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Fixed Point Theorems and the BCS Gap Equation for Superconductivity

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1 Introduction

In this paper we study the temperature dependence of the nonzero solution to the BCS gap equation for superconductivity [2, 4]:

$$u(T,\,x)=\int_0^{\hbar\omega_D}\frac{U(x,\,\xi)\,u(T,\,\xi)}{\sqrt{\xi^2+u(T,\,\xi)^2}}\,\tanh\frac{\sqrt{\xi^2+u(T,\,\xi)^2}}{2T}\,d\xi.$$

Here, the solution u is a function of the absolute temperature $T\geq 0$ and the energy x ($0\leq x\leq\hbar\omega_D$), and ω_D stands for the Debye angular frequency. The potential U satisfies $U(x,\,\xi)>0$.

The integral with respect to the energy ξ in the BCS gap equation is sometimes replaced by the integral over $\mathbb{R}^3$ with respect to the wave vector of an electron. Odeh [9], and Billard and Fano [3] established the existence and uniqueness of the positive solution to the BCS gap equation in the case $T=0$. For $T\geq 0$, Vansevenant [10] showed that there is a unique positive solution. Bach, Lieb and Solovej [1] dealt with the gap equation in the Hubbard model for a constant potential and showed that the solution is strictly decreasing with respect to the temperature. Recently, Frank, Hainzl, Naboko and Seiringer [5] gave a rigorous analysis of the asymptotic behavior of the transition temperature at weak coupling. Hainzl, Hamza, Seiringer and Solovej [6] proved that the existence of a positive solution is equivalent to the existence of a negative eigenvalue of a certain linear operator to show the existence of a transition temperature. Moreover, Hainzl and Seiringer [7] derived upper and lower bounds on the transition temperature and the energy gap for the BCS gap equation.

Since the existence and uniqueness of the solution were established for each fixed T in the previous literature, the temperature dependence of the solution is not covered except for the work by Bach, Lieb and Solovej [1]. It is well known that studying the temperature dependence of the solution to the BCS gap equation is very important in condensed matter physics. This is because, by dealing with the thermodynamical potential, this study leads to a mathematical proof of the statement that the transition to a superconducting state is a second-order phase transition in the BCS model. So it is highly desirable to study the temperature dependence of the solution to the BCS gap equation.

To this end we define a nonlinear integral operator A by

$$Au(T,\, x) = \int_0^{\hbar\omega_D} \frac{U(x,\, \xi)\, u(T,\, \xi)}{\sqrt{\xi^2 + u(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u(T,\, \xi)^2}}{2T}\, d\xi.$$

Here the right side of this equality is exactly the right side of the BCS gap equation. Our nonlinear integral operator A is defined on the sets V_T and V specified later. Since the solution to the BCS gap equation is a fixed point of the operator A , we apply fixed point theorems to the operator A and study the temperature dependence of the nonzero solution to the BCS gap equation.

2 The simple gap equation with a constant potential

We first deal with the case where the potential of the BCS gap equation is a positive constant. Let $U(x,\, \xi) = U_1$ at all $(x,\, \xi) \in [0,\, \hbar\omega_D]^2$, where $U_1 > 0$ is a positive constant. Then the solution to the BCS gap equation depends on the temperature T only. So we denote the solution by Δ_1 in this case, i.e., $\Delta_1 : T \mapsto \Delta_1(T)$. Then the BCS gap equation reduces to the simple gap equation [2]

$$1 = U_1 \int_0^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_1(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_1(T)^2}}{2T}\, d\xi.$$

The following is the definition of the temperature $\tau_1 > 0$.

Definition 2.1 (See Bardeen, Cooper and Schrieffer [2]).

$$1 = U_1 \int_0^{\hbar\omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_1}\, d\xi.$$

See also Niwa [8] and Ziman [14]. The implicit function theorem implies the following.

Proposition 2.2 ([11, Proposition 2.2]). *Set*

$$\Delta = \frac{\hbar\omega_D}{\sinh \frac{1}{U_1}}.$$

Then there is a unique nonnegative solution $\Delta_1 : [0,\, \tau_1] \to [0,\, \infty)$ to the simple gap equation such that the solution Δ_1 is continuous and strictly decreasing with respect to the temperature on the closed interval $[0,\, \tau_1]$:

$$\Delta_1(0) = \Delta > \Delta_1(T_1) > \Delta_1(T_2) > \Delta_1(\tau_1) = 0, \qquad 0 < T_1 < T_2 < \tau_1.$$

Moreover, the solution Δ_1 is of class C^2 on the interval $[0,\, \tau_1)$ and satisfies

$$\Delta_1'(0) = \Delta_1''(0) = 0 \quad \textit{and} \quad \lim_{T \uparrow \tau_1} \Delta_1'(T) = -\infty.$$

Proof. Set $Y = \Delta_1(T)^2$. Then the simple gap equation above becomes

$$1 = U_1 \int_0^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + Y}} \tanh \frac{\sqrt{\xi^2 + Y}}{2T}\, d\xi.$$

Note that the right side of this equality is a function of the two variables T and Y after integration with respect to the variable ξ. We moreover see that there is a unique function $T \mapsto Y$ implicitly defined by this equality. The implicit function theorem thus implies the result. □

Remark 2.3. We set $\Delta_1(T) = 0$ for $T > \tau_1$.

We introduce another positive constant $U_2 > 0$. Let $0 < U_1 < U_2$. When $U(x, \xi) = U_2$ at all $(x, \xi) \in [0, \hbar\omega_D]^2$, an argument similar to that in the proposition above gives that there is a unique nonnegative solution $\Delta_2 : [0, \tau_2] \to [0, \infty)$ to the simple gap equation

$$1 = U_2 \int_0^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_2(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_2(T)^2}}{2T}\, d\xi, \qquad 0 \le T \le \tau_2.$$

Here, $\tau_2 > 0$ is defined by

$$1 = U_2 \int_0^{\hbar\omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_2}\, d\xi.$$

We again set $\Delta_2(T) = 0$ for $T > \tau_2$. A straightforward calculation gives the following.

Lemma 2.4 ([12, Lemma 1.5]). (a) *The inequality* $\tau_1 < \tau_2$ *holds.*
(b) *If* $0 \le T < \tau_2$, *then* $\Delta_1(T) < \Delta_2(T)$. *If* $T \ge \tau_2$, *then* $\Delta_1(T) = \Delta_2(T) = 0$.

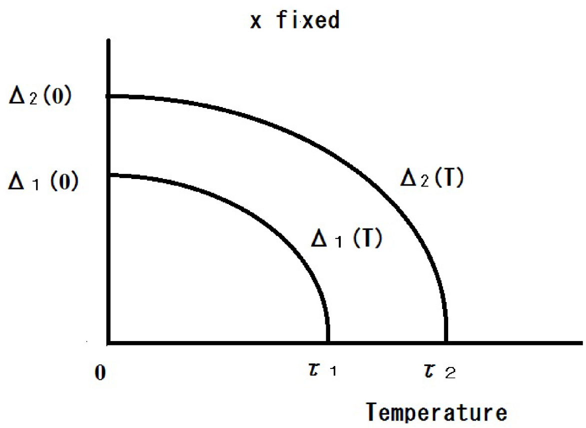

Figure 1: The graphs of the functions Δ_1 and Δ_2.

3 The BCS gap equation with a nonconstant potential

We assume the following condition on $U(\cdot, \cdot)$:

$$U_1 \le U(x, \xi) \le U_2 \quad \text{at all} \quad (x, \xi) \in [0, \hbar\omega_D]^2, \qquad U(\cdot, \cdot) \in C([0, \hbar\omega_D]^2).$$

Let $0 \le T \le \tau_2$ and fix T. We consider the Banach space $C[0, \hbar\omega_D]$ consisting of continuous functions of x only, and deal with the following temperature dependent subset V_T:

$$V_T = \{u(T, \cdot) \in C[0, \hbar\omega_D] : \ \Delta_1(T) \le u(T, x) \le \Delta_2(T) \text{ at } x \in [0, \hbar\omega_D]\}.$$

The Schauder fixed-point theorem implies the following.

Theorem 3.1 ([12, Theorem 2.2]). *Assume the condition above on $U(\cdot, \cdot)$. Let $T \in [0, \tau_2]$ be fixed. Then there is a unique nonnegative solution $u_0(T, \cdot) \in V_T$ to the BCS gap equation ($x \in [0, \hbar\omega_D]$)*

$$u_0(T, x) = \int_0^{\hbar\omega_D} \frac{U(x, \xi)\, u_0(T, \xi)}{\sqrt{\xi^2 + u_0(T, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T, \xi)^2}}{2T}\, d\xi.$$

Consequently, the solution is continuous with respect to x and varies with the temperature as follows:

$$\Delta_1(T) \le u_0(T, x) \le \Delta_2(T) \quad \text{at} \quad (T, x) \in [0, \tau_2] \times [0, \hbar\omega_D].$$

Proof. Clearly, V_T is a bounded, closed and convex subset of the Banach space $C[0, \hbar\omega_D]$. A straightforward calculation gives that our nonlinear integral operator $A : V_T \to V_T$ is compact. The Schauder fixed-point theorem thus implies the result. We can show the uniqueness of the nonzero fixed point of A defined on V_T by deriving a contradiction. $\square$

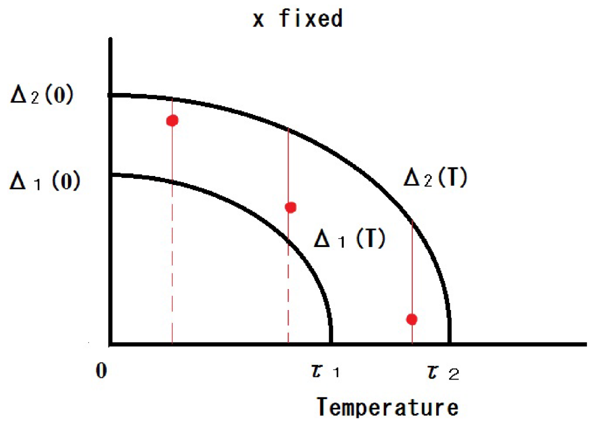

Figure 2: For each T, the solution $u_0(T, x)$ lies between $\Delta_1(T)$ and $\Delta_2(T)$.

4 Continuity of the solution with respect to the temperature

Let $U_0 > 0$ be a positive constant satisfying $U_0 < U_1 < U_2$. An argument similar to that in the proposition above gives that there is a unique nonnegative solution $\Delta_0 : [0, \tau_0] \to [0, \infty)$ to the simple gap equation

$$1 = U_0 \int_0^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_0(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_0(T)^2}}{2T} \, d\xi, \qquad 0 \le T \le \tau_0.$$

Here, $\tau_0 > 0$ is defined by

$$1 = U_0 \int_0^{\hbar\omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_0} \, d\xi.$$

We set $\Delta_0(T) = 0$ for $T > \tau_0$. A straightforward calculation gives the following.

Lemma 4.1 ([13, Lemma 1.1]). (a) $\tau_0 < \tau_1 < \tau_2$.
(b) *If* $0 \le T < \tau_0$, *then* $0 < \Delta_0(T) < \Delta_1(T) < \Delta_2(T)$.
(c) *If* $\tau_0 \le T < \tau_1$, *then* $0 = \Delta_0(T) < \Delta_1(T) < \Delta_2(T)$.
(d) *If* $\tau_1 \le T < \tau_2$, *then* $0 = \Delta_0(T) = \Delta_1(T) < \Delta_2(T)$.
(e) *If* $\tau_2 \le T$, *then* $0 = \Delta_0(T) = \Delta_1(T) = \Delta_2(T)$.

Remark 4.2. Let the functions Δ_l $(l = 0, 1, 2)$ be as above. For each Δ_l, there is the inverse $\Delta_l^{-1} : [0, \Delta_l(0)] \to [0, \tau_l]$. Here,

$$\Delta_l(0) = \frac{\hbar\omega_D}{\sinh \frac{1}{U_l}}$$

and $\Delta_0(0) < \Delta_1(0) < \Delta_2(0)$. See [13] for more details.

We introduce another temperature T_1. Let T_1 satisfy $0 < T_1 < \Delta_0^{-1}\left(\frac{\Delta_0(0)}{2}\right)$ and

$$\frac{\Delta_0(0)}{4\,\Delta_2^{-1}\left(\Delta_0(T_1)\right)} \tanh \frac{\Delta_0(0)}{4\,\Delta_2^{-1}\left(\Delta_0(T_1)\right)} > \frac{1}{2}\left(1 + \frac{4\hbar^2\omega_D^2}{\Delta_0(0)^2}\right).$$

Consider the following subset V of the Banach space $C([0, T_1] \times [0, \hbar\omega_D])$ consisting of continuous functions of both the temperature T and the energy x:

$$\begin{aligned} V &= \{u \in C([0, T_1] \times [0, \hbar\omega_D]) : \Delta_1(T) \le u(T, x) \le \Delta_2(T) \\ &\qquad at \ (T, x) \in [0, T_1] \times [0, \hbar\omega_D]\}. \end{aligned}$$

The Banach fixed-point theorem implies the following.

Theorem 4.3 ([13, Theorem 1.2]). *Assume the condition above on* $U(\cdot, \cdot)$. *Let* u_0, T_1 *and* V *be as above. Then* $u_0 \in V$. *Consequently, the solution* u_0 *to the BCS gap equation is continuous on* $[0, T_1] \times [0, \hbar\omega_D]$.

Proof. Clearly, V is a closed subset of our Banach space $C([0, T_1] \times [0, \hbar\omega_D])$. A straightforward calculation gives that our nonlinear integral operator $A : V \to V$ is contractive as long as T_1 satisfies the conditions mentioned before. The Banach fixed-point theorem thus implies the result. □

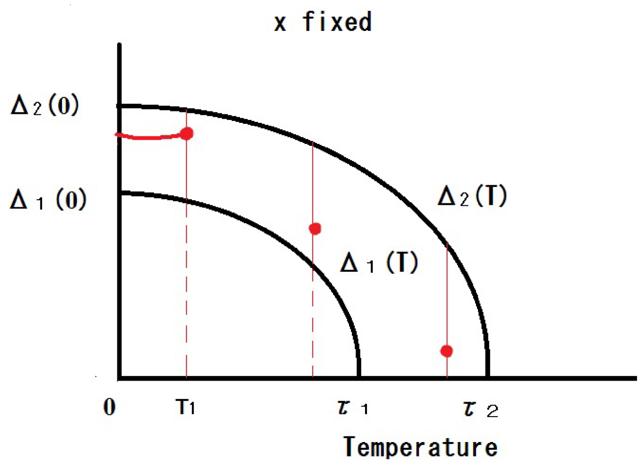

Figure 3: The solution u_0 is continuous on $[0,\, T_1] \times [0,\, \hbar\omega_D]$.

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