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Triangle Constructive Trice 2

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This is continuation of "Triangle constructive trice 1". We will investigate whether the subset of the algebraic system can be considered to be originally the same algebra system. Consequently, there is a remarkable difference between trices and lattices. We report on it.

1 Introduction

Let $(V,\ \leq)$ be an ordered set, and U be a subset of V. If the original order is preserved, the $(U,\ \leq)$ is an ordered set. That is, any subset of the ordered set is an ordered set. A semilattice $(V,\ *)$ is a set V with a single binary, idempotent, commutative and associative operation $*$. Under the relation defined by $a \leq_* b \Longleftrightarrow a*b=b$, any semilattice $(V,\ *)$ is a partially ordered set $(V,\ \leq_*)$. For $U\subset V$, the $(U,\ \leq_*)$ is ordered set. If there exist $\inf\{x\in U \mid a\leq x \text{ and } b\leq x\}$ for $a, b\in U$, then we define it as $a*b$. This $a*b$ doesn't necessarily exist. That is, U is not necessarily semilattice.

$$\begin{array}{ccc} (V,\ *)\text{: semilattice} & & (U,\ *)\text{: semilattice ?} \\ \Downarrow & & \Uparrow ? \\ (V,\ \leq_*)\text{: ordered set} & \Longrightarrow & (U,\ \leq_*)\text{: ordered set} \end{array}$$

Now, we chiefly think about the set that deletes one point from the algebraic system related to the order. We named this operation **one point deletion**. Let A be a nonempty set, let $F\ (=*_1, *_2, ..., *_n)$ be finitary semilattice operations on A.

$$(A,\ F)\text{: algebra} \quad \Longrightarrow \quad (A-\{b\},\ F)\text{: algebra ?}$$

If $(A-\{b\},\ F)$ is algebra, we call that b **is one-point-deletable** in $(A,\ F)$.

Please see the following figure 1 as an example. Let A be a seven points ordered set and $b\in A$. This $(A,\ \leq)$ is an ordered set and the $(A-\{b\},\ \leq)$ is ordered set too. We can consider $(A,\ *)$ is semilattice, but the $(A - \{b\},\ *)$ is not semilattice. Hence, this b is not one-point-deletable in $(A,\ *)$.

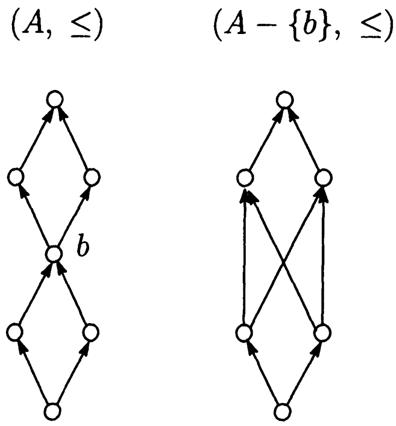


Figure 1:

We observed lattices. There is a lattice that does not become lattice even if which one point is deleted. Please see Figure 2. This doesn't have one-point-deletable point. Such lattices are infinite lattices.

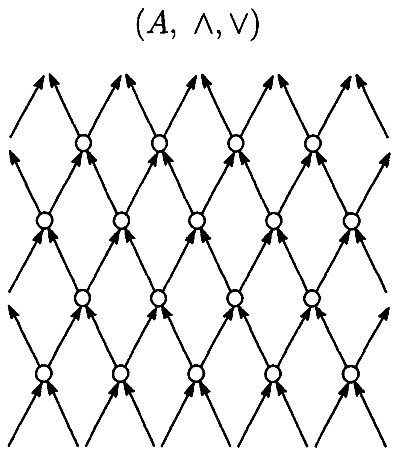
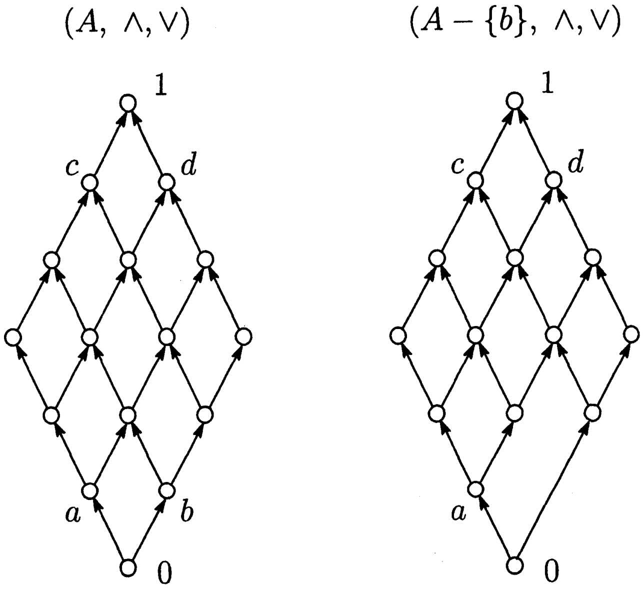


Figure 2: Example of no one-point-deletable point (infinite lattice)

In ordered sets, a covers b or b is covered by a (in notation $a \succ b$ or $b \prec a$) iff $b < a$ and, for no x, $b < x < a$. In bounded lattices (0 is the least element and 1 is greatest element) , an element a is called an atom if $a \succ 0$ and a dual atom, $a \prec 1$. Every atom point and dual atom point in bounded lattice can be one-point-deletable. Finite lattices have an atom point and a dual atom point. Hence,

Proposition 1 *Every finite lattice have one-point-deletable point.*



a and b are atoms, c and d are dual atoms. Hence,
a, b, c and d are one-point-deletable points in $(A, \wedge, \vee)$.

Figure 3: Example of finite lattice

We will consider about trice.

2 Preliminaries of trice

Please see [8] about the definition of the following terms; **n-semilattice, triple-semilattice, n-roundabout-absorption law, trice, sub-triple-semilattice, subtrice, triangular situation, triangle constructive law, triangle constructive trices (t-c trices), triangle natural law, hyper-interval, distributive trice, bounded trice.**
In [8], we prove three fundamental theorems of triangle constructive trices. We write them again here. The first theorem is that two elements trice which composes the distributive trice doesn't exist in a triangle constructive trice at all. The second is a theorem to compose the order reversing operation on triangle constructive trices.

The third is a theorem concerning homomorphism of triangle constructive trices. The homomorphism of two operations leads the homomorphism of another operation in triangle constructive trices.

Theorem 1 *Let T be a t-c trice. For $x, y \in T$ and $m, n \in \{1,2,3\}$ $(m \neq n)$,*

$$\text{if} \quad x \leq_m y \quad \text{and} \quad x \leq_n y \quad \text{then} \quad x = y.$$

Theorem 2 *Let T be a t-c trice. For $m, n, k \in \{1,2,3\}$ $(m \neq n \neq k \neq m)$,*

if $b \leq_m c \leq_m d$ and $d \leq_n c \leq_n b$, then $b \leq_k b *_k c \leq_k b *_k d$ and $b *_k d \leq_n b *_k c \leq_n b$.

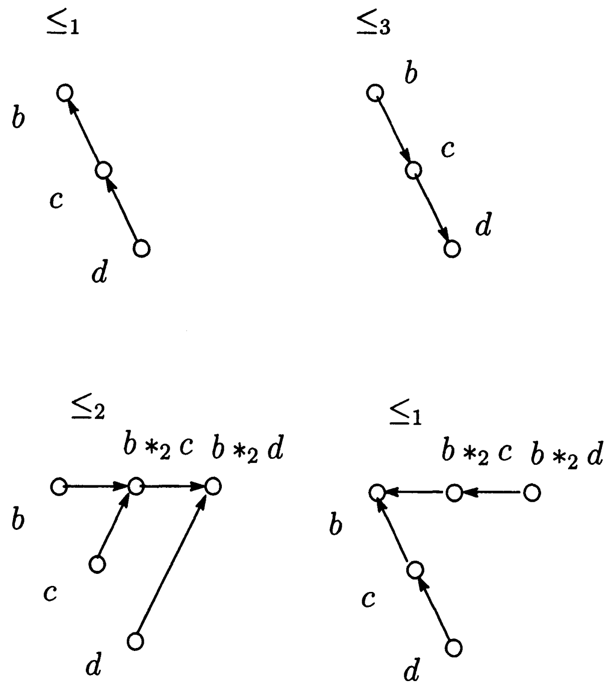


Figure 4: In case of $m = 3$, $n = 1$ and $k = 2$

Theorem 3 *Let T and T' be t-c trices. And let f be a function T to T'. For $m, n, k \in \{1,2,3\}$ $(m \neq n \neq k \neq m)$, if $f(a *_m b) = f(a) *'_m f(b)$ for any $a, b \in T$ and $f(a *_n b) = f(a) *'_n f(b)$ for any $a, b \in T$, then $f(a *_k b) = f(a) *'_k f(b)$ for any $a, b \in T$.*

And, we prepare the following theorem. It was used in "Triangle constructive trice 1",

Theorem 4 *Suppose $(T,\ *_1, *_2, *_3)$ is a bounded t-c trice. Let $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ be the maximum of $\leq_1$, $\leq_2$ and $\leq_3$, respectively. Then,*

$$\mathbf{1} *_3 \mathbf{2} = \mathbf{3} \tag{1}$$
$$\mathbf{2} *_1 \mathbf{3} = \mathbf{1} \tag{2}$$
$$\mathbf{3} *_2 \mathbf{1} = \mathbf{2} \tag{3}$$

that is, $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ is a triangular situation.

3 One point deletion on trices

First of all, the following proposition is clear from Theorem 4.

Lemma 1 *Let $(T, *_1, *_2, *_3)$ be a bounded t-c trice. And $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ be the maximum of $\leq_1$, $\leq_2$ and $\leq_3$, respectively. Then, neither $(T - \{\mathbf{1}\}, \leq_1)$, $(T - \{\mathbf{2}\}, \leq_2)$ nor $(T - \{\mathbf{3}\}, \leq_3)$ are semilattices, respectively.*

For bounded t-c trice $(T, *_1, *_2, *_3)$, $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ are not one-point-deletable. But, we can not remove the condition of triangle constructive from this lemma. Please see the next figure. This is a trice, it is not triangle constructive. The $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ are one-point-deletable.

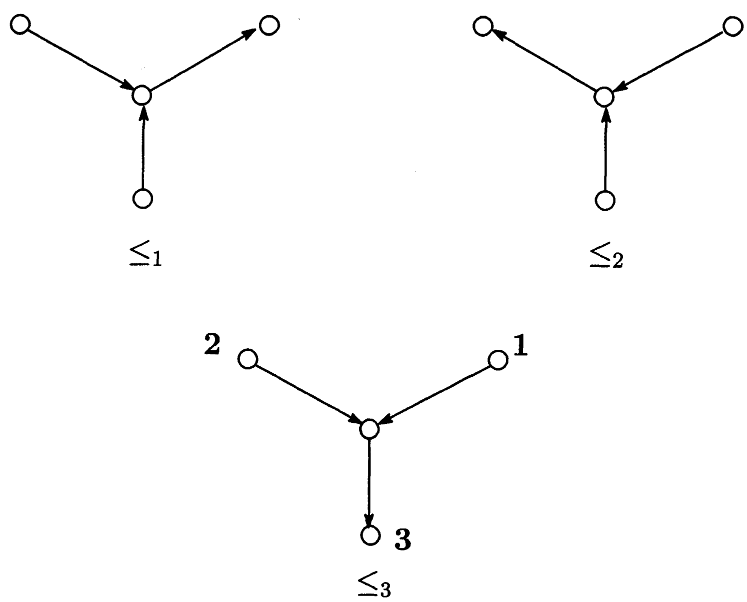


Figure 5: Example of non t-c trice

We can obtain next lemma from Theorem 2.

Lemma 2 *Let $(T, *_1, *_2, *_3)$ be a t-c trice. Suppose that $c \leq_1 b \leq_1 a$ and $a \leq_3 b \leq_3 c$. And let $d = a *_2 b$, $e = b *_2 c$ and $f = a *_2 c$. Then $d *_2 e = f$, $f \leq_1 d$, $f \leq_3 e$ and The (a, d, b), (b, e, c) and (a, f, c) are in a triangular situation, respectively.*

We expect that the b (when $c <_1 b <_1 a$ and $a <_3 b <_3 c$) in lemma 2 is not one-point-deletable. But, it is not easy to prove it directly. We obtain the following by adding the condition.

Lemma 3 *Let $(T, *_1, *_2, *_3)$ be a t-c trice. Suppose that $c <_1 b <_1 a$ and $a <_3 b <_3 c$. In addition, suppose that $b \prec_1 a$ or $b \prec_3 c$. Then $T - \{b\}$ is not a trice.*

We prepare the following lemma 4.

Lemma 4 *Let $(T, *_1, *_2, *_3)$ be a t-c trice. If $b \prec_1 a$, then $a \leq_2 b$ or $a \leq_3 b$.*

After all, the following lemma have been approved.

Lemma 5 *Let $(T, *_1, *_2, *_3)$ be a finite t-c trice. Suppose that $c <_1 b <_1 a$ and $a <_3 b <_3 c$. Then $T - \{b\}$ is not a trice.*

When it is detailed, it is necessary to divide to prove. We can obtain the next.

Lemma 6 *Let $(T, *_1, *_2, *_3)$ be a finite t-c trice. Suppose that $b \neq \mathbf{1}$, $b \neq \mathbf{2}$ and $b \neq \mathbf{3}$. Then $T - \{b\}$ is not a trice.*

We obtain next theorem by lemma 1 and lemma 6.

Theorem 5 *Let $(T, *_1, *_2, *_3)$ be a finite t-c trice. For any point $b \in T$, $T - \{b\}$ is not a trice. That is, all the points of T are not one-point-deletable.*

Please compare this theorem with the proposition 1. This is a remarkable difference between trices and lattices.

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