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Continuous semigroup structures on $\mathbb{R}$

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Let I be a real interval. A *semigroup* S *on* I is a semigroup $S=(I, *)$ such that the operation $*: I\times I\rightarrow I$ is continuous with respect to the ordinary topology and compatible with the ordinary order in $\mathbb{R}$;

$$x\leq y \Rightarrow x*z\leq y*z, z*x\leq z*y$$

for $x, y.z\in S$.

The following is classical.

Theorem 1 (Abel 1826 [1], Aczél 1949 [2]). *Any group on* $\mathbb{R}$ *is isomorphic to* $(\mathbb{R}, +)$.

Two topological ordered semigroups $(S, *)$ and $(S', *')$ are *equivalent* if there is a homeomorphism $f: S\rightarrow S'$ which is a homomorphism or an anti-homomorphism and is order-preserving or order-reversing.

Question 1. *How many non-equivalent semigroups on* $\mathbb{R}$ *?*

For $a\in \mathbb{R}\cup\{-\infty\}$ and $b\in \mathbb{R}\cup\{+\infty\}$, $I(a, b)$ (resp. $I[a, b]$, $I(a, b]$, $I[a, b)$) denotes the open (resp. closed, half-open) interval between a and b in $\mathbb{R}$.

The following result means that there are exactly three non-equivalent cancellative semigroups on $\mathbb{R}$.

Theorem 2 (Craigen and Pales 1989 [5]). *There are exactly three non-equivalent cancellative semigroups on* $\mathbb{R}_{+}=I(0, +\infty)$. *They are* $(\mathbb{R}_{+}, \times)$, $(\mathbb{R}_{+}, +)$ *and* $(\mathbb{R}_{+}, \star)$, *where* $\star$ *is given by*

$$x\star y=x+y+1.$$

Let $S=(S, *)$ be a semigroup. For $x\in S$ and $n\in \mathbb{N}$, x^{n*} denotes the n-th power of x with respect to $*$. The *order* of x (denoted by $\mathrm{ord}(x)$) is the least n such that $x^{n*}=x^{(n+1)*}$. If there is no such n, $\mathrm{ord}(x)=\infty$.

A semigroup S is *nil* if it has a zero z ($z*x=x*z=0$ for all $x\in S$) and for every $x\in S$ there is $n>0$ such that $x^{n*}=z$.

Let S be an ordered semigroup. An element $x \in S$ is *positive* (resp. *negative, idempotent*) if

$$x * x > x \quad \text{(resp. } x * x < x, \; x * x = x).$$

A subset of S is *positive* (resp. *negative*) if all its elements are positive (resp. negative). Let P (resp. Q, E) denotes the set of positive (resp. negative, idempotent) elements of S. Clearly,

$$S = P \cup Q \cup E \quad \text{(disjoint union).}$$

S is *positively* (resp. *negatively*) *Archimedean*, if it is positive (resp. negative) and for any $x, y \in S$ there is $n > 0$ such that $y < x^{n*}$ (resp. $y > x^{n*}$).

A nil semigroup is *positive* (resp. *negative*), if all elements other than zero are positive (resp. negative).

From now on, $S = (\mathbb{R}, *)$ is a semigroup on $\mathbb{R}$.

Lemma 3. *P and Q are open subsets of $\mathbb{R}$ and E is a closed subset of $\mathbb{R}$.*

Lemma 4. *For $x \in P$ (resp. $x \in Q$), the limit $\lim_{n\to\infty} x^{n*}$ is $+\infty$ (resp. $-\infty$) if $\mathrm{ord}(x) = \infty$, and the limit converges to an idempotent of S if $\mathrm{ord}(x) < \infty$.*

Let $e \in E \cup \{-\infty\}$ and $f \in E \cup \{+\infty\}$ such that $e < f$.

Lemma 5. *$I[e, f] = \{x \in \mathbb{R} \mid e \leq x \leq f\}$ is a subsemigroup of S.*

The open interval $I(e, f)$ is called *tube* if it contains no idempotent.

Lemma 6. *A tube is either positive or negative.*

Proposition 7. *Let $I(e, f)$ be a tube with $e \in E \cup \{-\infty\}$ and $f \in E \cup \{+\infty\}$.*

(1) If it is positive, then either $I(e, f)$ is a positively Archimedean semigroup, or $f \in E$ and $I(e, f) \cup \{f\}$ is a nil semigroup with zero f.

(2) If it is negative, then either $I(e, f)$ is a negatively Archimedean semigroup, or $e \in E$ and $I(e, f) \cup \{e\}$ is a nil semigroup with zero e.

Suppose that $I = I(e, f)$ is positively Archimedean, that is, for any $x, y \in I$ there is $n > 0$ such that $y < x^{n*}$.

For a fixed $a \in S$ define a real function $\phi_a : I \to \mathbb{R}$ by

$$\phi_a(x) = \inf\{m/n \mid m, n \in \mathbb{N}, m > 0, n > 0, x^{n*} \leq a^{m*}\}.$$

for $x \in S$. We call ϕ_a the *standard function based on a*, and is classical for Archimedean ordered semigroups (see Fuchs [6], Hölder [7]).

Theorem 8. *The function ϕ_a is an order-preserving continuous homomorphism of semigroups from $(I, *)$ into $(\mathbb{R}_+, +)$.*

Define

$$\mu_a = \inf\{\, \phi_a(x) \mid x \in I \,\}.$$

Lemma 9. *We have*

$$0 \leq \mu_a \leq 1.$$

Lemma 10. $\mu_a = 0$ *if and only if* $e \neq -\infty$ *or* $\inf\{x * x \mid x \in I\} = -\infty$.

Define
$$\tau = \inf\{x * x \mid x \in I, \phi_a(x) > \mu_a\}.$$

Lemma 11. $\tau = e$ *if and only if* $\mu_a = 0$.

Theorem 12. ϕ_a *is strictly increasing on* $I(\tau, f)$.

Corollary 13. *If* $e \neq -\infty$, *or* $\inf\{x*x \mid x \in I\} = -\infty$, *then* $(I, *)$ *is isomorphic to* $(\mathbb{R}_+, +)$.

More generally, we can characterize Archimedean tubes using Theorems 8 and 12 (a different approach is given in Storey [8]). Using the characterization, we can show that there are uncountably many non-equivalent Archimedean semigroups on $\mathbb{R}$. There are uncountably many non-equivalent nil semigroups on $\mathbb{R}$ too, but a complete characterization of nil semigroups on $\mathbb{R}$ seems to be difficult.

A closed interval $I = I[e, f]$ for $e, f \in E$ with $e \leq f$ is called a *joint* if it is connected, included in E and maximal (no such J strictly includes I). If I is a joint, $(I, *)$ is an idempotent semigroup (band).

$S = (\mathbb{R}, *)$ consists of positive tubes, negative tubes and joints. To classify semigroups on $\mathbb{R}$, we need to describe all possible combinations of them, but it seems a very hard problem. Similar problems for more general structures called threads are studied by Clifford [3, 4]. The following is a certain special result in this context.

An ordered group G with identity element e is *Archimedean*, if $\{x \in G \mid x > e\}$ is a positively Archimedean semigroup and $\{x \in G \mid x < e\}$ is a negatively Archimedean semigroup.

Theorem 14. *Let* $e, f, g \in E \cup \{-\infty, +\infty\}$ *with* $f < e < g$. *Suppose that* $I(e, g)$ *is a positive tube and* $I(f, e)$ *is a negative tube. Then,* $(I(f, g), *)$ *is an Archimedean group with the identity element* e, *and it is isomorphic to* $(\mathbb{R}, +)$.

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