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On the Structure of Siphons of Petri Nets

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Abstract

Siphons play an essential role for the analysis of the reachability of Petri nets. In this paper we study the structure of siphons of Petri nets and show that every siphon of a net is obtained by deleting some kinds of places in the net successively.

1 Introduction

In the study of Petri net, siphons play an essential role [4]. A siphon is a set of places P such that any transition having arcs to a place in P, also have arcs from a place in P. So, if a siphon P loses all tokens it never get tokens and all transitions having arcs from the places in P become dead.

In this paper, we mainly study the structure of siphons of a free choice Petri net. A pair of transitions is "in conflict" if they have a common input place, and a net is "free choice" if every transitions in conflict has only one input place [1, 3].

Let P be a siphon of a free choice net, and T be a set of transitions having arcs to the places in P. We show that

1. if P has an "end" place p that has no path to any other places, then $P-\{p\}$ is also a siphon, and
2. if P has two places p, q that have arcs to a common transition, then $P-\{p\}$ is also a siphon.

Deleting the places satisfying above conditions one by one, we can obtain all siphons of a given free choice Petri net.

Moreover, let R be a set of places not in P that has a path of length 2 to a place in P. Then we show that for any $R'\subseteq R$, there exists a siphon P' such that $P'\supseteq P$ and $P'\cap R=R'$. It means that concerning the inclusion relation, the structure of all subsets of R is embedded in the structure all siphons containing P.

In order to make this paper self-contained, we also include some related results discussed in earlier papers [1, 3, 4], and present them with refined proofs. In section 2, we give basic definitions of Petri nets using the notation of multisets. In section 3, we show that any Petri net N is emulated by a Petri net in a regular form, i.e. whose arcs are single weighted, has no loops, in(out)-degree of whose vertexes are less than 3 and is free choice. In section 4, we introduce a preorder on transitions, and show that every transition

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sequence is rearranged without losing fireability to the sequence which is consistent with the preorder. It means that we may concentrate the analysis to the strongly connected Petri nets. In section 5, we study siphons of a Petri net and give the main results of the paper.

2 Preliminaries

In this section, we give basic definitions and notations used in this paper. After giving notation for multisets, we define Petri nets and reachability of their markings on the framework of multisets.

2.1 Multisets and Strings

The notion of *multiset* is a generalization of the notion of set in which the finite multiplicities of elements are allowed. We use brackets [,] for multisets to distinguish them from sets, and na to denote n-multiples of an element a. If no confusion occurs, we may consider a set as a multiset whose multiplicities are less than 2. Hence, for example, $[a,b,b] = [a,2b] \neq [a,b] = \{a,b\}$.

For a multiset A, $|A|$ denotes the number of elements in A and $\underline{A} := \{a | a \in A\}$ is called the *underlying set* of A. Thus, $|[a,2b]| = 3$ and $\underline{[a,2b]} = \{a,b\}$.

For a finite set X, the class X° of all finite multisets over X is defined by $X^\circ := \{A \mid \underline{A} \subseteq X \text{ and } |A| < \infty\}$. For $A \in X^\circ$ and $x \in X$, $A(x)$ denotes the multiplicity of x in A. That is, $A \in X^\circ$ is identified with the function $A : X \to \{0,1,2,\cdots\}$.

Let A and B in X°. We define, $A \subseteq B, A \cup B, A \cap B, A - B, A + B$ and $A \cdot B$ by

$$\begin{aligned}
A \subseteq B \quad &\text{if} \quad A(x) \leq B(x), \\
(A \cup B)(x) \quad &:= \quad \max(A(x), B(x)), \\
(A \cap B)(x) \quad &:= \quad \min(A(x), B(x)), \\
(A - B)(x) \quad &:= \quad \max(0, A(x) - B(x)), \\
(A + B)(x) \quad &:= \quad A(x) + B(x), \\
(A \cdot B)(x) \quad &:= \quad A(x) \times B(x),
\end{aligned}$$

for any $x \in X$, respectively. Note that $A \cup B, A \cap B$ and $A - B$ are sets if A and B are sets. If B is a set, then $A \cdot B(x) = \begin{cases} A(x) & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$.

For an alphabet (finite set) X, X^* denotes the set of strings of finite length over X including the empty string ε of length 0. For $x_1 x_2 \cdots x_n \in X^*$ $(n \geq 0)$, we define $(x_1 x_2 \cdots x_n)^R := x_n \cdots x_2 x_1$. Note that $\varepsilon^R = \varepsilon$.

The Parikh map $\psi : X^* \to X^\circ$ is recursively defined by

$$\begin{cases} \psi(\varepsilon) & := \quad \emptyset, \\ \psi(xw) & := \quad [x] + \psi(w) \quad \text{for any } x \in X \text{ and } w \in X^*. \end{cases}$$

Hence, $\psi(x_1 x_2 \cdots x_n) = [x_1, x_2, \ldots, x_n]$. We also define $\underline{x_1 x_2 \cdots x_n} := \underline{\psi(x_1 x_2 \cdots x_n)} = \{x_1, x_2, \ldots, x_n\}$. Moreover we define $\psi(W) := \{\psi(w) | w \in W\}$ for $W \subseteq X^*$. Thus $\psi(X^*) = X^\circ$.

2.2 Petri Nets

A *Petri net* (or simply a *net*) is a bipartite multigraph $N = (P_N, T_N, F_N)$ where P_N is a finite nonempty set of *places*, T_N is a finite set of *transitions*, and $F_N \in (P_N \times T_N \cup T_N \times P_N)^\circ$ is a multiset of *arcs*. For $x \in P_N \cup T_N$, we define *postmultiset* of x by $x^N := [y|(x,y) \in F_N]$ and *premultiset* of x by ${}^N x := [y|(y,x) \in F_N]$. Note that $t^N, {}^N t \subseteq P_N^\circ$ for any $t \in T_N$.

Graphically, places, transitions and arcs are represented by circles, boxes and arrows respectively. For example, Fig. 1 and 2 are graphical representation of $N_1 = (P_1, T_1, F_1) = (\{p,q,r\}, \{s,t,u\}, [(p,s),(p,t),(s,q),(t,r),(q,u),(r,u),(u,p)])$ and $N_2 = (P_2, T_2, F_2) = (\{p\}, \{u\}, [2(p,u),(u,p)])$, respectively. Here, the label 2 of arc (p,u) of N_2 represents its multiplicity $F_2(p,u)$. Thus, ${}^{N_2}u = [2p]$ and $u^{N_2} = [p]$.

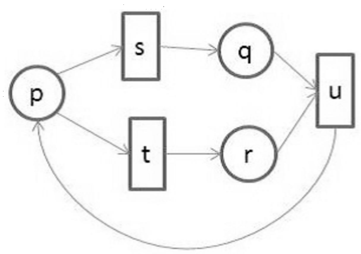

Fig. 1

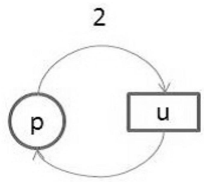

Fig. 2

A *marking* of a Petri net $N = (P_N, T_N, F_N)$ is a finite multiset over P_N, i.e. an element in P_N°. For a transition sequence $\tau \in T_N^*$, we recursively define the partial function $[\tau\rangle_N : P_N^\circ \to P_N^\circ$ as follows.

$$\begin{cases} \mu[\varepsilon\rangle_N & := \mu \\ \mu[t\tau\rangle_N & := \begin{cases} (\mu - {}^N t + t^N)[\tau\rangle_N & \text{if } {}^N t \subseteq \mu \\ \text{undefined} & \text{otherwise} \end{cases} \end{cases} \quad \text{for } t \in T_N \text{ and } \tau \in T_N^*.$$

We say that $\tau \in T_N^*$ is *fireable* at a marking $\mu \in P_N^\circ$ of N if $\mu[\tau\rangle_N \in P_N^\circ$, i.e. is defined. For a $W \subseteq T_N^*$, we define $\mu[W\rangle_N := \{\mu[\tau\rangle_N | \tau \in W\}$. The set $\mu[*\rangle_N := \mu[T_N^*\rangle_N$ is called the set of *reachable* markings of N from μ.

We define ${}^N U := \sum_{t \in T_N} U(t) \cdot {}^N t$ and $U^N := \sum_{t \in T_N} U(t) \cdot t^N$ for $U \in T_N^\circ$, and ${}^N \tau := {}^N \psi(\tau)$ and $\tau^N := \psi(\tau)^N$ for $\tau \in T_N^*$. From the definition of $[\tau\rangle_N$, it is easy to see that if $\mu[\tau\rangle_N = \nu$ then $\nu = \mu + \tau^N - {}^N \tau$

For example, $[2p][stu\rangle_{N_1} = [p,q][tu\rangle_{N_1} = [q,r][u\rangle_{N_1} = [p]$, ${}^{N_1}(stu) = [2p,q,r]$ and $(stu)^{N_1} = [p,q,r]$.

The set $\widehat{w}$ of *rearrangements* of $w \in X^*$ is defined by $\widehat{w} := \psi^{-1}(\psi(w)) = \{v \in X^* | \psi(v) = \psi(w)\}$. If $\mu[\tau\rangle_N \in P^\circ$, then $\mu + \tau^N \supseteq {}^N \tau$, $\mu[\tau\rangle_N = \mu + \tau^N - {}^N \tau$ and $\mu[\widehat{\tau}\rangle_N = \{\mu + \tau^N - {}^N \tau\}$.

The *reversed Petri net* of $N = (P_N.T_N, F_N)$ is defined by $N^{-1} := (P_N, T_N.F_N^{-1})$ where $F_N^{-1} := [(x,y)|(y,x) \in F_N]$, i.e. N^{-1} is the net obtained by reversing the direction of all arcs in N. Then it is easy to see that $\mu[\tau\rangle_N = \nu$ if and only if $\nu[\tau^R\rangle_{N^{-1}} = \mu$

3 Emulation

In this section, we study the notion of emulation on Petri nets, and show that any Petri net is emulated by the net which is ordinary (i.e. has no multiple arcs), pure (i.e. has no loops), max-degree 2 (i.e. has no vertexes of in(out)-degree greater than 2), and free choice (i.e. has no arcs from out-degree 2 places to in-degree 2 transitions).

Let $N_0 = (P_0, T_0, F_0)$ and $N_1 = (P_1, T_1, F_1)$. For $\tau_0 \in T_0^*$ and $\tau_1 \in T_1^*$, we denote $[\tau_0\rangle_{N_0} = [\tau_1\rangle_{N_1}$ if $\mu[\tau_0\rangle_{N_0} = \mu[\tau_1\rangle_{N_1}$ for any $\mu \in (P_0 \cap P_1)^\circ$. Informally, $[\tau_0\rangle_{N_0} = [\tau_1\rangle_{N_1}$ means that the firing effects τ_0 on N_0 and τ_1 on N_1 are equivalent.

We say that $N_0 = (P_0, T_0, F_0)$ is emulated by $N_1 = (P_1, T_1, F_1)$, if $P_0 \subseteq P_1$ and there exists a homomorphism $h : T_0 \to T_1^*$ such that $[t\rangle_{N_0} = [h(t)\rangle_{N_1}$ for every $t \in T_0$. For example, N_1 of Fig. 1 emulates N_2 of Fig. 2 through $h(u) = stu$.

Lemma 1 If N_1 emulates N_0 and N_2 emulates N_1, N_2 emulates N_0.

(Proof) If $N_1 = (P_1, T_1, F_1)$ emulates $N_0 = (P_0, T_0, F_0)$ through $h_1 : T_0 \to T_1^*$ and $N_2 = (P_2, T_2, F_2)$ emulates N_1 through $h_2 : T_1 \to T_2^*$, then N_2 emulates N_0 through $h_2(h_1(\cdot)) : T_0 \to T_2^*$. (Q.E.D)

A Petri net $N = (P, T, F)$ is *ordinal* if N has no multiple arcs, i.e. $F \subseteq P \times T \cup T \times P$.

Lemma 2 Every Petri net is emulated by an ordinal Petri net.

(Proof) Let $N_0 = (P_0, T_0, F_0)$.

Assume $F_0(t_0, p_0) = m + 1$ for some $m \geq 1$. To construct a Petri net $N = (P, T, F)$ emulating N_0 with $F(t_0, p_0) = 1$, we add new places $\{p_i | i = 1, 2, \ldots, m\}$ to P_0, new transitions $\{t_i | i = 1, 2, \ldots, m\}$ to T_0, and new arcs $[(t_i, p), (p_i, t_i), (t_{i-1}, p_i) | i = 1, 2, \ldots, m]$ to $F_0 - [m(t_0, p_0)]$. It is easy to see that $F(t_0, p_0) = 1$ and the multiplicity of every adding arcs is 1. Since ${}^N(t_0 t_1 t_2 \cdots t_m) = [p1, p2, \ldots, p_m] + {}^{N_0}t_0$ and $(t_0 t_1 t_2 \cdots t_m)^N = [mp_0, p1, p2, \ldots, p_m] + t^{N_1} - [mp_0]$, $[t\rangle_{N_0} = [t_0 t_1 t_2 \cdots t_m\rangle_N$.

Assume $F_0(p_0, t_0) = m + 1$ for some $m \geq 1$. To construct a Petri net $N = (P, T, F)$ emulating N_0 with $F(p_0, t_0) = 1$, we apply the above process to the reversed Petri net N_0^{-1} of N_0, and take the reverse of the resulted net.

Repeating the above processes, we eventually have $N_1 = (P_1, T_1, F_1)$ emulating N_0 such that $F_1(p, t) \leq 1$ and $F_1(t, p) \leq 1$ for all $p \in P_1$ and $t \in T_1$. (Q.E.D)

A Petri net $N = (P, T, F)$ is *pure* if N has no loops, i.e. ${}^N t \cap t^N = \emptyset$ for any $t \in T$.

Lemma 3 Every Petri net is emulated by a pure Petri net.

(Proof) Assume $N_0 = (P_0, T_0, F_0)$ is ordinal and $\{(p_0, t_0), (t_0, p_0)\} \subseteq F_0$. To construct $N = (P, T, F)$ emulating N_0 with ${}^N t_0 \cap t_0^N = \emptyset$, we add new place p, new transition t and new arcs $(t_0, p), (p, t), (t, p_0)$ to P_0, T_0 and $F_0 - [(t_0, p_0)]$, respectively. Clearly ${}^N t_0 \cap t_0^N = \emptyset$. Since ${}^N(t_0 t) = {}^N t_0 + [p]$ and $(t_0 t)^N = [p, p_0] + t_0^{N_0} - [p_0]$, $[t_0\rangle_{N_0} = [t_0 t\rangle_N$. Repeating this process, we eventually have the net $N_1 = (P_1, T_1, F_1)$ emulating N_0 and ${}^{N_1}t \cap t^{N_1} = \emptyset$ for any $t \in T_1$. (Q.E.D)

A Petri net $N = (P, T, F)$ is *max-degree* 2 if $|v^N| \leq 2$ and $|{}^N v| \leq 2$ for any $v \in P \cup T$.

Lemma 4 Every Petri net is emulated by an ordinary pure Petri net of max-degree 2.

(Proof) Assume $N_0 = (P_0, T_0, F_0)$ is ordinary and pure.

Let $p_0^{N_0} = [t_0, t_1, \ldots, t_m]$ with $m \geq 2$. To construct $N = (P, T, F)$ emulating N_0 such that $|p_0^N| = 2$, we add new places $\{p_i | i = 1, 2, \ldots, m\}$ to P_0, new transitions $\{s_i | i = 1, 2, \ldots, m\}$ to T_0, and new arcs $[(p_i, t_i), (p_{i-1}, s_i), (s_i, p_i) | i = 1, 2, \ldots, m]$ to $F_0 - [(p_0, t_i) | i = 1, 2, \ldots, m]$. Then it is easy to see that $|p_0^N| = 2$, $|v^N|, |{}^N v| \leq 2$ for every adding vertex v, $[t_0\rangle_{N_0} = [t_0\rangle_N$ and $[t_i\rangle_{N_0} = [s_1 s_2 \cdots s_i t_i\rangle_N$ for every $1 \leq i \leq m$.

Let $t_0^{N_0} = [p_0, p_1, \ldots, p_m]$ with $m \geq 2$. To construct $N = (P, T, F)$ emulating N_0 such that $|t_0^N| = 2$, we add new places $\{q_i | i = 1, 2, \ldots, m\}$ to P_0, new transitions $\{t_i | i = 1, 2, \ldots, m\}$ to T_0, and new arcs $[(t_i, p_i), (t_{i-1}, q_i), (q_i, t_i) | i = 1, 2, \ldots, m]$ to $F_0 - [(t_0, p_i) | i = 1, 2, \ldots, m]$. Then it is easy to see that $|t_0^N| = 2$, $|v^N|, |{}^N v| \leq 2$ for every adding vertex v, and $[t_0\rangle_{N_0} = [t_0 t_1 \cdots t_m\rangle_N$.

Other two cases where ${}^{N_0}t_0 = [p_0, p_1, \ldots, p_m]$ and ${}^{N_0}p_0 = [t_0, t_1, \ldots, t_m]$ with $m \geq 2$, are processed similarly by considering the reversed net N_0^{-1}.

Repeating the above processes, we eventually have a net $N_1 = (P_1, T_1, F_1)$ emulating N_0 and $|v^{N_1}| \leq 2$ and $|{}^{N_1}v| \leq 2$ for any $v \in P_1 \cup T_1$. (Q.E.D)

A Petri net $N = (P_N, T_N, F_N)$ is a *free choice* net [1] if $|p^N| = 1$ or $|{}^N t| = 1$ for any $(p, t) \in F_N \cap (P_N \times T_N)$. A Petri net $N = (P, T, F)$ is *in a regular form* if it is ordinal, pure, max-degree 2 and free choice.

Theorem 5 Every Petri net is emulated by a Petri net in a regular form.

(Proof) Assume $N_0 = (P_0, T_0, F_0)$ is ordinal, pure and max-degree 2.

Let $[(p, s), (p, t), (q, t)] \subseteq F_0$ with $p, q \in P_0$ and $s, t \in T_0$. To construct $N = (P, T, F)$ emulating N_0 such that $(p, t) \notin F$, we add new places r to P_0, new transitions u to T_0, and new arcs $(p, u), (u, r), (r, t)$ to $F_0 - [(p, t)]$. Then it is easy to see that $(p, t) \notin F$ and $[t\rangle_{N_0} = [ut\rangle_N$.

Repeating the above processes, we eventually have a net $N_1 = (P_1, T_1, F_1)$ emulating N_0 and $|p^{N_1}| + |{}^{N_1}t| \leq 3$ for any $(p, t) \in F_1 \cap (P_1 \times T_1)$. (Q.E.D)

Considering the reversed net, we can show that any Petri net is emulated by a Petri net $N_1 = (P_1, T_1, F_1)$ such that $|u^{N_1}| + |{}^{N_1}v| \leq 3$ for any $(u, v) \in F_1$. But, for the study of siphons, it is enough to assume that a net is free choice.

4 Preorder of transitions

For a relation R over a set A, we recursively define the relations

$$\begin{cases} R^0 & := \{(x, x) | x \in A\}, \\ R^{n+1} & := \{(x, z) | (x, y) \in R^n \text{ and } (y, z) \in R\} \quad \text{for any } n \geq 0. \end{cases}$$

Moreover we define $R^* := \bigcup_{n \geq 0} R^n$ and $R^{-n} := \{(y, x) | (x, y) \in R^n\}$ for $n = *, 1, 2, \cdots$. Since R^* is the reflexive transitive closure of R, R^* is a preorder (i.e. a reflexive transitive relation) of A. For a relation R, we also define $R(x) := \{y | (x, y) \in R\}$, and $R(X) := \bigcup_{x \in X} R(x)$.

Let $N = (P_N, T_N, F_N)$ be an ordinal Petri net. Since $F_N \subseteq P_N \times T_N \cup T_N \times P_N$, F_N is considered to be a relation over $P_N \cup T_N$, and $F_N^*(x)$ denotes the set of vertexes (places and transitions) that have a path from x in N.

For $\tau \in T_N^*$ and $T \subseteq T_N$, $\tau|_T$ denotes a homomorphic image of $h : T_N^* \to T^*$ such that $h(t) = \begin{cases} t & \text{if } t \in T \\ \varepsilon & \text{if } t \in T_N - T \end{cases}$. For $\tau, \sigma \in T_N^*$, we write $[\tau\rangle_N \preceq [\sigma\rangle_N$ if $\mu[\tau\rangle_N = \mu[\sigma\rangle_N$ or undefined for any marking μ of N. Then we have the following lemma.

Lemma 6 Let $N = (P_N, T_N, F_N)$ and $T \subseteq T_N$ be a set of transitions such that $F_N^{-1}(T) \cap F_N(T_N - T) = \emptyset$. Then $[\tau\rangle_N \preceq [(\tau|_T)(\tau|_{T_N-T})\rangle_N$.

(Proof) If $T = \emptyset$ or $T = T_N$, it is clear since $(\tau|_T)(\tau|_{T_N-T}) = \tau$. Assume $\emptyset \subset T \subset T_N$ and $F_N^{-1}(T) \cap F_N(T_N - T) = \emptyset$. Note that $(\tau|_T)(\tau|_{T_N-T}) \in \hat{\tau}$.

Let $t \in T_N - T$ and μ is a marking of N. Since $t^N \times F_N^{-1}(T) = \emptyset$, $\mu \times F_N^{-1}(T) = (\mu + t^N) \times F_N^{-1}(T)$. Thus, if $\mu[\tau\rangle_N$ is defined then $\mu[\tau|_T\rangle_N$ is defined.

Let $t \in T$ and μ be a marking of N. Since $F_N(T_N - T) \times {}^N t = \emptyset$, $\mu \times F_N^{-1}(T_N - T) = (\mu - {}^N t) \times F_N^{-1}(T_N - T)$ and $\mu \times F_N^{-1}(T_N - T) = (\mu - {}^N \sigma) \times F_N^{-1}(T_N - T)$ for any $\sigma \in T^*$. Thus, if $\mu[\tau\rangle_N$ is defined then $\mu[\tau|_T\rangle_N[\tau|_{T_N-T}\rangle_N = \mu[\tau\rangle_N$. (Q.E.D)

A sequence $\sigma \in X^*$ is *consistent* with a preorder R of X if $\sigma \in X^*xX^*yX^*$ means $(x, y) \in R$ or $(x, y) \notin R^{-1}$ for any $x, y \in X$.

Theorem 7 Let $N = (P_N, T_N, F_N)$. There exists a rearrangement σ of $\tau \in T_N^*$ consistent with F_N^* and $[\tau\rangle_N \preceq [\sigma\rangle_N$.

(Proof) We will show that if τ is not consistent with F_N^*, τ has a rearrangement σ such that $[\tau\rangle_N \preceq [\sigma\rangle_N$. Assume $\tau = \tau_0 s \tau_1 t \tau_2$ and $(s, t) \in F_N^{-*} - F_N^*$. We also may assume (s, u) and (u, t) are not in F_N^{-*} or in F_N^* for any $u \in \psi(\tau_1)$. Let $T := F_N^{-*}(t)$. Since $F_N^{-1}(T) = T$, $F^{-1}(T) \cap F(T_N - T) = \emptyset$. Moreover $s \in T_N - T$ and $F_N^*(T_N - T) = T_N - T$. It means that $\underline{\psi(s\tau_1)} \subseteq T_N - T$ and $[\tau_0 s \tau_1 t \tau_2\rangle_N \preceq [\tau_0 t s \tau_1 \tau_2\rangle_N$ from Lemma 6. (Q.E.D)

Let $N = (P_N, T_N, F_N)$. We define the following types of subnets of N.

$N(P, T) := (P, T, F_N \cap (P \times T \cup T \times P))$ for $P \subseteq P_N$ and $T \subseteq T_N$,
$N(T) := N(P_N, T)$ for $T \subseteq T_N$, and
$N(\tau) := N(\underline{\tau})$ for $\tau \in T_N^*$.

Informally, $N(P, T)$ is the subnet of N obtained by deleting vertexes not in $P \cup T$. $N(T)$ and $N(\tau)$ are the subnet of N obtained by deleting transitions not (appearing) in T and τ respectively.

Then the following proposition is clear from the definition.

Proposition 8 Let $N = (P_N, T_N, F_N)$. For any $\tau \in T_N^*$, $[\tau\rangle_N = [\tau\rangle_{N(\tau)}$.

From Theorem 7, it is easy to see that for any $\tau \in T_N^*$, there exists a rearrangement $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ of τ such that $\underline{\sigma_i}$ $(1 \le i \le k)$ is a set of transtions in a strongly connected component of $N(\underline{\sigma_i \cdots \sigma_k})$ that has no input arcs from any othe trasitions.

Thus we can concentrate our study on strongly connected Petri nets.

5 Siphons

5.1 Basic Facts about Siphons

Let $N = (P_N, T_N, F_N)$. A set of places $P \subseteq P_N$ is *marked* at a marking μ of N if $P \cap \mu \neq \emptyset$. P is a *siphon* of N if $F_N^{-1}(P) \subseteq F_N(P) \neq \emptyset$. A siphon P of N is *minimal* if it properly contains no siphons of N [3].

We define $N[P] := N(P, F_N^{-1}(P))$ for $P \subseteq P_N$. $N[P]$ is the subnet of N consisting places in P, transitions having arcs to places in P and arcs connecting them. Since $N[P]$ plays an essential role in the study of siphon, we call $N[P]$ a *siphon net* when P is a siphon. A siphon P of N is *strongly connected* if the siphon net $N[P]$ is strongly connected.

Let $N = (P_N, T_N, F_N)$ be strongly connected. From the definition of a siphon, for any set P of places, a siphon $R \supseteq P$ of N is constructed by the following nondeterministic algorithm. (See [2] for the linear time algorithm.)

```
R := P;
repeat
  for t ∈ F_N^{-1}(R) − F_N(R), add some r ∈ F_N^{-1}(R) to R;
until F_N^{-1}(R) ⊆ F_N(R);
```

Since N is strongly connected, P_N is a siphon of N. Hence, the above algorithm eventually stops and gets a siphon.

Note that $F_N^{-1}(P)$ is the set of transitions which marks places in P, and $F_N(P)$ is the set of transition that needs marks in P to fire. Thus, if P is unmarked at a marking μ of N, then transition in $F_N(P)$ are dead, i.e. not fireable at any marking in $\mu[*\rangle_N$.

Proposition 9 Let $N = (P_N, T_N, F_N)$, $P \subseteq P_N$, and μ is a marking of N.

1. If a siphon P is unmarked at μ, then it is unmarked at any marking in $\mu[*\rangle_N$ and $\mu[*\rangle_N = \mu[(T_N - F_N(P))^*\rangle_N = \mu[*\rangle_{N(T_N - F_N(P))}$.
2. If no transitions in a nonempty set $T \subseteq T_N$ are fireable at μ, then $N(T)$ has a siphon unmarked at μ.
3. If $\mu[\tau\rangle_N$ is defined, then any siphon of $N(\tau)$ is marked at μ.

(Proof)

1. If a siphon P is unmarked at μ, then no transitions in $F_N(P)(\supseteq F_N^{-1}(P))$ are fireable at μ. Thus, at any marking $\nu \in \mu[T_N\rangle_N = \mu[T_N - F_N(P)\rangle_N$, P is unmarked and no transitions in $F_N(P)$ are fireable. Repeating this argument, P is unmarked at any $\nu \in \mu[T_N^*\rangle_N = \mu[(T_N - F_N(P))^*\rangle_N$.
2. Let $N(T) = (P_N, T, F)$ and $P := \{p | \mu(p) = 0\}$. Since, no transitions in $T \neq \emptyset$ are fireable at μ in N and $N(T)$, $F(P) = T \supseteq F^{-1}(P)$.
3. It is clear from 1 and 2. (Q.E.D)

5.2 Structural properties of siphons

In the previous section, we have seen that siphons play an essential role in the reachability analysis. In this section we study the structural properties of siphons.

Proposition 10 Let $N = (P_N, T_N, F_N)$ and $P \subseteq P_N$.

1. Every siphon of $N[P]$ is also a siphon of N.

2. If $|P| \geq 2$ and $N[P]$ is strongly connected, then P is a siphon of N.

(Proof) Let $N[P] = (P, F_N^{-1}(P), F)$. Since $F \cap F_N^{-1}(P) \times P = F_N \cap F_N^{-1}(P) \times P$, $F^{-1}(Q) = F_N^{-1}(Q)$ for every $Q \subseteq P$.

1. If $Q \subseteq P$ and $F^{-1}(Q) \subseteq F(Q)$, then $F_N^{-1}(Q) = F^{-1}(Q) \subseteq F(Q) \subseteq F_N(Q)$.
2. If $|P| \geq 2$ and $N[P]$ is strongly connected, $F^{-1}(P) \subseteq F(P) \neq \emptyset$. Thus P is a siphon of $N[P]$ and N from 1. (Q.E.D)

Note that if $N[\{p\}]$ is strongly connected, then $F(\{p\}) = \emptyset$ since N has no loops. Thus, the condition $|P| > 2$ is essential in the statement 2 of the above Theorem.

Proposition 11 Let $N = (P_N, T_N, F_N)$ and $F \subseteq P_N \times T_N$.

1. Any siphon of $(P_N, T_N, F_N - F)$ is a siphon of N.
2. If $(F_N - F)(P) = F_N(P)$, then P is a siphon of N if and only if P is a siphon of $(P_N, T_N, F_N - F)$.

(Proof) Since $F \subseteq P_N \times T_N$, $(F_N - F) \cap T_N \times P_N = F_N \cap T_N \times P_N$. Thus $(F_N - F)^{-1}(P) = F_N^{-1}(P)$ and $(F_N - F)(P) \subseteq F(P)$ for any $P \subseteq P_N$.

1. If P is a siphon of $(P_N, T_N, F_N - F)$, i.e. $(F_N - F)^{-1}(P) \subseteq (F_N - F)(P) \neq \emptyset$, then $F_N^{-1}(P) = (F_N - F)^{-1}(P) \subseteq (F_N - F)(P) \subseteq F_N(P) \neq \emptyset$.
2. If P is a siphon of N, since $(F_N - F)^{-1}(P) = F_N^{-1}(P) \subseteq F_N(P) = (F_N - F)(P)$, P is a siphon of $(P_N, T_N, F_N - F)$. Thus the result follows from 1.

Now we show that some kinds of places in a siphon can be deleted preserving the set siphon, and any siphon of a strongly connected net is obtained by this way.

Proposition 12 Let $N = (P_N, T_N, F_N)$ be in a regular form and P_N be a siphon of N.

1. For any place $p \in P_N$, any siphon of $(P_N - \{p\}, T_N)$ is a siphon of N.
2. If $F_N^2(p) = \emptyset$, then $P_N - \{p\}$ is a siphon of N.
3. If $|F_N^{-1}(F_N(p))| > 1$, then $P_N - \{p\}$ is a siphon of N.

(Proof)

1. Let $F = F_N \cap ((P_N - \{p\}) \times T_N \cup T_N \times (P_N - \{p\}))$ be the set of arcs $(P_N - \{p\}, T_N)$. Then for any $P' \subseteq P_N - \{p\}$, $F_N^{-1}(P') = F^{-1}(P')$ and $F_N(P') = F(P')$. Thus, any siphon of $(P_N - \{p\}, T_N)$ is a siphon of N.
2. Since $F_N^2(p) = \emptyset$, $F_N(p) \cap F_N^{-1}(P_N) = \emptyset$. Since $F_N^{-1}(P_N) \subseteq F_N(P_N)$, $F_N^{-1}(P_N) \subseteq F_N(P_N) - F_N(p) \subseteq F_N(P - \{p\})$. Thus, $F_N^{-1}(P_N - \{p\}) \subseteq F_N^{-1}(P_N) \subseteq F_N(P_N - \{p\})$.
3. Since N is in a regular form, $|F_N(p)| = 1$ and $F_N(P_N) = F_N(P_N - \{p\})$. Thus, $F_N^{-1}(P_N - \{p\}) \subseteq F_N^{-1}(P_N) \subseteq F_N(P_N) = F_N(P_N - \{p\})$. (Q.E.D)

Theorem 13 Let $N = (P_N, T_N, F_N)$ be a Petri net in a regular form and assume that P_N is a siphon of N. Every siphon of a N is obtained by repeating the following operation.

1. Delete a place such that $F_N^2(p) = \emptyset$ and arcs from/to p.
2. Delete a place such that $|F_N^{-1}(F_N(p))| > 1$ and arcs from/to p.
3. Take some strongly connected components from disjoint union of them.

(Proof) Let P be a siphon of N. Any arcs outside into $N[P]$ is in $P_N \times T_N$ and can be deleted by the operation 1.

Consider the out-going path π from $N[P]$. If π ends some siphon net $N[P']$ containing more than 1 vertexes, we can delete the arc of π into $N[P']$ by the operation 1. Thus we may assume π has an end vertex. Then we can delete all places in π by repeating the operation 2.

Let N' be the results of the above operations. Then we can get $N[P]$ as a finite union of strongly connected components of N'. (Q.E.D)

Finally, we give a theorem concerning about the inclusion structure of a siphons of a given net.

Theorem 14 Let P be a siphon of a strongly connected Petri net $N = (P_N, T_N, F_N)$, and $R := \{r \in P_N - P \mid F_N(r) = F_N(p) \subseteq F_N^{-1}(P)$ for some $p \in P\}$

1. For any $R' \subseteq R$, there exists a siphon $P' \supseteq P$ such that $P' \cap R = R'$.
2. The structure of all subsets of R under the inclusion relation is embedded in the class of siphons including P.

(Proof)

1. Let $r \in R'$, $F := F_N \cap ((R - \{r\}) \times F_N^{-1}(P))$ and $N' := (P_N, T_N, F_N - F)$. Since N is strongly connected, there exists a path from a place in P to r in N'. Thus applying the sihon constructing algorithm in 5.1 to N', we have a siphon $P_p \supseteq P \cup \{p\}$ of N' such that $P_p \cap R = \{p\}$. P_p is also a siphon of N from Propsotion 11. Repeating this process, for any $R' \subseteq R$, we can obtain a siphon $P' \supseteq P$ such that $P' \cap R = R'$.
2. It is clear from 1.

(Q.E.D)

5.3 Example

We give an example of Theorem 13 and 14.

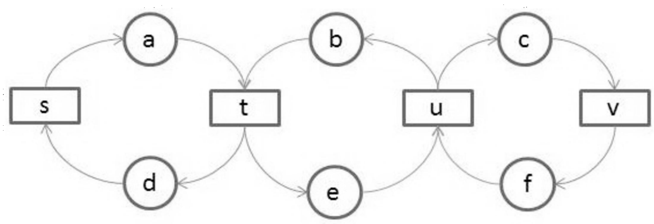

Fig. 3

Let N be the Petri net represented in the Fig. 3. N is strongly connected and $\{a,b,c,d,e,f\}$ is a siphon of N. By the operation 2 of Theorem 13, $\{b,c,d,e,f\}$, $\{a,c,d,e,f\}$, $\{a,b,c,d,f\}$, $\{a,b,c,d,e\}$, $\{b,c,d,f\}$, $\{b,c,d,e\}$, $\{a,c,d,f\}$ and $\{a,c,d,e\}$ are siphons of N. Then by the operation 1 of Theorem 13, $\{b,c,e,f\}$, $\{a,b,d,e\}$, $\{b,c,f\}$, $\{c,f\}$, $\{b,d,e\}$, $\{b,c,e\}$, $\{b,e\}$, $\{a,d,e\}$ and $\{a,d\}$ are siphons of N.

Among the siphons of N, siphon nets of $\{a,b,c,d,e,f\}$, $\{a,c,d,f\}$, $\{b,c,e,f\}$, $\{a,b,d,e\}$, $\{a,d\}$, $\{b,e\}$ and $\{c,f\}$ are strongly connected, and $\{a,d\}$, $\{b,e\}$, $\{c,f\}$ are minimal siphons of N. The strongly connected siphons including $\{b,e\}$ are $\{b,e\}$, $\{b,c,e,f\}$, $\{a,b,d,e\}$ and $\{a,b,c,d,e,f\}$.

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