数理解析研究所講究録
第 1917 巻 2014 年 172-178

Discrete population model

defined by linear recurrence sequence

By

Yukio Hagiwara* , Masari Ishigaki, Shougo Matukata***,**
Osamu Suzuki ****and J.Tamura***

**Department of Earth Sciences,s College of Humanities and Sciences, Nihon University156 Setagaya,Tokyo,Japan E-mail: hagiwarau@8f.adep.or.jp*

***Department of Information Science, College of Humanities and Sciences, Nihon University156 Setagaya,Tokyo,Japan E-mail: zumechanmokyu@yahoo.co.jp*

****Department of Information Science, College of Humanities and Sciences, Nihon University156 Setagaya,Tokyo,Japan E-mail: . matsukata@.kthrlab.ac.jp*

*****Department of Information Science, College of Humanities and Sciences, Nihon University156 Setagaya,Tokyo,Japan E-mail: osuzuki@.chs. nihon-u .ac.jp*

******Department o***Department off Compute rand System Analysis College of Humanities and Sciences, Nihon University156 Setagaya,Tokyo,JapanE-mail: armatiju@ezweb.ne.jp*

INTRODUCTION

In this paper, we consider a population model which is defined by Fibonacci type at first. Namely we consider the population model of well known Fibonacci rabbits. Then we encounter with the difficulties to obtain realistic models. Namely we have the following two problems:.

(1) **The infinity population numbers problem:** The population numbers become infinity (as the exponential order).

(2) **The life time length problem:** Each rabbit can live forever.

We discuss these problems by introducing a concept of "degeneration" of population numbers and give the final answer by introducing the theory of linear recurrence sequences. Finally we compare the given discrete population models with well known Malthus model, Verhulst model and Voltera-Lotka model and suggest possibilities of description on mass extinctions.

1. Fibonacci rabbits model

In this section we recall some basic facts on Fibonacci rabbit population model. The generation rules are given as follows:

The generation rules of Fibonacci rabbits:

(1) Every pair of male and female rabbits bear a pair of rabbits of the same type every month, after two months they are born.

(2) Each rabbit never dies and lives forever.

The sequence of number of n-generation Fibonacci rabbits $\{a_n\}$ is called Fibonacci sequence and it is given by the following recurrence formula:

$$a_n = a_{n-1} + a_{n-2} (a_0 = a_1 = 1)$$

We recall well known facts on Fibonacci sequence:

(i) The generation function of the sequence is given as follows:

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} a_n x^n$$

(ii) By use of the decomposition of the rational functions, we have

$$a_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

where $\alpha, \beta (\alpha > \beta)$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. Hence we see that the numbers of rabbits tend to infinity by the fact: $\alpha > 1, |\beta| < 1$. By this fact we have proposed the problems in Introduction.

2. Degenerations of Fibonacci rabbits sequence

We introduce a concept of degeneration for generation of Fibonacci rabbits and discuss the problems in Introduction:

Definition(Degeneration)

The generations are called of N-generation life time type, when each member can live at most N generations. The N-generation is called Type I, when it makes babies untill just before it dies. Otherwise it is called of Type II. by the generation rule Namely it makes babies successively, but it does not make baby just before it dies.

At first we notice that degeneration sequences produce several sequences which are interesting in the theory of Fibonacci sequences:

(1) 3-generation of Type I: We can obtain the following sequence $\{b_n\}$ for 3-generation of Type 1 by the following recurrence formula:

$$b_n = b_{n-2} + b_{n-3} + b_{n-4} (b_1 = b_2 = b_b = 1)$$

(2)**(Padovan sequence)** We can obtain the Padovan sequence $\{c_n\}$ for 3-generation of type II. The Padovan sequence $\{c_n\}$ is a sequence of natural numbers which is defined by the following recurrence formula ([2]):

$$c_n = c_{n-2} + c_{n-3} (c_0 = c_1 = c_2 = 1)$$

(3)2-degeneration model of Type II: We can obtain the sequence: 1,2,1,2,....: This gives the simplest solution to the problem (2) in Introduction. This example shows that we may expect to obtain the realistic model by successive degenerations processes from Fibonacci rabbits.

Here we can propose the following problems connected to the two problems (1) and (2):

(1)' Can we obtain sequences with increases of polynomial type by making degeneration of the Fibonacci sequence ?

(2)'' Can we obtain the sequences of periodic bounded sequences by making degenerations ?

We notice that the Padovan rabbits still divergent to infinity as exponential order when time tends to infinity.

3. Degeneralization for general sequences

We can introduce a concept of degeneration for more general sequences and discuss the same problems for them: For example, we can perform for Tribonacci sequence([2]). The Triboonacci sequence $\{t_n\}$ is a sequence which is defined by the following recurrence formula([2]):

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} (t_0 = t_1 = t_2 = 1)$$

This sequence divergent as exponential order. We can associate a population model for the sequence and consider the degeneration of the sequence. Then we may expect to obtain sequences systematically which are already known in its theory. On the basis of this fact, we can find a hierarchy structure in sequences of Fibonacci type.

4.Construction of population model by use of linear recurrence sequence

In order to solve the problems (1) and (2) in Introduction, we introduce the theory of linear recurrence sequences([4]).

DEFINITION

A sequence $\{a_n\}$ is called linear recurrence sequence, when it is defined by the following recurrence formula:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$$

where $c_1, c_2, \dots, c_r$ are constants. Then the sequence can be determined by the initial condition; $\{a_1, a_2, \dots, a_r\}$.

We give some basic notations

(1) **(Generating function)**

The generating function is given by a rational function $F(x)$:

$$F(x)=\sum_{n=0}^{\infty} a_n x^n$$

Conversely, when a rational function is given, then the coefficient of Taylor expansion gives a linear recurrence sequence.

(2) **(Characteristic polynomial and characteristic roots)**

The characteristic polynomial of the linear recurrence sequence is given as follows:

$$f(x)=x^r - a_1 x^{r-1} - \ldots - a_r$$

(ii) $f(x)=0$ is called characteristic equation and its roots are called characteristic roots . The different characteristic roots are denoted by $\alpha_1,\alpha_2,\ldots,\alpha_l(|\alpha_1|>|\alpha_2|>\ldots>|\alpha_l|)$ and we assume the following decomposition:

$$f(x)=(x-a_1)^{m_1}(x-a_2)^{m_2}\ldots(x-a_l)^{m_l}$$

Then we can discuss the problems depending on the absolute values of the roots and the initial conditions. Namely we can prove the following theorem:

THEOREM

Let $\{a_n\}$ be a linear recurrence sequence. Then we can obtain the following assertions:

(1) When there exists one root whose absolute value is bigger than 1, the sequences divergent as exponential degree for suitable initial values.

(2) When there exists one root whose absolute value is one, the sequences divergent as polynomial degree for suitable initial values with/out periodic perturbations or they are periodic(see Figure 4).

(3) The classification can be given.

The proofs can be given by the following Lemma:

Lemma

Let $\{a_n\}$ be a linear recurrence sequence. Then we have the following assertions:

(1) The following elements are linearly independent:

$$\{a_k^{m_k}, na_k^{m_k}, n^2 a_k^{m_k},\ldots, n^{m_k-1}a_k^{m_k},\}(k=1,2,\ldots,l)$$

(2) Each $\{a_n\}$ can be expressed as a linear combination of the elements of (1):

$$a_n=\sum_{k=1}^{l}\sum_{t=1}^{m_k} c_{k,t} n^t a_k^{m_t}$$

The coefficients can be determined by the initial conditions

5. Examples

We give several examples and their computer simulations:

(1) Choosing $c_1 = c_2 = 1 (r = 2)$ and $a_1 = 1, a_2 = 1$, we can obtain Fibonacci sequence (see Figure 1). We see that the characteristic polynomial is $f(x) = x^2 - x - 1$. Hence we see easily that it divergent infinity as exponential order

(2) Choosing $c_1 = 3, c_2 = -3, c_3 = 1 (r = 3)$ and $a_1 = 3, a_2 = 1, a_3 = 0$, we can obtain the sequence which is called Euler's trigonometrical sequence (see Figure 2). We see that the characteristic polynomial is $f(x) = (x-1)^3$. Hence we see easily that it divergent to infinity as polynomial order

(3) Choosing $c_1 = 1, c_2 = -2, c_3 = 2, c_4 = 1 (r = 4)$ and $a_1 = 1, a_2 = 2, a_3 = a_4 = 1$ we can obtain a periodic sequence (see Figure 3). We see that the characteristic polynomial is $f(x) = (x-1)(x^2+1)$.

(4) Choosing $c_1 = 2, c_2 = -2, c_3 = 2, c_4 = 1 (r = 4\}$ and $a_1 = 1, a_2 = 2, a_3 = a_4 = 1$, we can obtain a sequence which has a divergence of polynomial type, but periodic along the polynomial divergence(see Figure 4). We see that the characteristic polynomial is $f(x) = (x-1)^2(x^2+1)$.

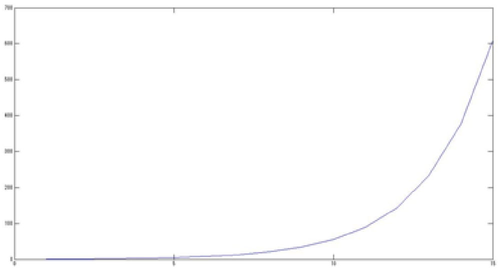

Figure 1

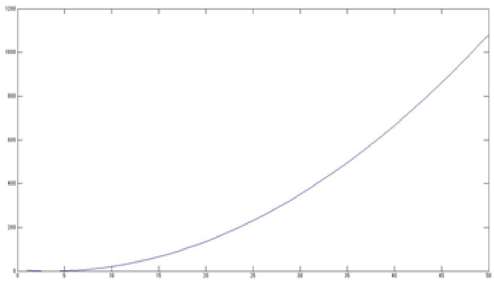

Figure 2

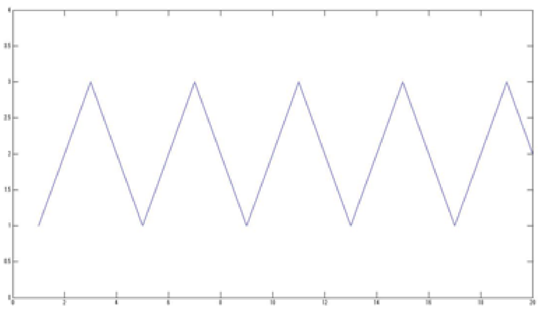

Figure 3

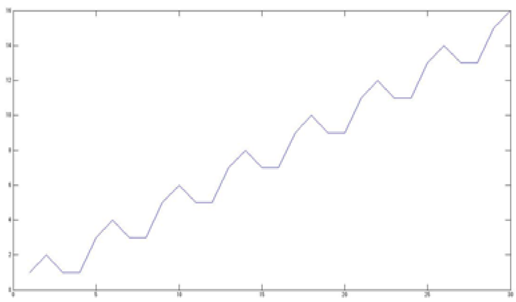

Figure 4

5. Future problem

We discuss our discrete population model and state several problems in the future.

Our discrete model vs continuous model

We compare our discrete population model with well known continuous model and state several problems. At first we have to examine the reason why we may use continuous model. We believe that we may use continuous models, when treated population models which constitute many population. This consideration describes the real population quite well, for example, Malthus model or Verhulst model. Especially Verhulst model describes them quite well([3]). We may say that our model of exponential type corresponds to the Malthus model. Still we have no models which correspond to Verhulst model. In order to realize the corresponding models, we have to introduce a concept of deformation of models of linear recurrence sequences. Then we can obtain the corresponding model. We can describe the deformations in terms of the deformations of characteristic polynomial: We explain deformation by use of the following examples: We consider

$$f(x) = (x - \alpha)(x^2 + 1)$$

where α is a deformation parameter. When α is bigger than 1, the model the model behaves as exponential model. When α tends to 1, the model becomes that of periodic type.

Our discrete model vs chaotic model

The well known chaotic dynamical system due to May is given by the discretization of Verhulst equation ([5]). The motivation of introduction of chaotic dynamical system is to describe a population model of certain insects which behaves a periodic model. This model can be also obtained from our mode by use of deformations (Figure 5).

Mass extinction

We know that we have the explosion in the Cambrian period and the mass extension in Permian period in the history of marine animals. Hence we can observe the process of the exponential increase at first and then mass extinction happens.

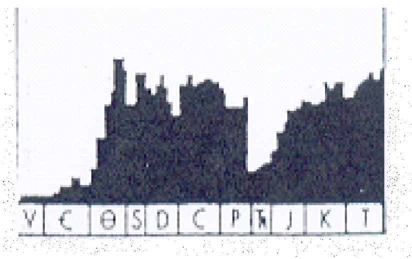

Figure 5

The first part can be described by use of Malthus model. But the mass extinction in the Permian period is difficult to describe by the continuous model. We can observe many examples of mass extinction in the usual world. For example, we see that fishes produce so many eggs, but only few fishes can survive at the final stage. Because this fact, the nature make a good valance and makes the stable populations. The main problem for population genetics can be stated as follows:

PROBLEM

How can we describe the total process in population from the explosion to the mass extinction ?

It is well known that the logistic equation can describe the former half part of the process. But it seems to us that it can not describe the mass extinctions. We may try to solve this problem by use of our results.

Other problems

We can discuss the Voltera-Lotka model of the coexsistence population model on lions and rabbits by use our model:

$$\begin{cases} dx/dt = ax - bxy \\ dy/dt = cxy - dy \end{cases}$$

where a,b,c,d are positive constants and x and y are the number of rabbits and lions respectively. Taking the fact that the Malthus equation corresponds to our non-degenerate model into account, we may discuss the equation in terms of our model. Namely we may consider the following system of linear recurrence sequences:

$$\begin{cases} a_{n+2} = a(a_{n+1} + a_n) - b(a_{n+1} + a_n)(b_{n+1} + b_n) \\ b_{n+2} = c(b_{n+1} + b_n) - d(a_{n+1} + a_n)(b_{n+1} + b_n) \end{cases}$$

The problem can be stated as follows: Find sequences $\{a_n\}, \{b_n\}$ so that they are periodic. We have never seen the study on a system of linear recurrence sequences.

REFERENCE

[1] A.S.Posamentier,and I.Lehmann: The Fabulous Fibonacci Numbers
[2] As for Padovan sequence and Tribonacci sequence, see A000931 in OEIS
[3] As for continuous population models, we have so many references.
[4] As for linear recurrence sequences, we have so many references. For example, A. Brousseau: Linear recurrence and Fibonacci sequences, Fibonacci Association (1971)
[5] As for chaotic population models, we have so many references. For example, Z. Elhadj: Frontiers in the study of chaotic dynamical system with open problems, World Scientific (2011)