which the uniqueness of the microlocal Cauchy problem near $(0, 0)$ fails to hold (see Theorem 2.5).

Theorem 3.6 (Polynomial estimate away from the pseudo-resonances). *For any $C > 0$ and $\varepsilon > 0$, there exists $M > 0$ such that if $E \in B_1(C, \varepsilon) \setminus (\widetilde{\operatorname{Res}(P)} + D(\frac{\varepsilon h}{|\log h|}))$, then, for $\theta = h|\log h|$, we have*

$$(P_\theta - E)^{-1} = \mathcal{O}(h^{-M}).$$

Recall that P_θ is the complex distorted operator obtained from P as in Section 1.1.

Proof. We prove this theorem by a contradiction argument (see Burq [7] for the limiting absorption principle). We assume that there exist $\varepsilon > 0$, $C > 0$, $E \in B_1(C, \varepsilon) \setminus \{\widetilde{\operatorname{Res}(P)} + D(\frac{\varepsilon h}{|\log h|})\}$, and $u \in L^2$ with $\|u\| = 1$ satisfying

$$(P_\theta - E)u = \mathcal{O}(h^\infty), \tag{3.7}$$

and we deduce $\|u\| = \mathcal{O}(h^\infty)$, which is a contradiction.

Step 1.

We first reduce the problem to a neighborhood of the trapped set.

Lemma 3.7. *If $u = 0$ microlocally at each point of $K(E_0)$, then $\|u\| = \mathcal{O}(h^\infty)$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi = 1$ near E_0. $P_\theta - E \approx P - E$ is elliptic outside $p^{-1}(E_0)$, we have

$$u = \varphi(P)u + \mathcal{O}(h^\infty).$$

Next, for (x, ξ) close to $p^{-1}(E_0)$, $-\operatorname{Im} p_\theta(x, \xi) \approx 2\theta E_0$ for large x and $\mathcal{O}(\theta)$ elsewhere. Hence, for $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ on a compact set, we get

$$\mathcal{O}(h^\infty) = -\operatorname{Im} \langle (P_\theta - E)u, u\rangle \geq \theta \|(1 - \chi)u\|^2 - \mathcal{O}(\theta)\|\chi u\|^2.$$

Then

$$\|(1 - \chi)u\| \leq C\|\chi u\| + \mathcal{O}(h^\infty).$$

On the other hand, for (x, ξ) close to $p^{-1}(E_0)$ with x in a compact set, the Hamiltonian flow $\exp(tH_p)(x, \xi)$ either diverges to ∞ or approaches the trapped set as $t \to -\infty$. In the former case, $\exp(tH_p)(x, \xi)$ belongs to the incoming region for negative large t. Now, a theorem of Michel and the first author [4] implies that the quasi mode u is microlocally 0 at such a point. Hence $u = 0$ microlocally at (x, ξ) by the propagation of singularities (Theorem 2.4). In the other case, if $\exp(tH_p)(x, \xi)$ approaches the trapped set as $t \to -\infty$, we obtain also $u = 0$ microlocally at (x, ξ) using the assumption of the lemma and propagation of singularities. □

Notice that at this point, we have not used the particular form of $K(E_0)$.

Step 2.

We now prove

Lemma 3.8. *$u = 0$ microlocally near each point of $\Lambda_- \setminus (\gamma \cup (0, 0))$.*

Proof. Let $\rho \in \Lambda_- \setminus (\gamma \cup \{(0, 0)\})$. Since $\rho \notin K(E_0)$ and $\exp(tH_p)(\rho) \to (0, 0)$ as $t \to +\infty$, then $\exp(tH_p)(\rho)$ should diverge to ∞ as $t \to -\infty$. Hence again by the theorem of Michel and the first author, and propagation of singularity, we get the lemma. □

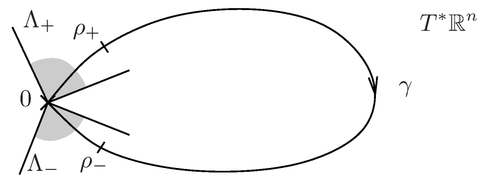

FIGURE 3. The geometrical setting of Section 3.

Step 3.
On Λ_-, u is supported only on $\gamma \cup \{(0,0)\}$, and hence, by (A4), we can apply the first part of Theorem 2.8 for (x,ξ) near each point $\rho_+ = (x_+,\xi_+) \in \Lambda_+ \cap \gamma$. In particular, u is of the form

$$u(x,h) = a_+(x,h)e^{i\phi_+(x)/h} \quad \text{microlocally near } \rho_+,$$

with $a_+(x,h) \in S(h^{-c})$ for some constant $c > 0$. The standard Maslov theory tells us that, after continuation along γ, where p is of real principal type, the solution u is always a Lagrangian distribution, and in particular of the form

$$u(x,h) = a_-(x,h)e^{i\tilde{\phi}_+(x)/h} \quad \text{microlocally near } \rho_- \tag{3.8}$$

with $a_-(x,h) \in S(h^{-c})$. Here $\tilde{\phi}_+$ is the generating function of the evolution of Λ_+ near ρ_-, vanishing at the origin:

$$\widetilde{\Lambda}_+ = \{(x,\xi);\ \xi = \partial_x\tilde{\phi}_+(x)\}, \quad \tilde{\phi}_+(0) = 0.$$

Step 4.
Now starting from the expression (3.8), we continue u one more tour along $\gamma \cup \{(0,0)\}$ using the latter part of Theorem 2.8 and the Maslov theory. Then we obtain another expression of u microlocally near $\rho_- := (x_-,\xi_-)$:

$$u(x,h) = \tilde{a}_-(x,h)e^{i\tilde{\phi}_+(x)/h} \quad \text{microlocally near } \rho_-. \tag{3.9}$$

At the "principal level", the value $\tilde{a}_-(x,h)$ is determined by the value $a(x_-,h)$. More precisely

Lemma 3.9. *We have*

$$\tilde{a}_-(x,h) = \mathcal{M}(x,x_-;\zeta,h)a_-(x_-,h) + S(h^{\operatorname{Re}\zeta - c + 1 - \mu}),$$

where $\mathcal{M}(x,x_-;\zeta,h) \in S(h^{\operatorname{Re}\zeta})$ *is a symbol satisfying*

$$\mathcal{M}(x_-,x_-;\zeta,h) = h^{\zeta}m(\zeta)e^{iA/h}. \tag{3.10}$$

Here A and m are given by (3.1) and (3.2), and μ is any positive constant.

Proof. By Theorem 2.8, we have, for each point $\rho' = (x',\xi') \in \Lambda_+$ with $g_+(x') \cdot g_-(x_-) \neq 0$,

$$u(x',h) = a_+(x',h)e^{i\phi_+(x)/h},$$

with

$$a_+(x',h) = J(x',x_-;\zeta,h)a_-(x_-,h) + S(h^{\operatorname{Re}\zeta - c + 1 - \mu}). \tag{3.11}$$

On the other hand, by Maslov theory, $\tilde{a}(x,h)$ can be expressed by the value $a_+(x,h)$ when $\rho=(x,\xi)=\exp(TH_p)(\rho')$ with $\rho'=(x',\xi')\in\Lambda_+$:

$$\tilde{a}_-(x,h)=e^{iA/h}e^{-\frac{\pi}{2}\nu i}e^{izT}\sqrt{\frac{D(t_{\rho'},y')}{D(t_\rho,y')}}\Big|_{y'=y'_0}\,a_+(x',h)+S(h^{\mathrm{Re}\,\zeta-c+1}),\tag{3.12}$$

where $D(t,y)$ was defined by (3.3) and $t_\rho, t_{\rho'}$ and y'_0 are such that $\rho=(x(t_\rho,y'_0),\xi(t_\rho,y'_0))$ and $\rho'=(x(t_{\rho'},y'_0),\xi(t_{\rho'},y'_0))$.

To obtain $\mathcal{M}(x_-,x_-;\zeta,h)$, we compute the product of (3.11) and (3.12). Taking into account the different parametrization by time on the curve γ. We have $g_+(\rho')=g_+(\rho_-)e^{-\lambda_1 T}$, and

$$I_{0\to\rho'}=\mathcal{I}_{0\to\rho'}e^{-(\frac{\lambda_1}{2}+L)T},\qquad I_{\rho_-\to 0}=\mathcal{I}_{\rho_-\to 0}$$

and finally

$$\sqrt{\frac{D(t_{\rho'},y')}{D(t_{\rho_-},y')}}\Big|_{y'=y'_0}=\mathcal{I}_{\rho'\to\rho_-}.$$

Then we get

$$\mathcal{M}(x_-,x_-;\zeta,h)=h^\zeta e^{iA/h}\frac{\Gamma(\zeta+\frac{1}{2})}{(i\lambda_1 g_+\cdot g_-)^{\zeta+\frac{1}{2}}}\sqrt{\frac{\lambda_1}{2\pi}}e^{-\frac{\pi}{2}(\nu+\frac{1}{2})i}|g_-|$$

$$\times\,\mathcal{I}_{0\to\rho'}\,\mathcal{I}_{\rho'\to\rho_-}\,\mathcal{I}_{\rho_-\to 0}\,\frac{e^{izT}e^{-(\frac{\lambda_1}{2}+L)T}}{e^{-\lambda_1 T(\zeta+\frac{1}{2})}},\tag{3.13}$$

and the lemma follows. □

Step 5.
From (3.8), (3.9) and Lemma 3.9, we obtain, microlocally near ρ_-,

$$a_-(x,h)=\mathcal{M}(x,x_-;\zeta,h)a_-(x_-,h)+S(h^{\mathrm{Re}\,\zeta-c+1-\mu})\tag{3.14}$$

At $x=x_-$, we obtain, using (3.10),

$$a_-(x_-,h)=\frac{\mathcal{O}(h^{\mathrm{Re}\,\zeta-c+1-\mu})}{1-\mathcal{M}(x_-,x_-;\zeta,h)}=\frac{\mathcal{O}(h^{-c+1-\mu})}{h^{-\zeta}-me^{iA/h}}.$$

Since $m\neq 0$, we have the following lemma.

Lemma 3.10. *If E is outside $\widetilde{\mathrm{Res}}(P)+D(\varepsilon\frac{h}{|\log h|})$, then $|h^{-\zeta}-me^{iA/h}|\geq\nu$ for an h-independent positive constant ν.*

Using this fact, we get

$$a_-(x_-,h)=\mathcal{O}(h^{-c+1-\mu}).$$

Substituting this in (3.14), we obtain

$$a_-(x,h)\in S(h^{\mathrm{Re}\,\zeta-c+1-\mu}).$$

On the other hand, $\mathrm{Im}\,E>(-L-\lambda_1+\varepsilon)h$ implies $\mathrm{Re}\,\zeta>-1+\frac{\varepsilon}{\lambda_1}$. Hence

$$a_-(x,h)\in S(h^{-c+\frac{\varepsilon}{\lambda_1}-\mu}).$$

Thus we obtain $a_-\in S(h^{-c+\frac{\varepsilon}{\lambda_1}-\mu})$ from the assumption $a_-\in S(h^{-c})$. Repeating this argument, we conclude that

$$a_-(x,h)\in S(h^\infty).$$

This means that u is microlocally 0 near ρ_-, and a last application of Theorem 2.5 leads us to the conclusion that $u = 0$ microlocally in a neighborhood of $(0,0)$ and hence all over $K(E_0)$ by the propagation of singularities. This implies $\|u\| = \mathcal{O}(h^\infty)$ by Lemma 3.7, which finishes the proof. □

3.3. **Existence of resonances.** In this section, we prove that the set of resonances and the set of pseudo-resonances are close to each other in $B_1(C,\varepsilon)$ in the following sense.

Definition 3.11. *Let A, B and C be subsets in $\mathbb{C}$. We say that*

$$\operatorname{dist}(A,B) \leq \varepsilon \quad \text{in } C,$$

if and only if

$$\forall a \in A \cap C, \quad \exists b \in B \quad \text{s.t. } |a-b| \leq \varepsilon, \text{ and}$$
$$\forall b \in B \cap C, \quad \exists a \in A \quad \text{s.t. } |a-b| \leq \varepsilon.$$

Theorem 3.12 (Asymptotics of resonances). *In $B_1(C,\varepsilon)$, it holds that*

$$\operatorname{dist}\big(\operatorname{Res}(P), \widetilde{\operatorname{Res}}(P)\big) = o\Big(\frac{h}{|\log h|}\Big) \quad \text{as } h \to 0.$$

Proof. The existence of a pseudo-resonance close to each resonance is already proved in Theorem 3.6. We prove the existence of a resonance close to each pseudo-resonance also by a contradiction argument. We assume that there exist $\varepsilon > 0$ and $\widetilde{E} \in \widetilde{\operatorname{Res}}(P) \cap B_1(C,\varepsilon)$ such that there is no resonance in the interior of the circle $\mathcal{C} := \{z \in \mathbb{C};\ |z - \widetilde{E}| = \frac{\varepsilon h}{|\log h|}\}$.

For $E \in \mathcal{C}$, we consider the inhomogeneous equation

$$(P_\theta - E)u = v, \tag{3.15}$$

where v is chosen as follows. Let H_- be a hyperplane containing $x = x_-$ and transversal to $\Pi_x\gamma$, and let $\tilde{v}(x) = \tilde{\chi}(x,h)e^{i\tilde{\phi}_+(x)/h}$ be the WKB solution of the microlocal Cauchy problem

$$\begin{cases} (P-E)\tilde{v} = 0, \\ \tilde{v} = \chi(x)e^{i\tilde{\phi}_+(x)/h} & \text{on } H_-, \end{cases}$$

where $\chi \in C_0^\infty$ and $\chi = 1$ near x_-. We define

$$v(x) = \psi[P,\varphi]\tilde{v},$$

where ψ and φ are smooth cutoff functions with small compact support such that φ is equal to 1 in a neighborhood of x_-, and that ψ is equal to 1 on $\operatorname{supp}[P,\varphi]\tilde{v} \cap \gamma_+$, where $\gamma_+ = \bigcup_{t<0} \exp tH_p(\rho_-)$. Obviously, v is holomorphic in E near E_0.

From Theorem 3.6, the solution $u = (P_\theta - E)^{-1}v$ of (3.15) is well-defined and satisfies $\|u\| = \mathcal{O}(h^{-c})$. Moreover, working as in (3.8), one can prove that u is of the form

$$u = a_-(x,h)e^{i\tilde{\phi}_+(x)/h}, \quad a_- \in S(h^{-c}), \tag{3.16}$$

microlocally near x_-. Thanks to (3.15), $(P-z)u = 0$ microlocally near $(0,0)$. Moreover, on Λ_-, u is supported microlocally on γ. Thus Theorem 2.8 implies that u is of the form

$$u_+(x',h) = a_+(x',h)e^{i\phi_+(x)/h},$$

with

$$a_+(x',h) = J(x',x_-;\zeta,h)a_-(x_-,h) + S(h^{\operatorname{Re}\zeta - c + 1 - \mu}), \tag{3.17}$$

microlocally on $\Lambda_+ \cap \gamma$. On the other hand, along γ, u satisfies the following microlocal Cauchy problem:

$$\begin{cases} (P-E)u = v & \text{microlocally along } \gamma, \\ u = u_+ & \text{microlocally near } S_+^\varepsilon \cap \gamma. \end{cases}$$

The solution of this linear problem can be written as the sum $u_1 + u_2$ of the solutions of the microlocal Cauchy problems

$$\begin{cases} (P-E)u_1 = 0 & \text{along } \gamma, \\ u_1 = u_+ & \text{near } S_+^\varepsilon \cap \gamma, \end{cases} \quad \text{and} \quad \begin{cases} (P-E)u_2 = 0 & \text{along } \gamma, \\ u_2 = 0 & \text{near } S_+^\varepsilon \cap \gamma. \end{cases}$$

As in (3.12), we have

$$u_1(x,h) = \left(e^{iA/h} e^{-\frac{\pi}{2}\nu i} e^{izT} \mathcal{I}_{\rho' \to \rho}\, a_+(x',h) + S(h^{\operatorname{Re}\zeta - c+1})\right) e^{i\tilde{\phi}_+(x)/h}.$$

Moreover, v has been constructed so that $u_2 = \varphi \tilde{v}$ microlocally near ρ_-. Summing up, we get

$$a_-(x,h) = e^{iA/h} e^{-\frac{\pi}{2}\nu i} e^{izT} \mathcal{I}_{\rho' \to \rho} a_+(x',h) + \tilde{\chi}(x,h) + S(h^{\operatorname{Re}\zeta - c+1-\mu}), \tag{3.18}$$

near x_-. Combining (3.17) and (3.18), we obtain

$$a_-(x,h) = \mathcal{M}(x, x_-; \zeta, h) a_-(x_-, h) + \tilde{\chi}(x,h) + S(h^{\operatorname{Re}\zeta - c+1-\mu}). \tag{3.19}$$

Lemma 3.13. $a_- \in S(1)$.

Proof. Substituting $x = x_-$ in (3.19), we have, since $\tilde{\chi}(x_-, h) = \chi(x_-) = 1$,

$$a_-(x_-) = \frac{1 + \mathcal{O}(h^{\operatorname{Re}\zeta - c+1-\mu})}{1 - \mathcal{M}(x_-, x_-; \zeta, h)} = \frac{h^{-\zeta} + \mathcal{O}(h^{-c+1-\mu})}{h^{-\zeta} - me^{iA/h}}$$

For $E \in \mathcal{C}$, one has $(h^{-\zeta} - me^{iA/h})^{-1} = \mathcal{O}(1)$ by Lemma 3.10 and also $h^{-\zeta} = \mathcal{O}(1)$. Hence

$$a_-(x_-) = \mathcal{O}(1) + \mathcal{O}(h^{-c+1-\mu}).$$

Using again (3.19), we obtain

$$a_-(x) = S(1) + S(h^{-c+1-\mu}),$$

under the assumption that $a_- \in S(h^{-c})$. Then we get $a_- \in S(1)$ by a boot-strap argument, and the lemma follows. □

By this lemma, (3.19) becomes

$$a_-(x) = \frac{h^{-\zeta}}{h^{-\zeta} - me^{iA/h}} \tilde{\chi}(x,h) + S(h^{1-\mu}).$$

We integrate this identity with respect to E along the contour $\mathcal{C}$, and we obtain

$$\int_{\mathcal{C}} a_-(x) dE = 2\pi \lambda_1 \frac{h}{\log h} \chi + S(h^{2-\mu}).$$

This is a contradiction because the left hand side should vanish if there is no resonance inside the contour $\mathcal{C}$. □

References

[1] J. Aguilar and J.-M. Combes, *A class of analytic perturbations for one-body Schrödinger Hamiltonians*, Comm. Math. Phys. **22** (1971), 269–279.

[2] J.-F. Bony, S. Fujiié, T. Ramond, and M. Zerzeri, *Microlocal kernel of pseudodifferential operators at a hyperbolic fixed point*, J. Funct. Anal. **252** (2007), no. 1, 68–125.

[3] J.-F. Bony, S. Fujiié, T. Ramond, and M. Zerzeri, *Resonances for homoclinic trapped sets*, in preparation.

[4] J.-F. Bony, L. Michel, *Microlocalization of resonant states and estimates of the residue of the scattering amplitude*, Comm. Math. Phys. 246 (2004), no. 2, 375–402.

[5] P. Briet, J.-M. Combes, and P. Duclos, *On the location of resonances for Schrödinger operators in the semiclassical limit I: Resonances free domains*, J. Math. Anal. Appl. **126** (1987), no. 1, 90–99.

[6] P. Briet, J.-M. Combes, and P. Duclos, *On the location of resonances for Schrödinger operators in the semiclassical limit II: Barrier top resonances*, Comm. in Partial Differential Equations **2** (1987), no. 12, 201–222.

[7] N. Burq, *Semi-classical estimates for the resolvent in non trapping geometries*, Int. Math. Res. Not. **5** (2002), 221–241.

[8] S. Fujiié, A. Lahmar-Benbernou and A. Martinez, *Width of shape resonances for non globally analytic potentials*, J. Math. Soc. Japan Volume 63, Number 1 (2011), 1-78.

[9] S. Fujiié and T. Ramond, *Matrice de scattering et résonances associées à une orbite hétérocline*, Ann. Inst. H. Poincaré Phys. Théor. **69** (1998), no. 1, 31–82.

[10] C. Gérard and J. Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type*, Comm. Math. Phys. **108** (1987),

[11] V. Guillemin and S. Sternberg, *Geometric asymptotics*, Mathematical Surveys, No. 14, American Mathematical Society, Providence, R.I (1977), xviii+474 pp.

[12] B. Helffer and A. Martinez, *Comparaison entre les diverses notions de résonances*, Helv. Phys. Acta **60** (1987), no. 8, 992–1003.

[13] B. Helffer and J. Sjöstrand, *Multiple wells in the semiclassical limit III: Interaction through nonresonant wells*, Math. Nachr. **124** (1985), 263–313.

[14] B. Helffer and J. Sjöstrand, *Résonances en limite semi-classique*, Mém. Soc. Math. France **24-25** (1986), iv+228.

[15] L. Hörmander, *Linear differential operators*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars, 1971, pp. 121–133.

[16] W. Hunziker, *Distortion analyticity and molecular resonance curves*, Ann. Inst. H. Poincaré Phys. Théor. **45** (1986), no. 4, 339–358.

[17] A. Martinez, *Resonance free domains for non globally analytic potentials*, Ann. Henri Poincaré **3** (2002), no. 4, 739–756.

[18] M. Rouleux, *Absence of resonances for semiclassical Schrödinger operators with Gevrey coefficients*, Hokkaido Math. J. Volume 30, Number 3 (2001), 475-517.

[19] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque, vol. 95, Soc. Math. France, 1982, pp. 1–166.

[20] J. Sjöstrand, *Semiclassical resonances generated by nondegenerate critical points*, Pseudodifferential operators, Lecture Notes in Math., vol. 1256, Springer, 1987, pp. 402–429.