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Petri Net Morphisms and Codes

静岡理工科大学・総合情報学部
國持良行 (Yoshiyuki Kunimochi)
Faculty of Comprehensive Informatics,
Shizuoka Institute of Science and Technology

abstract The motivation for this work is an extension of an automorphism which we used to introduce to a net in [5][6]. In the extension, we focused on the connection of edges which come in or go out a place. So we give our definition of morphims between Petri nets based on place connectivity and investigate the properties of these morphisms, in particular, surjective morphisms and automorphisms [11, 12, 13]. In this paper we summarize these results and add new results related to ideals of surjective morphisms and families of prefix codes.

In the first chapter, we introduce morphims between two Petri nets. The set S of all morphisms of a Petri net forms a semigroup with zero element and the multiplication expressed by a semi-direct product[11]. The second chapter deals with the pre-order induced by surjective morphisms. Two diamond properties are proved[11]. Moreover, these expression has been revised in terms of ideals in the semigroup $\mathcal{S}$[12]. In the third chapter especially, the group $\mathbf{Aut}(\mathcal{P})$ of all automorphisms of a Petri net $\mathcal{P}$ forms a group. We investigate the inclusion relations among such monoids and groups[11]. Moreover, we investigate the decomposition of automorphism group $G = \mathbf{Aut}(\mathcal{P})$ into $G = KN = NK$, where N is a kind of normal subgroup of G[13]. In the last chapter we show the properties of languages and codes generated by two Petri nets ordered by a surjective morphism. The languages generated by them and their reachability sets have close correspondence to each other [12].

1 Preliminaries

Here we introduce a morphism of a Petri net and show the properties of the semigroup composed of these morphisms. We denote the set of all nonnegative integers by $\boldsymbol{N}_0$.

DEFINITION 1.1 (Petri net) A *Petri net* is a 4-tuple (P, T, W, μ_0) where
(1) $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of *places*,
(2) $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of *transitions*,
(3) $W : \boldsymbol{E}(P,T) \to \{0, 1, 2, 3, \ldots\}$ is a *weight function*, where $\boldsymbol{E}(P,T) = (P \times T) \cup (T \times P)$,
(4) $\mu_0 : P \to \{0, 1, 2, 3, \ldots\}$ is the *initial marking*,
(5) $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.
A Petri net structure (net, for short) $N = (P, T, W)$ without any specific initial marking is denoted by N, a Petri net with a given initial marking μ_0 is denoted by (N, μ_0). □

A Petri net morphism based on place connectivity is introduced in the following way. We denote the set of all positive rational numbers by $\boldsymbol{Q}_+$.

DEFINITION 1.2 Let $\mathcal{P}_1 = (P_1, T_1, W_1, \mu_1)$ and $\mathcal{P}_2 = (P_2, T_2, W_2, \mu_2)$ be Petri nets. Then a triple $(f, (\alpha, \beta))$ of maps is called a *morphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if the maps $f : P_1 \to \boldsymbol{Q}_+$, $\alpha : P_1 \to P_2$ and $\beta : T_1 \to T_2$ satisfy the condition that for any $p \in P_1$ and $t \in T_1$,

$$\begin{aligned} W_2(\alpha(p), \beta(t)) &= f(p) W_1(p, t), \\ W_2(\beta(t), \alpha(p)) &= f(p) W_1(t, p), \\ \mu_2(\alpha(p)) &= f(p) \mu_1(p). \end{aligned} \tag{1.1}$$

In this case we write $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$. □

The morphism $(f,\ (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ is called *injective* (resp. *surjective*) if both α and β are injective (resp. surjective). In particular, it is called an *isomorphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if it is injective and surjective. Then $\mathcal{P}_1$ is said to be *isomorphic* to $\mathcal{P}_2$ and we write $\mathcal{P}_1 \simeq \mathcal{P}_2$. Moreover, in case of $\mathcal{P}_1 = \mathcal{P}_2$, an isomorphism is called an *automorphism* of $\mathcal{P}_1$. By $\mathbf{Aut}(\mathcal{P})$ we denote the set of all the automorphisms of $\mathcal{P}$.

For Petri nets $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ if there exists a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order is regarded as a partial order by identifying isomorphisms.

PROPOSITION 1.1 ([11]) Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be Petri nets. Then,
(1) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_1$.
(2) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_1 \iff \mathcal{P}_1 \simeq \mathcal{P}_2$.
(3) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$ imply $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$. □

We introduce a composition of morphisms; all the morphisms between Petri nets form a semigroup $\mathcal{S}$ under semid composition.

PROPOSITION 1.2 The set ${\boldsymbol{Q}_+}^P$ of all maps from a nonempty set P to $\boldsymbol{Q}_+$ forms a commutative group under the operation $\otimes_P$:

$$(\forall f, g \in {\boldsymbol{Q}_+}^P)[f \otimes_P g : p \mapsto f(p)g(p)].$$

$\mathbf{1}_{\otimes_P} \in {\boldsymbol{Q}_+}^P : p \mapsto 1$ is the identity and $f^{-1} \in {\boldsymbol{Q}_+}^P : p \mapsto 1/f(p)$ is the inverse of $f \in {\boldsymbol{Q}_+}^P$. □

Whenever it does not cause confusion, we write $\otimes$ instead of $\otimes_P$. The composition $f \circ g$ of maps f and g is written by the form gf of multiplication. For example, $(gf)(x) \stackrel{\text{def}}{=} (f \circ g)(x) = f(g(x))$. Immediately we obtain the following lemma.

LEMMA 1.1 ([12]) Let α and β be arbitrary transformations on P and $f, g : P \to \boldsymbol{Q}_+$. Then the following equations are true.

(1) $(\alpha\beta)f = \alpha(\beta f)$.
(2) $\alpha(f \otimes g) = (\alpha f) \otimes (\alpha g)$.
(3) $\alpha \mathbf{1}_\otimes = \mathbf{1}_\otimes$.
(4) $(\alpha f) \otimes (\alpha f^{-1}) = \mathbf{1}_\otimes$.
(5) $(\alpha f)^{-1} = \alpha f^{-1}$. □

For a surjective morphim $x : \mathcal{P}_1 \to \mathcal{P}_2$, $\mathcal{P}_1$ is called the domain of x, denoted by $Dom(x)$, and $\mathcal{P}_2$ is called the image(or range) of x, denoted by $Im(x)$.

PROPOSITION 1.3 ([11]) The set $\mathcal{S}$ of all surjective morphisms between two Petri nets and the zero element 0 forms a semigroup under the following multiplication of $x = (f,\ (\alpha, \beta))$ and $y = (g,\ (\gamma, \delta))$:

$$x \cdot y \stackrel{\text{def}}{=} \begin{cases} (f \otimes_P \alpha g, (\alpha\gamma, \beta\delta)) & \text{if} \quad Im(x) = Dom(y), \\ 0 & \text{if} \quad x = y = 0 \,\text{or}\, Im(x) \neq Dom(y), \end{cases}$$

where P is the set of places in the Petri net $Im(x) = Dom(y)$. □

2 Ideals in the semigroup $\mathcal{S}$

In this section we consider ideals and Green's relations on the semigroup $\mathcal{S}$.

2.1 Green's equivalences on the semigroup $\mathcal{S}$

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a monoid M, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$\begin{aligned} &x\mathcal{L}y \iff Mx = My, \\ &x\mathcal{R}y \iff xM = yM, \\ &x\mathcal{J}y \iff MxM = MyM, \\ &\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \\ &\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*, \end{aligned}$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. Mx (resp. xM) is called the *principal left* (resp. *right*) *ideal generated by* x and MxM the it principal (two-sided) ideal generated by x. Then, the following facts are generally true[2, 1].

FACT 1 *The following relations are true.*

$$\begin{aligned} &(1)\, \mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L} \\ &(2)\, \mathcal{H} \subset \mathcal{L} \ (\textit{resp.}\, \mathcal{R}) \subset \mathcal{D} \subset \mathcal{J} \end{aligned}$$

FACT 2 *An $\mathcal{H}$-class of a monoid M is a group if and only if it contains an idempotent.*

Now we consider the case of $M = \mathcal{S}^1$ in the rest of the manuscript, where $\mathcal{S}^1 = \mathcal{S} \cup \{1\}$ is the monoid obtained from the semigroup $\mathcal{S}$ by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in \mathcal{S}$ and $1 \cdot 1 = 1$.

LEMMA 2.1 *Let $x : \mathcal{P}_1 \to \mathcal{P}_2$, $y : \mathcal{P}_3 \to \mathcal{P}_4 \in \mathcal{S}$. Then,*
(1) $x\mathcal{S} \subset y\mathcal{S} \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \sqsubseteq \mathcal{P}_4$.
(2) $\mathcal{S}x \subset \mathcal{S}y \Longrightarrow \mathcal{P}_1 \sqsubseteq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$.
(3) $x\mathcal{S} = y\mathcal{S} \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \simeq \mathcal{P}_4$.
(4) $\mathcal{S}x = \mathcal{S}y \Longrightarrow \mathcal{P}_1 \simeq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$. □

Note that any reverses of the implications above are not necessarily true.

PROPOSITION 2.1 ([12]) The following conditions are equivalent.
(1) H is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}(\mathcal{P})$ for some Petri net $\mathcal{P}$. □

PROPOSITION 2.2 ([12]) On the monoid $\mathcal{S}^1$, $\mathcal{J} = \mathcal{D}$. □

2.2 Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}$ we find a elements z such that $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$ (resp. $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$). $x\mathcal{S}^1 \cap y\mathcal{S}^1 = \{0\}$ (resp. $\mathcal{S}^1x \cap \mathcal{S}^1y = \{0\}$) is a trivial case(i.e., $z = 0$). We should only consider the non-trivial case.

For two maps $f : X_1 \to Y$ and $g : X_2 \to Y$, the relations $\leq_{\mathrm{R}}$ and $=_{\mathrm{R}}$ are defined by

$$\begin{aligned} f \leq_{\mathrm{R}} g &\overset{\mathrm{def}}{\iff} (\forall y \in Y)(|f^{-1}(y)| \leq |g^{-1}(y)|), \\ f =_{\mathrm{R}} g &\overset{\mathrm{def}}{\iff} (\forall y \in Y)(|f^{-1}(y)| = |g^{-1}(y)|), \end{aligned}$$

respectively.

LEMMA 2.2 Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1, 2, 3)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_3$, $y = (g, (\gamma, \delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. If $\gamma \leq_{\mathrm{R}} \alpha$ and $\delta \leq_{\mathrm{R}} \beta$, then $\mathcal{S}^1x \subset \mathcal{S}^1y$. □

For a map $f : X \to Y$, the equivalence relation $\ker f$ on X is defined by $\ker f \overset{\mathrm{def}}{=} \{(a, b) | f(a) = f(b)\}$.

LEMMA 2.3 Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0,1,2)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_0 \to \mathcal{P}_1$, $y = (g, (\gamma, \delta)) : \mathcal{P}_0 \to \mathcal{P}_2$ be elements of $\mathcal{S}$. If $\ker\gamma \subset \ker\alpha$ and $\ker\delta \subset \ker\beta$, then $x\mathcal{S}^1 \subset y\mathcal{S}^1$. □

Summarizing these two lemmas, we get the following property.

PROPOSITION 2.3 Let $x = (f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$, $y = (g, (\gamma, \delta)) : \mathcal{P}_3 \to \mathcal{P}_4$ be elements of $\mathcal{S}^1$, where $\mathcal{P}_i (i = 0,1,2,3)$ are Petri nets.
(1) $\mathcal{S}^1 x \subset \mathcal{S}^1 y \iff \mathcal{P}_2 = \mathcal{P}_4, \gamma \leq_{\mathrm{R}} \alpha$ and $\delta \leq_{\mathrm{R}} \beta$.
(2) $x\mathcal{S}^1 \subset y\mathcal{S}^1 \iff \mathcal{P}_1 = \mathcal{P}_3, \ker\gamma \subset \ker\alpha$ and $\ker\delta \subset \ker\beta$.
(3) $\mathcal{S}^1 x = \mathcal{S}^1 y \iff \mathcal{P}_2 = \mathcal{P}_4, \gamma =_{\mathrm{R}} \alpha$ and $\delta =_{\mathrm{R}} \beta$.
(4) $x\mathcal{S}^1 = y\mathcal{S}^1 \iff \mathcal{P}_1 = \mathcal{P}_3, \ker\gamma = \ker\alpha$ and $\ker\delta = \ker\beta$. □

The following propositions claim that the intersection of finite principal left (resp. right) ideals is also a principal are left (resp. right) ideal.

PROPOSITION 2.4 (Intersection of Principal Left Ideals) ([11]) Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1,2,3)$ be Petri nets, $x : \mathcal{P}_1 \to \mathcal{P}_3$ and $y : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. Then, there exist a Petri net P and a surjective morphism z such that $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$. □

COROLLARY 2.1 (Diamond Property I) ([11]) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2,3)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. Then there exists a Petri net $\mathcal{P}$ such that $\mathcal{P} \sqsupseteq \mathcal{P}_1$ and $\mathcal{P} \sqsupseteq \mathcal{P}_2$. □

PROPOSITION 2.5 (Intersection of Principal Right Ideals) ([11]) Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0,1,2)$ be Petri nets, $x : \mathcal{P}_1 \to \mathcal{P}_3$ and $y : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. Then, there exist a Petri net $\mathcal{P}$ and a surjective morphism z such that $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$. □

COROLLARY 2.2 (Diamond Property II) ([11]) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0,1,2)$ be Petri nets with $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$. Then there exists a Petri net $\mathcal{P}_3$ such that $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. □

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$ and show the uniqueness of them up to isomophism.

DEFINITION 2.1 (Irreducible) A Petri net $\mathcal{P}$ is called $\sqsupseteq$*-irreducible* if $\mathcal{P} \sqsupseteq \mathcal{P}'$ implies $\mathcal{P} \simeq \mathcal{P}'$ for any Petri net $\mathcal{P}'$. Then $\mathcal{P}$ is called an $\sqsupseteq$-irreducible form. □

COROLLARY 2.3 ([11]) Let $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}''$ be Petri nets with $\mathcal{P} \sqsupseteq \mathcal{P}'$ and $\mathcal{P} \sqsupseteq \mathcal{P}''$. If $\mathcal{P}'$ and $\mathcal{P}''$ are $\sqsupseteq$-irreducible, then $\mathcal{P}' \simeq \mathcal{P}''$. □

3 Structure of the automorphism group of a Petri net

In this section, we give a fixed Petri net $\mathcal{P} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, \mu)$ and consider some properties of the structure of the automorphism group of this Petri net. Our aim in this section is to decompose the automorphism group $G = \mathbf{Aut}(\mathcal{P})$ of $\mathcal{P}$ into the form $G = KN = NK$, where N is a kind of normal subgroup of G.

3.1 The group of automorphisms of a Petri net

Let ${\boldsymbol{Q}_+}^P \rtimes (P^P \times T^T)$ be the semi-direct product of the group ${\boldsymbol{Q}_+}^P$ and the monoid $P^P \times T^T$, equipped with the multiplication defined by

$$(f, (\alpha, \beta))(g, (\alpha', \beta')) \stackrel{\text{def}}{=} (f \otimes \alpha\, g, (\alpha\alpha', \beta\beta')), \tag{3.1}$$

where P^P is the set of all maps from P to P and T^T is the set of all maps from T to T. ${\boldsymbol{Q}_+}^P \rtimes (P^P \times T^T)$ forms a monoid with the identity $(\mathbf{1}_\otimes, (1_P, 1_T))$, where $\mathbf{1}_\otimes$ is the identity of the group ${\boldsymbol{Q}_+}^P$, 1_P and 1_T are the identity maps on P and T respectively.

Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Now we consider the following set related to the Petri net $\mathcal{P}$.

$$\begin{array}{rl} \mathbf{Mor}(\mathcal{P}) : & \text{the set of all the morphisms of } \mathcal{P}. \\ \mathbf{Aut}(\mathcal{P}) : & \text{the set of all the automorphisms of } \mathcal{P}. \end{array}$$

By changing the weight function and the markings of $\mathcal{P}$, we can construct another Petri net $\mathcal{P}_0 = (P, T, 0^{\boldsymbol{E}(P,T)}, 0^P)$ be Petri nets, where 0^P denotes the special marking with $0^P : P \to \boldsymbol{N}_0, p \mapsto 0$ and $0^{\boldsymbol{E}(P,T)}$ the special weight function with $0^{\boldsymbol{E}(P,T)} : \boldsymbol{E}(P,T) \to \boldsymbol{N}_0, e \mapsto 0$. Then the following inclusion relation holds.

PROPOSITION 3.1 ([11]) Let $\mathcal{P} = (P, T, W, \mu)$ and $\mathcal{P}_0 = (P, T, 0^{\boldsymbol{E}(P,T)}, 0^P)$ be Petri nets. And let S_P and S_T be the symmetric groups of P and T, respectively.
(1) The subset $\boldsymbol{Q}_+{}^P \rtimes (S_P \times S_T)$ of $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ forms a group with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$.
(2) $\mathbf{Mor}(\mathcal{P}_0) = \boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$.
(3) $\mathbf{Mor}(\mathcal{P})$ is a submonoid of $\mathbf{Mor}(\mathcal{P}_0)$.
(4) $\mathbf{Aut}(\mathcal{P}_0) = \boldsymbol{Q}_+{}^P \rtimes (S_P \times S_T)$.
(5) $\mathbf{Aut}(\mathcal{P})$ is a subgroup of $\mathbf{Aut}(\mathcal{P}_0)$. □

3.2 Similarity and automorphism

We state the decomposition the automorphism group $G = \mathbf{Aut}(\mathcal{P})$. Recall that $(\boldsymbol{Q}_+{}^P, \otimes_P)$ is an abelian group.

LEMMA 3.1 ([13]) Let P be a nonempty set and P_1, P_2 be subsets of P.
(1) $\boldsymbol{Q}_+{}^{P_1} = \{f \in \boldsymbol{Q}_+{}^P \mid f(p) = 1, p \in P \setminus P_1\}$ is a subgroup of $(\boldsymbol{Q}_+{}^P, \otimes_P)$.
(2) $\boldsymbol{Q}_+{}^{P_1} \otimes_P \boldsymbol{Q}_+{}^{P_2} = \boldsymbol{Q}_+{}^{P_1 \cup P_2}$. □

DEFINITION 3.1 (Similar) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Two places $p, q \in P$ are said to be *similar* if there exists some positive rational number r such that $\mu(q) = r\mu(p)$, $W(q,t) = rW(p,t)$ and $W(t,q) = rW(t,p)$ for all $t \in T$. Two transitions $s, t \in T$ are said to be *similar* if $W(p,s) = W(p,t)$ and $W(s,p) = W(t,p)$ for all $p \in P$. □

The similarity defined above is obviously an equivalence relation on $P \cup T$. We denote this relation by $\sim_{\mathcal{P}}$ or simply $\sim$ and the $\sim_{\mathcal{P}}$-class of a place or a transition u by $C(u)$. A place (resp. a transition) is said to be *isolated* if it has no connection to any transitions (resp. any places). Especially, a place p is *0-isolated* if it is isolated and $\mu(p) = 0$. Note that two any 0-isolated places p and q are similar obviously.

LEMMA 3.2 (Transposition-type automorphisms) ([13]) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net, $p, q \in P$ be two distinct similar places in P and $s, t \in T$ be two distinct similar transitions in T. Then
(1) If p is not 0-isolated, $N_{\{p,q\}} = \langle (f_{p,q}, ((p\ q), \mathbf{1}_T)) \rangle$ is a subgroup of $\mathbf{Aut}(\mathcal{P})$ and its order is 2, where $(p\ q)$ is the transposition of p and q, $f_{p,q}(p) = r$, $f_{p,q}(q) = 1/r$, $f_{p,q}(x) = 1$ for $x \in P \setminus \{p,q\}$, and r is the rational number such that $\mu(p) = r\mu(q)$, $W(p,t) = rW(q,t)$ and $W(t,p) = rW(t,q)$ for all $t \in T$.
(2) If p is 0-isolated, $N_{\{p,q\}} = \boldsymbol{Q}_+{}^{\{p,q\}} \times \langle ((p\ q), \mathbf{1}_T) \rangle$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P})$.
(3) $N_{\{t,s\}} = \langle (\mathbf{1}_{\otimes_P}, (\mathbf{1}_P, (s\ t))) \rangle$ is a subgroup of $\mathbf{Aut}(\mathcal{P})$ and its order is 2. □

For a $\sim_{\mathcal{P}}$-class $C(u)$ of u, the subgruop $N_{C(u)}$ of $\mathbf{Aut}(\mathcal{P})$ is defined as follows:

$$N_{C(u)} = \begin{cases} \langle S_{\{a,b\}} | a, b \in C(u), a \neq b \rangle & \text{if } |C(u)| \geq 2, \\ \{(\mathbf{1}_{\otimes_P}, (\mathbf{1}_P, \mathbf{1}_T))\} & \text{if } |C(u)| = 1. \end{cases}$$

If u is a 0-isolated place, the $\sim_{\mathcal{P}}$-class $Z = C(u)$ is the set of all 0-isolated places in P and we can easily verify that $N_Z = \boldsymbol{Q}_+{}^Z \times (S_Z \times \{\mathbf{1}_T\})$, where S_Z is the symmetric group of Z. The following proposition holds with respect to N_Z.

PROPOSITION 3.2 (Separation of 0-isolated places) ([13]) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net, $Z \subset P$ be the $\sim_{\mathcal{P}}$-class of all the 0-isolated places, $N_Z = {\boldsymbol{Q}_+}^Z \times (S_Z \times \{1_T\})$, $H = \{(f, (\alpha, \beta)) \in \mathbf{Aut}(\mathcal{P}) | \, f|Z = 1_{\otimes_Z}, \alpha|Z = 1_Z\}$. Then, $\mathbf{Aut}(\mathcal{P}) = N_Z \times H$.

LEMMA 3.3 ([13]) Let $\mathcal{P} = (P, T, W, \mu)$,$\{p, q\} \subset P$,$\{s, t\} \subset T$ and $C(u)$ be the $\sim_{\mathcal{P}}$-class of $u \in P \cup T$. If $(f, (\alpha, \beta))$ is an automorphism of $\mathcal{P}$, then
(1) $p \sim_{\mathcal{P}} q \iff \alpha(p) \sim_{\mathcal{P}} \alpha(q)$,
(1') $s \sim_{\mathcal{P}} t \iff \beta(s) \sim_{\mathcal{P}} \beta(t)$,
(2) $\alpha(C(p)) = \{\alpha(q) | q \sim_{\mathcal{P}} p\} = C(\alpha(p))$,
(2') $\beta(C(t)) = \{\beta(s) | s \sim_{\mathcal{P}} t\} = C(\beta(t))$,
(3) $\min\{i | C(\alpha^i(u)) = C(u)\} = \min\{i | C(\beta^i(v)) = C(v)\}$ if $u, v \in P \cup T$ are connected,. □

Note that $|C(\alpha(p))| = |C(p)|$ for all $p \in P$ and $|C(\beta(t))| = |C(t)|$ for all $t \in T$.
Let $C_1, C_2, \ldots, C_k$ be the all $\sim_{\mathcal{P}}$-classes on $P \cup T$ and $\pi = \{C_1, C_2, \ldots, C_k\}$ be the partition of $P \cup T$ determined by $\sim_{\mathcal{P}}$. Then we introduce the permutation group $S_\pi = \{\sigma \in S_{P \cup T} \,|\, \forall X \in \pi, X^\sigma = X\} = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$, which does not move any elements of π.

PROPOSITION 3.3 (Embedding into a symmetric group) ([13]) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net without 0-isolated places.
(1) $\phi : \mathbf{Aut}(\mathcal{P}) \to S_{P \cup T}, (f, (\alpha, \beta)) \mapsto (\alpha, \beta)$ is a monomorphims, i.e. $\mathbf{Aut}(\mathcal{P}) \simeq \phi(G) \subset S_{P \cup T}$.
(2) $S_\pi \subset \phi(G)$.
(3) $X \in \pi \Longrightarrow g(X) \in \pi$ for any $g \in \phi(G)$.
(4) S_π is a normal subgroup of $\phi(G)$, that is, $S_\pi \lhd \phi(G)$.
(5) Let $a_1, a_2, \ldots, a_k$ be a system of representatives for S_π of $\phi(G)$ and $A = \langle a_1, a_2, \ldots, a_k \rangle$. Putting $K = \phi^{-1}(A), N = \phi^{-1}(S_\pi)$, $\mathbf{Aut}(\mathcal{P}) = KN = NK$.

THEOREM 3.1 ([13]) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net and $C_1, C_2, \ldots, C_k$ be the all $\sim_{\mathcal{P}}$-classes on $P \cup T$. $N = N_{C_1} \times N_{C_2} \times \cdots \times N_{C_k}$ is a normal subgroup of $G = \mathbf{Aut}(\mathcal{P})$ and $K = \langle \{a_i | i \in \Lambda\} \rangle$ is a subgroup generated by $\{a_i | i \in \Lambda\}$ with $G = \bigcup_{i \in \Lambda} a_i N$.
(1) If P has no 0-isolated places, $G = KN = NK$.
(2) Otherwise, $G = {\boldsymbol{Q}_+}^Z \times (KN) = (KN) \times {\boldsymbol{Q}_+}^Z$, where $Z \subset P$ be the $\sim_{\mathcal{P}}$-class of a 0-isolated place.

LEMMA 3.4 (1-step reduction) ([13]) Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net.
(1) $p, q \in P$ be two distinct similar places in P. Then $\mathcal{P} \sqsupseteq \mathcal{P}' = (P', T, W', \mu')$, where $P' = P - \{q\}$, $W' = W|(P' \times T) \cup (T \times P')$, $\mu' = \mu|P'$.
(2) $s, t \in T$ be two distinct similar transitions in T. Then $\mathcal{P} \sqsupseteq \mathcal{P}' = (P, T', W', \mu)$, where $T' = T - \{s\}$, $W' = W|(P \times T') \cup (T' \times P)$. □

In the lemma above, $|P' \cup T| = |P \cup T'| = |P \cup T| - 1$ holds. So we call such a relation *1-step reduction*, denoted by $\sqsupseteq_1$.

PROPOSITION 3.4 ([13]) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i) (i = 1, 2)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$, $(f, (\alpha, \beta))$: $\mathcal{P}_1 \to \mathcal{P}_2$ be a surjective morphism. If $\mathcal{P}_2$ is a normal form, then
(1) For any $p, q \in P$, $p \sim_{\mathcal{P}} q \iff \alpha(p) = \alpha(q)$,
(2) For any $t, s \in T$, $t \sim_{\mathcal{P}} s \iff \beta(t) = \beta(s)$. □

4 Petri net morphisms and codes

4.1 Behavior of Petri nets

The behavior of many systems can be described in terms of system states and their changes. In order to simulate the dynamic behavior of a system, a state or marking in a Petri net $\mathcal{P} = (P, T, W, \mu)$ is changed according to the following transition (firing) rule:
(1) A transition $t \in T$ is said to be *enabled* (under the marking μ or under the Petri net $\mathcal{P}$) if $W(p, t) \leq \mu(p)$ for every place $p \in P$, where $W(p, t)$ is the weight of the arc from p to t. Then each input place p of

t is marked with at least $W(p,t)$ tokens. An enabled transition may or may not fire (depending on whether or not the event actually takes place).
(2) A *firing* of an enabled transition t removes $W(p,t)$ tokens from each input place p of t, and adds $W(t,p)$ tokens to each output place p of t. As a consequence of the firing, the current marking μ is replaced with the following new marking μ':

$$\mu'(p) = \mu(p) - W(p,\, t) + W(t,\, p) \;\text{ for } \forall p \in P. \tag{4.1}$$

Then we define the transition function $\delta_{\mathcal{P}}$ by $\delta_{\mathcal{P}}(\mu,\, t) = \mu'$.
(3) A sequence $w = t_1 t_2 \ldots t_n$ of transitions is said to be a *firing sequence* in a Petri net $\mathcal{P} = (P,\, T,\, W,\, \mu)$ if $\mu_0 = \mu$, $\mu_n = \mu'$, and $\mu_i = \delta_{\mathcal{P}}(\mu_{i-1},\, t_i)$ for each i $(1 \le i \le n)$. Then μ' is called a *reachable* from $\mathcal{P}$, and we extend $\delta_{\mathcal{P}}$ from T to T^* by $\delta_{\mathcal{P}}(\mu,\, w) = \mu'$. By assuming that $\delta_{\mathcal{P}}(\mu,\, w) = \perp$ if w is not a firing sequence from $\mathcal{P}$ or $\mu = \perp$, the transition function $\delta_{\mathcal{P}} : ({N_0}^P \cup \{\perp\}) \times T^* \to ({N_0}^P \cup \{\perp\})$ is regarded as a total function. The set of all reachable markings from $\mathcal{P}$ is called the *reachability set* of $\mathcal{P}$, denoted by $R(\mathcal{P})$.

LEMMA 4.1 ([12]) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2)$ be Petri nets. $(f,\, (\alpha,\beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$. Then,
(1) $t \in T_1$ is enable in $\mathcal{P}_1 \iff \beta(t) \in T_2$ is enable in $\mathcal{P}_2$. More precisely, $\mu_1' = \delta_{\mathcal{P}_1}(\mu_1,\, t)\,(\neq \perp)$ and $f \otimes \mu_1 = \alpha\mu_2$ if and only if $\mu_2' = \delta_{\mathcal{P}_2}(\mu_2,\, \beta(t))\,(\neq \perp)$ and $f \otimes \mu_1' = \alpha\mu_2'$.

(2) w is a firing sequence in $\mathcal{P}_1 \iff \beta(w)$ is a firing sequence in $\mathcal{P}_2$. More precisely, $\mu_1' = \delta_{\mathcal{P}_1}(\mu_1,\, w)\,(\neq \perp)$ and $f \otimes \mu_1 = \alpha\mu_2$ if and only if $\mu_2' = \delta_{\mathcal{P}_2}(\mu_2,\, \beta(w))\,(\neq \perp)$ and $f \otimes \mu_1' = \alpha\mu_2'$.

LEMMA 4.2 ([12]) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2)$ be Petri nets. $(f,\, (\alpha,\beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$. Then,

(1) $\varphi : R(\mathcal{P}_1) \to R(\mathcal{P}_2), \mu_1' \mapsto \mu_2'$, where μ_1' and μ_2' are markings in LEMMA 4.1 (2), is a bijection.

(2) Let $R_i \subset R(\mathcal{P}_i)$ with $\varphi(R_1) = R_2$ and $K_i = \{w \in {T_i}^* \mid \delta_{\mathcal{P}_i}(\mu_i,\, w) \in R_i\}$ $(i = 1,2)$. Then $K_2 = \beta(K_1)$.

4.2 Petri net languages and codes

Let $\mathcal{P} = (P,\, T,\, W,\, \mu_0)$ be a Petri net, Σ be an alphabet, $\sigma : T \to \Sigma$ be a labeling of the transitions and $F \subseteq {N_0}^P$ be a finite set of final markings. Then we define the languages $\mathcal{L}_{\mathrm{L}}(\mathcal{P},\, \sigma,\, F)$, $\mathcal{L}_{\mathrm{G}}(\mathcal{P},\, \sigma,\, F)$, $\mathcal{L}_{\mathrm{T}}(\mathcal{P},\, \sigma)$ and $\mathcal{L}_{\mathrm{P}}(\mathcal{P},\, \sigma)$ as follows:

$$\begin{aligned}
&\mathcal{L}_{\mathrm{L}}(\mathcal{P},\, \sigma,\, F) \stackrel{\text{def}}{=} \{\sigma(w) \mid w \in T^*, \mu = \delta_{\mathcal{P}}(\mu_0, w) \text{ and } \mu \in F\},\\
&\mathcal{L}_{\mathrm{G}}(\mathcal{P},\, \sigma,\, F) \stackrel{\text{def}}{=} \{\sigma(w) \mid w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \ge \mu_f \text{ for some } \mu_f \in F\},\\
&\mathcal{L}_{\mathrm{T}}(\mathcal{P},\, \sigma) \stackrel{\text{def}}{=} \{\sigma(w) \mid w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \neq \perp \text{ but for all } t \in T, \delta_{\mathcal{P}}(\mu, wt) = \perp\},\\
&\mathcal{L}_{\mathrm{P}}(\mathcal{P},\, \sigma) \stackrel{\text{def}}{=} \{\sigma(w) \mid w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \neq \perp\}.
\end{aligned}$$

Languages $\mathcal{L}_{\mathrm{L}}(\mathcal{P},\, \sigma,\, F)$, $\mathcal{L}_{\mathrm{G}}(\mathcal{P},\, \sigma,\, F)$, $\mathcal{L}_{\mathrm{T}}(\mathcal{P},\, \sigma)$ and $\mathcal{L}_{\mathrm{P}}(\mathcal{P},\, \sigma)$ for some Petri net $\mathcal{P}$, some labeling σ and some set F of markings are called *L-type, G-type,T-type* and *P-type Petri net languages* respectively.

PROPOSITION 4.1 ([12]) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2)$ be Petri nets. $(f,\, (\alpha,\beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$.
For any $L_1 = \mathcal{L}_X(\mathcal{P}_1,\, \sigma_1,\, F_1)$, $X \in \{\mathrm{L}, \mathrm{G}\}$ (resp. $L_1 = \mathcal{L}_X(\mathcal{P}_1,\, \sigma_1)$, $X \in \{\mathrm{T}, \mathrm{P}\}$), there exists some $L_2 = \mathcal{L}_X(\mathcal{P}_2,\, \sigma_2,\, F_2)$ (resp. $L_2 = \mathcal{L}_X(\mathcal{P}_2,\, \sigma_2)$) such that $L_1 = \sigma_1(\beta^{-1}({\sigma_2}^{-1}(L_2)))$. Then L_1 is regular (resp. linear, context-free) if and only if L_2 is regular (resp. linear, context-free). □

Let $\mathcal{P} = (P,\, T,\, W,\, \mu_0)$ be a Petri net, δ be the transition function of $\mathcal{P}$, Then we can define a prefix code C of $\mathcal{P}$ as follows:

$$\begin{aligned}
&C = L \setminus LT^+ \neq \emptyset\\
&L = \{w \mid w \in T^+, \mu = \delta(\mu_0, w) \in F\},
\end{aligned}$$

for some (possibly infinite) set $F \subseteq {N_0}^P$ of final markings.

PROPOSITION 4.2 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)\,(i = 1, 2)$ be Petri nets. $(f, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$.
(1) If C is a prefix code of $\mathcal{P}_1$, then $\beta(C)$ is also a prefix code of $\mathcal{P}_2$.
(2) If C is a prefix code of $\mathcal{P}_2$, then $\beta^{-1}(C)$ is also a prefix code of $\mathcal{P}_2$.
$\beta : T_1 \to T_2$ is extended to the homomorphism $\beta : {T_1}^* \to {T_2}^*$, that is, $\beta(1) = 1$, $\beta(ua) = \beta(u)\beta(a)$, where 1 is the empty word, $u \in {T_1}^*$ and $a \in T_1$. □

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