数理解析研究所講究録
第 1918 巻 2014 年 152-160

ON CAPORASO'S CONJECTURE ON BRILL-NOETHER LOCI FOR TRIVALENT GRAPHS

TAKEO NISHINOU

ABSTRACT. We report on our result on Brill-Noether theory on graphs. In particular, we prove Caporaso's conjecture which states that the existence of graphs on which there are no special divisors if the Brill-Noether number is negative. In fact, we can prove full Brill-Noether theorem for a class of trivalent graphs. This, combined with Baker's specialization lemma, gives a purely combinatorial proof of classical Brill-Noether theorem.

1. INTRODUCTION

This is a report on the author's recent result concerning the divisor theory of graphs. The divisor theory of graphs has undergone big progress after the seminal work of Baker and Norine [2] about Riemann-Roch theory of graphs appeared. Since then, this theory has been developed in several directions. One of remarkable results is the Brill-Noether theory of graphs developed by Cools, Draisma, Payne and Robeva [4], which is a deepening of the Riemann-Roch theory. They showed (among other results) the existence of particular four-valent graphs which are Brill-Noether general, that is, the set of divisors of given Brill-Noether number (see Section 2) has the expected dimension. However, graph theoretically four-valent graphs are not general: general graphs are trivalent. So the natural question is whether there exist trivalent graphs which are Brill-Noether general.

In this context, L. Caporaso conjectured [3] that there should be trivalent graphs which has the property that the set of effective divisors whose Brill-Noether number is negative is empty. In this report, we give an example of a class of trivalent graphs which confirms Caporaso's conjecture. In fact, these graphs are Brill-Noether general (that is, the Brill-Noether theorem holds even when the Brill-Noether number is nonnegative).

2. REVIEW ON DIVISOR THEORY ON GRAPHS

In this section we recall the divisor theory on metric graphs very briefly. See for example [2] for more information.

2.1. **Divisors on graphs.** Let Γ be a compact connected metric graph. A divisor on Γ is a finite $\mathbb{Z}$-linear combination of points of Γ, and we write $Div(\Gamma)$ for the additive group of all divisors. A general element of $Div(\Gamma)$ is written as

$$D = a_1 v_1 + \cdots + a_r v_r,$$

where $a_i \in \mathbb{Z}$ and $v_i \in \Gamma$. We define the degree of D by

$$deg(D) = a_1 + \cdots + a_r.$$

A rational function on Γ is a piecewise linear function on Γ with integral slopes. If f is a rational function of Γ and $v \in \Gamma$, define $ord_v(f)$ by the sum of incoming slopes of f at v. Then define $div(f) \in Div(\Gamma)$ by

$$div(f) = \sum_{v \in \Gamma} ord_v(f) \cdot v$$

and call it the divisor of f. The divisors of rational functions on Γ compose a subgroup $Prin(\Gamma)$ of $Div(\Gamma)$, the subgroup of principal divisors.

The Picard group of Γ is defined by

$$Pic(\Gamma) = Div(\Gamma)/Prin(\Gamma).$$

One sees every principal divisor has degree 0, so there is a well-defined map

$$deg : Pic(\Gamma) \to \mathbb{Z}.$$

An element of $Pic(\Gamma)$ is called a divisor class. Denote by $Pic_d(\Gamma)$ the subset $deg^{-1}(d)$ of $Pic(\Gamma)$. In particular, $Pic_0(\Gamma)$ is a subgroup of $Pic(\Gamma)$.

2.2. **Linear system of divisors.** A divisor $D = a_1 v_1 + \cdots + a_r v_r \in Div(\Gamma)$ is effective if each coefficient a_i is nonnegative. A divisor D' is linearly equivalent to D if $D - D' \in Prin(\Gamma)$.

The rank $r(D)$ of an effective divisor D is the largest integer r such that, for every effective divisor E of degree r, $D - E$ is linearly equivalent to an effective divisor. If D is not linearly equivalent to an effective divisor, then set $r(D) = -1$.

For nonnegetive integers r and d, the Brill-Noether locus

$$W_d^r(\Gamma) \subset Pic_d(\Gamma)$$

is the set of divisor classes of degree d and rank at least r.

In classical theory of divisors on Riemann surfaces, for an Riemann surface of genus g, the Brill-Noether locus $W_d^r(C)$ is similarly defined. Let

$$\rho(g, r, d) = g - (r+1)(g - d + r)$$

be the Brill-Noether number. Then it is known that for a general curve, if $\rho(g, r, d)$ is nonnegetive, the dimension of $W_d^r(C)$ is equal to $\rho(g, r, d)$, and if $\rho(g, r, d)$ is negative, then $W_d^r(C)$ is an empty set.

For graphs however, there are open sets in the moduli space of metric graphs of genus g, where $\dim W^r_d(\Gamma)$ is strictly larger than $\rho(g,r,d)$ (see [3]). One of the conjectures of Caporaso is the following.

Conjecture 2.1. *Assume $g \geq 2$ and $\rho(g,r,d) < 0$. Then there exists a 3-regular graph Γ of genus g for which $W^r_d(\Gamma) = \emptyset$.*

Theorem 2.2. *Caporaso's conjecture holds for the following graphs.*

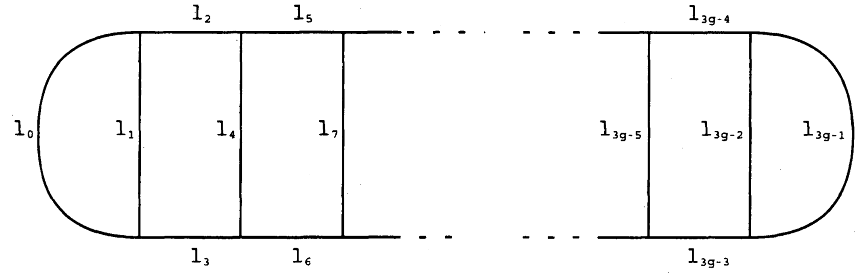

FIGURE 1. The graph Γ_g

Here the edge lengths of the edges l_i, $1 \leq i \leq 3g-3$ are given by

$$\mathit{length}(l_i) = \varepsilon^i,$$

with ε small positive constant.

Remark 2.3. *The edge length need not strictly take these values. If we perturb the lengths of edges slightly (compared to its original length, for example, we can change the length of the edge l_i to $\varepsilon^i + O(\varepsilon^{i+1})$), then the resulting graph still satisfies the conclusion of Theorem 2.2. Thus, we actually have the open subset of the moduli space of the metric graphs where the conclusion of Theorem 2.2 holds.*

Remark 2.4. *The same line of argument proves that in fact the full Brill-Noether theorem is true for these graphs. That is, when the Brill-Noether number ρ is nonnegative, then the corresponding Brill-Noether locus has dimension ρ. In this note, we concentrate on the case $\rho < 0$.*

3. IDEA OF THE PROOF

Let us write the graph in the statement of Theorem 2.2 by Γ_g. Recall that Brill-Noether number is given by

$$\rho = g - (r+1)(g-d+r).$$

One sees that $\rho = -1$ if and only if the genus g of Γ_g, the degree d of the divisor and rank r of the linear system satisfy the relation

$$(g,d) = (r(p-1)+p-2, pr-1)$$

for some positive integer p. The cases when ρ is smaller than -1 are easier than the case $\rho = -1$, so we consider the cases where $\rho = -1$.

An important tool for the proof is the *chip-firing deformation* which we introduce below. It gives a convenient way to construct linearly equivalent divisors of a given divisor.

Recall the *chip-firing move* on a graph [1, 2]. Let Γ be a graph and D a divisor on it. Let v be a k-valent vertex of Γ and $v_1, \cdots, v_k$ be the neighboring vertices of v. Then applying a chip-firing move at v to D gives the new divisor D'

$$\begin{cases} D' = D + kv - v_1 - v_2 - \cdots - v_k \text{ ('borrowing' move), or} \\ D' = D - kv + v_1 + v_2 + \cdots + v_k \text{ ('giving' move).} \end{cases}$$

Any divisor linearly equivalent to D can be obtained as a result of a sequence of chip-firing moves.

In the case of a metric graph, a straightforward generalization of chip-firing move can be defined, which we call *chip-firing deformation.* To define it, we prepare some notation. Let Γ be a compact connected metric graph. Let v be a k-valent point of Γ (a point on the interior of an edge is a divalent point. The valency at the vertices of Γ is defined as usual). Consider the open subset $\Gamma \setminus \{v\}$ of Γ. This is a graph with k open ends. Add one valent vertices to each of these ends. This gives a closed graph Γ' which is not necessarily connected.

Definition 3.1. We call the graph Γ' the graph obtained from Γ by *cutting* Γ *at* v.

Let D be a divisor on Γ whose summand at the point v is lv with $l > 0$.

Definition 3.2. A divisor D' on Γ' is obtained from D by *cutting the pair* (Γ, D) *at* v if

$$D' = D - lv + \sum_{i=1}^{k} c_i v_i$$

with $e_i \geq 0$ and $\sum_{i=1}^{k} c_i = l$.

Note that $D - lv$ does not have a summand at the point v, so we can think of it as a divisor on the graph Γ'. Thus, the expression of D' in Definition 3.2 makes sense.

Now we define the chip-firing deformation. For simplicity, we only define it for the case of *positive* ends. Let Γ be a metric graph and D an effective divisor on Γ. Let

$$v_1, \cdots, v_a$$

be points of Γ and Γ' be the graph obtained from Γ be cutting it at each of the points $v_1, \cdots, v_a$. Let

$$\Gamma'_1, \cdots, \Gamma'_b$$

be the connected components of Γ'. Assume that by cutting the pair (Γ, D) suitably at each of the points $v_1, \cdots, v_a$, we obtain a component of Γ' (let it be Γ'_1 for simplicity) and an effective divisor D'_1 on it which have the following property:

- Each one-valent vertex is contained in the support of the divisor D'_1.

Let

$$w_1, \cdots, w_c$$

be the one-valent vertices of Γ'_1 and write D'_1 as

$$D'_1 = D''_1 + \sum_{i=1}^{c} a_i w_i$$

with $a_i > 0$ for all i. Here D''_1 is the subdivisor of D'_1 whose support is disjoint from the one-valent vertices of Γ'_1.

Take a positive number ε so that it is not greater than the length of any leaf. Here a *leaf* is an edge of Γ'_1 which has one of $w_1, \cdots, w_c$ as one of its ends. Let $\Gamma'_{1,\varepsilon}$ be the graph obtained from Γ'_1 by shortening each leaf f_i, $i = 1, \cdots, c$ by the length $\frac{\varepsilon}{a_i}$. Here f_i has w_i as one of its ends and a_i is the coefficient of w_i in D'_1 (see the above expression for D'_1).

Let $w_{i,\varepsilon}$ be the point of $\Gamma'_{1,\varepsilon}$ corresponding to w_i in the obvious way. If the length of the leaf f_i is greater than ε, then $w_{i,\varepsilon}$ is again a one-valent vertex, and if the length of f_i is equal to ε, then $w_{i,\varepsilon}$ coincides with another end of f_i. We point out that in the latter case, some $w_{i,\varepsilon}$ and $w_{i',\varepsilon}$ may give the same point even if $i \neq i'$. So we have a divisor

$$\sum_{i=1}^{c} a_i w_{i,\varepsilon}$$

on $\Gamma'_{1,\varepsilon}$.

Note that the graph $\Gamma'_{1,\varepsilon}$ is naturally a sub metric graph of Γ although Γ'_1 is not in general. So the divisor $\sum_{i=1}^{c} a_i w_{i,\varepsilon}$ can be seen as a divisor on Γ. Also note that

$$\sum_{i=1}^{c} a_i w_i$$

naturally gives an effective divisor on γ by mapping w_i to one of the points from $v_1, \cdots, v_a$ where it is cut. Let

$$\sum_{i=1}^{c} a_i \bar{w}_i$$

be this divisor on Γ.

Summarizing, we replaced the subdivisor $\sum_{i=1}^{c} a_i \bar{w}_i$ of D to a divisor $\sum_{i=1}^{c} a_i w_{i,\varepsilon}$, and obtained a new effective divisor

$$\tilde{D} = D - \sum_{i=1}^{c} a_i \bar{w}_i + \sum_{i=1}^{c} a_i w_{i;\varepsilon}.$$

Definition 3.3. We call the above process to obtain $\tilde{D}$ from D a *chip-firing deformation.*

The following results hold for chip-firing deformation, analogously to the case of chip-firing move.

Lemma 3.4. *If $\tilde{D}$ is obtained from D by a chip-firing deformation, then $\tilde{D}$ is linearly equivalent to D.* □

Proposition 3.5. *If D' is an effective divisor which is linearly equivalent to D, then D' is obtained from D by a sequence of chip-firing deformations.* □

As an example of the use of the notion of chip-firing deformation, we prove the following result.

Lemma 3.6. *Let l_i, $i \in \{0, 1, 4, \cdots, 3g-2\}$ be a vertical edge of Γ_g. Let D be an effective divisor on Γ_g whose degree is bounded by $3g$. Assume that the intersection*

$$Supp(D) \cap int(l_i),$$

where $int(l_i) = l_i \setminus \partial l_i$, is one point whose coefficient is 1 and which is contained in a medium neighborhood of the middle point of l_i. Then, if D' is any effective divisor linearly equivalent to D, the intersection

$$Supp(D') \cap int(l_i)$$

is not empty.

Remark 3.7. *Note that the right most vertical edge l_{3g-1} is excluded from the statement of the lemma.*

Proof. Let p be the point $Supp(D) \cap int(l_i)$. Let a, b be the two end points of ∂l_i. For any divisor

$$E = \sum_{j \in J} b_j q_j$$

on Γ_g, consider the following real valued function f_i. Namely, let

$$E' = \sum_{k \in K} b_k q_k$$

be the subdivisor of E where

$$Supp(E) \cap l_i = \{q_k\}_{k \in K}.$$

Then

$$f_i(E) = \sum_{k \in K} b_k(d(a, q_k) - d(b, q_k)),$$

where d is the distance function on Γ_g. By assumption,

$$f_i(D) \equiv \frac{1}{2}\varepsilon^i + O(\varepsilon^{i+1}) \mod \varepsilon^i.$$

If the statement of the lemma is not true, there is an effective divisor D' linearly equivalent to D such that

$$f_i(D') \equiv 0 \mod \varepsilon^{i+1}.$$

Consider a chip-firing deformation which changes the value of f_i. Obviously the relevant graph Γ'_1 must contain a part of the edge l_i. Then there are two cases:

(1) There are two vertices of Γ'_1 contained in l_i.
(2) There is only one vertex of Γ'_1 contained in l_i.

In the case (1), the graph Γ'_1 is a subsegment of l_i, or contains the complement of a subsegment of l_i. In each case, it is clear that the corresponding chip-firing deformation does not change f_i. In the case (2), the graph Γ'_1 contains a part of some edge l_j with $j > i$. Then, the corresponding chip-firing deformation can change the value of f_i at most $O(\varepsilon^{i+1})$. Since the degree of the divisor D is bounded by $3g$, it follows that we can change the value of f_i at most $O(\varepsilon^{i+1})$, proving the lemma. □

We name the vertices of the graph Γ_g as follows:

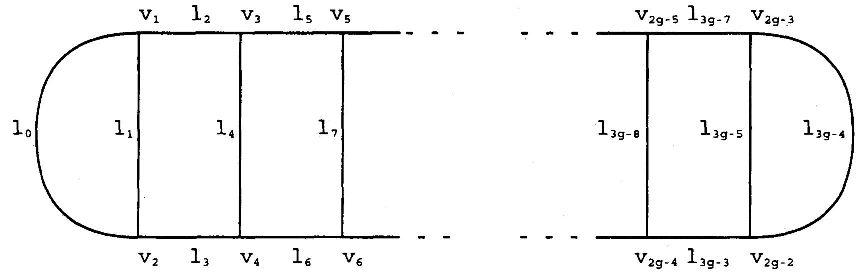

FIGURE 2

Consider the following open graph γ_{k+1} of the graph Γ_g:

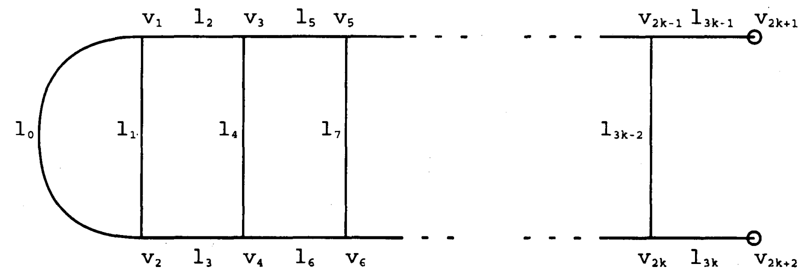

FIGURE 3

It has $k+1$ vertical edges $l_0, l_1, l_4, \cdots, l_{3k-2}$. Let $a_0, a_1, a_4, \cdots, a_{3k-2}$ be the middle points of these edges. Let S_{k+1} be the set of ordered sequences of $k+1$ nonnegetive integers $(z_0, z_1, z_4, \cdots, z_{3k-2})$ satisfying

$$z_0 + z_1 + z_4 + \cdots + z_{3k-2} = k + 1.$$

Let $\mathcal{D}_{k+1}$ be the following set of effective divisors

$$\mathcal{D}_{k+1} = \{z_0a_0+z_1a_1+z_4a_4+\cdots+z_{3k-2}a_{3k-2} \mid (z_0, z_1, z_4, \cdots, z_{3k-2}) \in \mathcal{S}_{k+1}\}.$$

Using chip-firing deformation, we can also prove the following result.

Proposition 3.8. *Let D be an effective divisor of degree d on Γ_g, where d is a positive integer not larger than $3g$. Let $E = D \cap \gamma_{k+1}$ be the subdivisor of D whose support is contained in γ_{k+1}, where k is a fixed nonnegative integer with $k \leq g-1$. Assume the divisor D satisfies the following two conditions:*

(a) *For any element α of $\mathcal{D}_{k+1}$, there exists a divisor linearly equivalent to D which contains α as a subdivisor.*

(b) *D is linearly equivalent to an effective divisor D' such that the subdivisor $E' = D' \cap \gamma_{k+1}$ satisfies*

$$\deg E' \leq k,$$

Then D is linearly equivalent to an effective divisor $\tilde{D}$ such that the subdivisor $\tilde{E} = \tilde{D} \cap \gamma_{k+1}$ satisfies

$$deg\tilde{E} \geq 2k+2.$$

□

Using this proposition, we can prove the following result.

Proposition 3.9. *Let D be an effective divisor on Γ_g with $\deg(D) \leq 3g$ such that for any member α of $\mathcal{D}_{k+1}$, $k \leq g-1$, there is a divisor D' linearly equivalent to D such that D' contains α as a subdivisor. Then for any effective divisor $\tilde{D}$ linearly equivalent to D, the degree of the intersection $\tilde{E} = \tilde{D} \cap \gamma_{k+1}$ is at least k.* □

Now we outline the proof of Theorem 2.2. In this note we give a detailed argument only for the case $p = 2$. Namely,

$$(g, d) = (r, 2r-1).$$

Assume that there is an effective divisor D on Γ_g with $\deg(D) = 2g-1$ which has rank at least r. Let a, b be points on the edge l_{3g-1} such that $d(v_{2g-3}, a) = \varepsilon^{3g-1} - 10\varepsilon^{3g-2}$ and $d(v_{2g-3}, b) = \varepsilon^{3g-1} - 5\varepsilon^{3g-2}$. Let $\overline{ab}$ be the interval on l_{3g-1} of length $5\varepsilon^{3g-2}$ whose end points are a, b. Let Γ'_g be the open subgraph of Γ_g defined by

$$\Gamma'_g = \Gamma_g \setminus \overline{ab}.$$

Apply Proposition 3.9 to the graph Γ'_g. Note that the graph Γ'_g is not the same as the graph γ_g in Proposition 3.9, the proposition is applicable to Γ'_g. Then any effective divisor D' linearly equivalent to D contains a subdivisor of degree at least g whose support is contained in Γ'_g. On the other hand, there must be an effective divisor linearly equivalent to D which has ga as a subdivisor, where a is a point in $\overline{ab}$. However, such a divisor has degree at least $2g$, a contradiction. This proves Theorem 2.2 for the case $p = 1$ $(\rho = -1)$. The other cases with $\rho = -1$ can be proved by similar argument. $\square$

References

[1] Biggs, N., *Chip-firing and the critical group of a graph.* J. Algebraic Combinatorics., 9 (1999), 25-45.

[2] Baker, M. and Norine, S., *Riemann-Roch and Abel-Jacobi Theory on a Finite Graph.*, Advances in Mathematics 215 (2007), 766–788.

[3] Caporaso,L., *Algebraic and combinatorial Brill-Noether theory.* ArXiv:1106.1140.

[4] Cools, F., Draisma, J, Payne, S. and Robeva, E. *A tropical proof of the Brill-Noether Theorem.* Advances in Mathematics. 230 (2012), 759-776.

Department of Mathematics, Rikkyo University, Tokyo, Japan
E-mail address: nishinou@rikkyo.ac.jp