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ON G-BI-ISOVARIANT EQUIVALENCE BETWEEN G-REPRESENTATION SPACES

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ABSTRACT. Let G be a compact Lie group. In this paper, we introduce a new equivalent relation between real G-representation spaces, that is, we say that G-representation spaces V and W are G-bi-isovariantly equivalent and write as $V \rightleftharpoons_G W$ if there exist G-isovariant maps $V \to W$ and $W \to V$. We show that G-bi-isovariant equivalence between real G-representations V, W with $V^G = W^G = \{O\}$ implies $\mathrm{Dim}\, V = \mathrm{Dim}\, W$ if G is finite, or $V \cong W$ if G has positive dimension.

1. Introduction and Main Theorem

Throughout this paper, all maps are thought to be continuous. Let G be a compact Lie group. Suppose X and Y are G-spaces. Clearly, every G-equivariant map $\varphi : X \to Y$ satisfies $G_x \subset G_{\varphi(x)}$, where G_x denotes the isotropy subgroup of G at x. A G-equivariant map $\varphi : X \to Y$ is called a *G-isovariant map* if $G_x = G_{\varphi(x)}$ holds for all $x \in X$. In other words, φ is a G-isovariant map if $\varphi|_{G(x)}$ is injective, where $G(x)$ denotes the G-orbit through x.

In this article, we will consider G-isovariant maps between real G-representation spaces. Let V and W be real G-representations with the G-fixed point sets V^G and W^G respectively. By using Wassermann's results proved in [6], we can easily show the following result.

Proposition 1.1 (Isovariant Borsuk-Ulam theorem). *Let G be a compact solvable Lie group. If there is a G-isovariant map $\varphi : V \to W$, then the Borsuk-Ulam inequality*

$$\dim V/V^G \leqq \dim W/W^G,$$

that is,

$$\dim V - \dim V^G \leqq \dim W - \dim W^G$$

holds.

Incidently, the reason why Propositon 1.1 is called Isovariant Borsuk-Ulam theorem is what it is originated from the Borsuk-Ulam theorem ([1]):

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Proposition 1.2 (The Borsuk-Ulam theorem). *Let C_2 be a cyclic group of order 2. Assume that C_2 acts on both S^m and S^n antipodally. If there exists a continuous C_2-map $f : S^m \to S^n$, then $m \leqq n$ holds.*

It is unknown whether similar statements as Propositon 1.1 hold for any compact Lie group. The group G is called a Borsuk-Ulam group (BUG) if whenever there is a G-isovariant map $\varphi : V \to W$, then the Borsuk-Ulam inequality

$$\dim V/V^G \leqq \dim W/W^G,$$

that is,

$$\dim V - \dim V^G \leqq \dim W - \dim W^G$$

holds. Wasserman conjectured that all compact Lie groups are BUGs. He gave a sufficient condition called the prime condition for being a BUG. In our previous work [2], we proved that it is not necessary, that is, we showed there are infinitely many finite groups which does not satisfy it. For the proof, we introduced a new sufficient condition called the Möbius condition.

In the present work, we introduce a new aspect. Namely, we give insight to the relationship between V and W when there exists an isovariant map from not only V to W but also W to V without the assumption that G is a BUG.

Definition 1.3. Let G be a compact Lie group. Let V and W be G-representations. We say that V and W are G-bi-isovariantly equivalent and write as $V \rightleftharpoons_G W$ if there exist G-isovariant maps $V \to W$ and $W \to V$.

Clearly G-bi-isovariant equivalence is an equivalent relation, and $V \rightleftharpoons_G W$ implies $V \rightleftharpoons_H W$ for any subgroup H.

Let $\mathcal{S}(G)$ be the set of all subgroup of G, V a real G-representation space. We define the dimension function

$$\operatorname{Dim} V : \mathcal{S}(G) \to \mathbb{Z} \quad \text{by} \quad H \mapsto \dim V^H.$$

Then, we have the following theorem :

Theorem 1.4. *Let G be a compact Lie group, and V, W real G-representations such that $V^G = W^G = \{O\}$. Assume $V \rightleftharpoons_G W$. Then, $\operatorname{Dim} V = \operatorname{Dim} W$, that is, $\dim V^H = \dim W^H$ holds for any $H \in \mathcal{S}(G)$. Moreover, if $\dim G > 0$ and G is connected, then V is isomorphic to W as G-representations.*

This article is constructed as follows. In section 2, we give a proof of our theorem when G is finite. In the last section, we explain that isovariant condition is essential in our result, and generalize our main theorem.

2. Proof of our theorem

In this section we prove our theorem when G is finite. The non finite case is shown by using Traczyk's result ([4]), which will be shown in our upcoming paper.

Let G be a finite group. For any $H \in \mathcal{S}(G)$, it holds that

$$\dim V = \frac{1}{|H|}\sum_{g\in H}\chi_V(1), \quad \dim V^H = \frac{1}{|H|}\sum_{g\in H}\chi_V(g).$$

Hence,

$$\dim W - \dim W^H - (\dim V - \dim V^H) = \frac{1}{|H|}\sum_{g\in H}(\chi_W(1) - \chi_W(g) - \chi_V(1) + \chi_V(g)).$$

Put

$$h(H) = \sum_{g\in H}(\chi_W(1) - \chi_W(g) - \chi_V(1) + \chi_V(g)).$$

Then, by [2] we see that

$$h(H) = \sum_{D\in \mathrm{Cycl}(H)} \Big(\sum_{D \leqq C:\text{cyclic} \leqq H} \mu(D, C)\Big) h(D),$$

where $\mathrm{Cycl}(H)$ denotes the set of all cyclic subgroups of H, and $\mu(\ , \ )$ is the Möbius function. Since $D \in \mathrm{Cycl}(H)$ is a BUG and $V \rightleftharpoons_G W$, we have

$$\dim V - \dim V^D = \dim W - \dim W^D,$$

that is, $h(D) = 0$ by Proposition 1.1. Thus, we have

$$\dim V - \dim V^H = \dim W - \dim W^H$$

for any subgroup H of G. Since $V^G = W^G = \{O\}$, by choosing G as H, we see that $\dim V$ must be equal to $\dim W$, and consequently $\mathrm{Dim}\, V = \mathrm{Dim}\, W$.

3. Remarks

Our theorem does not hold without the assumption that the maps are isovariant. Waner gave a necessary and sufficient condition for the exixtence of a G-map from $S(V) \to S(W)$ with $V \supset W$, where $S(V)$ and $S(W)$ denote the unit spheres ([5]). By using Waner's criterion, we see the following :

Example 3.1. Let $G = C_{pq}$ a cyclic group of order pq where p and q are distinct prime numbers. For $i = 1, p, q$, let (T_i, ρ_i) be the complex 1-dimensional representation of G such that $\rho_i(g)(z) = \zeta^i z$, where $z \in \mathbb{C}$ and $\zeta = \exp \dfrac{2\pi\sqrt{-1}}{pq}$. Put

$$V = T_1 \oplus T_p \oplus T_q \quad \textit{and} \quad W = T_p \oplus T_q.$$

Then, they satisfy Waner's criterion, thereby, there exist a G-map from $S(V)$ to $S(W)$.

As is stated in Theorem 1.4, if G is finite, $\operatorname{Dim} V = \operatorname{Dim} W$ holds. Do there exist finite groups such that $V \rightleftharpoons_G W$ imply $V \cong W$? At the last of this article, we give insight to the problem.

Decompose V and W into the direct sums of irreducible representations as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \quad \text{and} \quad W = W_1 \oplus W_2 \oplus \cdots \oplus W_s.$$

Then, according to tom Dieck's book [3], $\operatorname{Dim} V = \operatorname{Dim} W$ if and only if $r = s$ and for each i, V_i is Galois conjugate to some $W_{\sigma(i)}$, where σ is a permutation of $\{1, 2, \ldots, r\}$, that is, there exists $\psi \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ such that $\psi(\chi_{V_i}) = \chi_{W_{\sigma(i)}}$, where $n = \mathrm{LCM}\{\ |g| \mid g \in G\}$. Thus, we obtain the following :

Proposition 3.2. *Let G be a finite group, and V, W real G-representations such that $V^G = W^G = \{O\}$. Under the above conditions, if the action of $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is trivial, $V \rightleftharpoons_G W$ implies $V \cong W$.*

As a corollary, we have :

Corollary 3.3. *Let G be a finite group. Let V and W be real G-representation spaces such that $V^G = W^G = \{O\}$. Assume $V \rightleftharpoons_G W$. Then, if $\chi_V \in \mathbb{Q}$, then $V \cong W$.*

We can illustrate some examples.

Example 3.4. Let G be one of the following groups. Let V and W be real G-representation spaces such that $V^G = W^G = \{O\}$. Then, the characters of all real G-representations take the value in $\mathbb{Q}$. Therefore, $V \rightleftharpoons_G W$ implies $V \cong W$.

- $\mathfrak{S}_n$: the symmetric group of degree n with $n \in \mathbb{N}$.
- C_n : the cyclic group of order n with $n = 2, 3, 4, 6$.
- $C_2^k \times C_3^\ell$: the direct product of C_2's and C_3's, where $k, \ell \geqq 0$.
- C_4^k : the direct product of C_4's, where $k \geqq 1$.
- Q_8^k : the direct product of the quaternion group Q_8's, where $k \geqq 1$.
- D_4^k : the direct product of the diheadral group D_4's, where $k \geqq 1$.

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