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GROUP ACTIONS ON COMPLEX PROJECTIVE SPACES VIA GROUP ACTIONS ON DISKS AND SPHERES

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Dedicated to Professors Mikiya Masuda and Masaharu Morimoto on the occasion of their 60th birthdays

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1. Two questions in transformation groups

When studying classification problems in the theory of transformation groups one usually focuses on smooth actions of compact Lie groups G on specific manifolds M, such as Euclidean spaces, disks, spheres, and complex projective spaces. Consider the following two basic related questions.

(1) *Which manifolds F are diffeomorphic to the corresponding fixed points sets M^G in M?*
(2) *Which G-vector bundles over F are isomorphic to the G-normal bundles of M^G in M?*

Our goal is to discuss results related to (1) obtained so far for actions on Euclidean spaces, disks, and spheres, and then to describe new results for actions on complex projective spaces obtained in [2]. Hence, every manifold F which occurs as the fixed point set is a second-countable space, i.e., F is paracompact and F has countably many connected components, possibly not of the same dimension.

We do not discuss the Smith theory and the converse related results. Except for Theorem 2.1 below, the acting group G is *always* a finite group not of prime power order.

2. Equivariant triangulation and thickening conclusion

For a finite dimensional countable CW-complex X, let $\widetilde{KO}(X)$ be the reduced real K-theory of X. More generally, if G is a compact Lie group and X is a G-CW-complex (i.e., a topological space built up from G-equivariant cells), we denote by $\widetilde{KO}_G(X)$ the G-equivariant reduced real K-theory of X.

Theorem 2.1. *Let G be a compact Lie group and let F be a smooth manifold such that F is compact (resp., $\partial F = \emptyset$). Let ν be a real G-vector bundle over F such that $\dim \nu^G = 0$. Then the following two statements are equivalent.*

(1) *There exists a finite (resp., finite dimensional, countable) contractible G-CW-complex X such that $X^G = F$, and the Whitney sum $\tau_F \oplus \nu$ stably extends to a real G-vector bundle over X, i.e., the class $[\tau_F \oplus \nu]$ lies in the image of the restriction map*
$$\widetilde{KO}_G(X) \to \widetilde{KO}_G(F).$$
(2) *There exists a smooth action of G on a disk (resp., Euclidean space) M such that* (i) *the fixed point set M^G is diffeomorphic to F, and* (ii) *the G-equivariant normal bundle of M^G in M is stably isomorphic to ν.*

For a smooth G-manifold M with fixed point set F, the tangent bundle τ_M has the structure of a real G-vector bundle over M such that $\tau_M|_F \cong \tau_F \oplus \nu$, where ν is the G-equivariant normal bundle of F in M. In particular, G acts trivially on the tangent bundle τ_F and $\dim \nu^G = 0$. Moreover, by the Equivariant Triangulation Theorem [1], M has the structure of a G-CW-complex containing F as a subcomplex.

Therefore, in Theorem 2.1, if (2) is true, so is (1). The converse implication, (1) implies (2), follows by the Equivariant Thickening Theorem [11].

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3. Six G-fixed point set bundle conditions

The discussion below is based on the work of Oliver [10]. For a finite group G, denote by $\mathcal{P}(G)$ the family of subgroups of G of prime power order. Two real (resp., complex) G-modules U and V are called *$\mathcal{P}(G)$-matched* if U and V are isomorphic as real (resp., complex) P-modules for every $P \in \mathcal{P}(G)$.

Definition 3.1. The *Oliver three-class series* $\mathcal{G}_{\mathbb{R}} \subset \mathcal{C}_{\mathbb{C}} \subset \mathcal{G}_{\mathbb{C}}$ of finite groups G not of prime power order, is defined as follows.

$G \in \mathcal{G}_{\mathbb{R}}$ if there exist two real $\mathcal{P}(G)$-matched G-modules U and V such that $\dim_{\mathbb{R}}(U^G) = 0$ and $\dim_{\mathbb{R}}(V^G) = 1$.

$G \in \mathcal{C}_{\mathbb{C}}$ if there exist complex $\mathcal{P}(G)$-matched G-modules U and V such that $\dim_{\mathbb{C}}(U^G) = 0$ and $\dim_{\mathbb{C}}(V^G) = 1$, and moreover U and V are self-conjugate.[1]

$G \in \mathcal{G}_{\mathbb{C}}$ if there exist complex $\mathcal{P}(G)$-matched G-modules U and V such that $\dim_{\mathbb{C}}(U^G) = 0$ and $\dim_{\mathbb{C}}(V^G) = 1$.

Lemma 3.2. *The following three statements are true.*

(1) *$G \in \mathcal{G}_{\mathbb{R}}$ if and only if there exist subgroups $N \trianglelefteq H \leq G$ such that H/N is isomorphic to the dihedral group of order $2pq$ for some two distinct primes p and q.*
(2) *$G \in \mathcal{C}_{\mathbb{C}}$ if and only if there exists an element $g \in G$ such that g is not of prime power order, and g is conjugate to its inverse g^{-1}.*
(3) *$G \in \mathcal{G}_{\mathbb{C}}$ if and only if G has an element g not of prime power order.*

Let $\mathcal{G}$ be the class of finite groups not of prime power order. Let $\mathcal{G}_2^{\triangleleft} \subset \mathcal{G}$ be the class of groups $G \in \mathcal{G}$ with a normal 2-Sylow subgroup G_2. Set $\mathcal{G}_2^{\ntriangleleft} = \mathcal{G} \setminus \mathcal{G}_2^{\triangleleft}$. Note that $\mathcal{C}_{\mathbb{C}} \subset \mathcal{G}_2^{\ntriangleleft}$, i.e., if G has an element g not of prime power order such that g is conjugate to its inverse, then G_2 is not normal in G.

Definition 3.3. The *Oliver six-class splitting* of the class $\mathcal{G}$ by the Oliver three-class series

$$\mathcal{G}_{\mathbb{R}} \subset \mathcal{C}_{\mathbb{C}} \subset \mathcal{G}_{\mathbb{C}} \subset \mathcal{G} = \mathcal{G}_2^{\triangleleft} \cup \mathcal{G}_2^{\ntriangleleft}$$

and the two classes $\mathcal{G}_2^{\triangleleft}$ and $\mathcal{G}_2^{\ntriangleleft}$, is the following decomposition of $\mathcal{G}$ into six mutually disjoint classes:

(1) $\mathcal{G}_{\mathbb{R}}$ and $\mathcal{C}_{\mathbb{C}} \setminus \mathcal{G}_{\mathbb{R}}$, both contained in $\mathcal{G}_2^{\ntriangleleft}$,
(2) $(\mathcal{G}_{\mathbb{C}} \setminus \mathcal{C}_{\mathbb{C}}) \cap \mathcal{G}_2^{\ntriangleleft}$ and $(\mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}) \cap \mathcal{G}_2^{\ntriangleleft}$,
(3) $(\mathcal{G}_{\mathbb{C}} \setminus \mathcal{C}_{\mathbb{C}}) \cap \mathcal{G}_2^{\triangleleft} = \mathcal{G}_{\mathbb{C}} \cap \mathcal{G}_2^{\triangleleft}$ and $(\mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}) \cap \mathcal{G}_2^{\triangleleft}$.

Consider the following maps (group homomorphisms):

– the complexification of real bundles

$$c_{\mathbb{R}} \colon \widetilde{KO}(F) \to \widetilde{KU}(F), \quad [\xi] \mapsto [\xi \otimes \mathbb{C}],$$

– the quaternization of complex bundles

$$q_{\mathbb{C}} \colon \widetilde{KU}(F) \to \widetilde{KSp}(F), \quad [\xi] \mapsto [\xi \otimes \mathbb{H}],$$

– the complexification of symplectic bundles

$$c_{\mathbb{H}} \colon \widetilde{KSp}(F) \to \widetilde{KU}(F), \quad [\xi] \mapsto [c_{\mathbb{H}}(\xi)],$$

– the realification of complex bundles

$$r_{\mathbb{C}} \colon \widetilde{KU}(F) \to \widetilde{KO}(F), \quad [\xi] \mapsto [r_{\mathbb{C}}(\xi)].$$

For an abelian group A, the subgroup $\operatorname{Div} A$ of *quasidivisible elements of* A is defined as

$$\operatorname{Div} A = \bigcap_{\varphi} \operatorname{Ker}(\varphi),$$

where φ varies within homomorphisms mapping A into free abelian groups. Note that if A is finitely generated then quasidivisible elements are simply torsion elements. In particular, if F is a compact smooth manifold, the K-theory groups of F are finitely generated, and therefore

$$\operatorname{Div} K(F) = \operatorname{Tor} K(F)$$

for the real, complex, and symplectic K-theory groups of F.

[1] A complex G-module is *self-conjugate* if it is isomorphic to its complex conjugate.

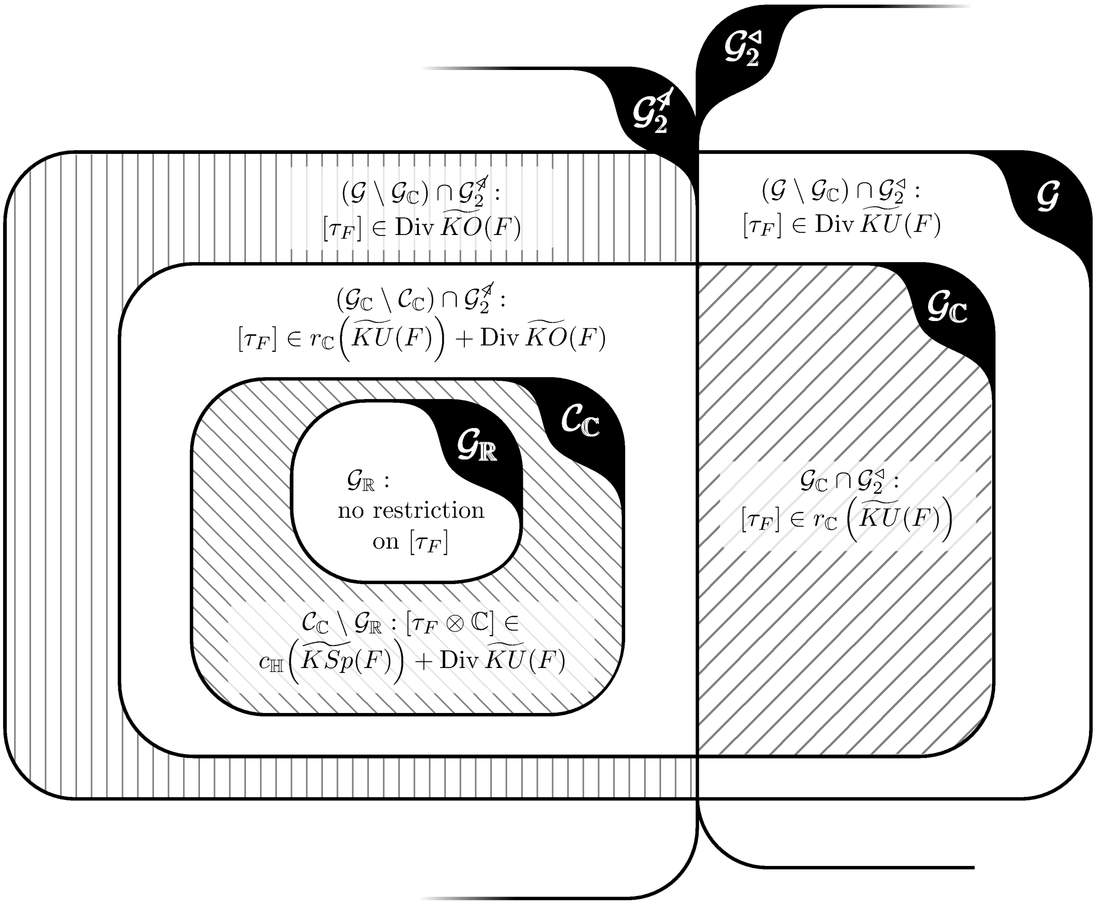

FIGURE 1. Oliver six-class splitting of $\mathcal{G}$ with six G-fixed point set bundle conditions

The six G-fixed point set bundle conditions defined below depend on the classes in the Oliver six-class splitting of $\mathcal{G}$, the class of finite groups G not of prime power order, described in Definition 3.3.

Definition 3.4. *The G-fixed point set bundle conditions.* Let $G \in \mathcal{G}$. Then the class $[\tau_F]$ of the tangent bundle τ_F of a smooth manifold F is said to be *well-G-located in* $\widetilde{KO}(F)$, provided:

(1) if $G \in \mathcal{G}_{\mathbb{R}}$: there is no restriction on the class

$$[\tau_F] \in \widetilde{KO}(F).$$

(2) if $G \in \mathcal{C}_{\mathbb{C}} \setminus \mathcal{G}_{\mathbb{R}}$:

$$[\tau_F \otimes \mathbb{C}] \in c_{\mathbb{H}}\Big(\widetilde{KSp}(F)\Big) + \operatorname{Div} \widetilde{KU}(F).$$

(3) if $G \in (\mathcal{G}_{\mathbb{C}} \setminus \mathcal{C}_{\mathbb{C}}) \cap \mathcal{G}_2^{\ntriangleleft}$:

$$[\tau_F] \in r_{\mathbb{C}}\Big(\widetilde{KU}(F)\Big) + \operatorname{Div} \widetilde{KO}(F).$$

(4) if $G \in (\mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}) \cap \mathcal{G}_2^{\ntriangleleft}$:

$$[\tau_F] \in \operatorname{Div} \widetilde{KO}(F).$$

(5) if $G \in \mathcal{G}_{\mathbb{C}} \cap \mathcal{G}_2^{\triangleleft}$:

$$[\tau_F] \in r_{\mathbb{C}}\Big(\widetilde{KU}(F)\Big).$$

(6) if $(G \in \mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}) \cap \mathcal{G}_2^{\triangleleft}$:

$$[\tau_F] \in r_{\mathbb{C}}\Big(\operatorname{Div} \widetilde{KU}(F)\Big).$$

4. Oliver number and equivariant extension of bundles

The results presented in this section have been obtained by Oliver [9, 10].

Proposition 4.1. *For a finite group G not of prime power order, the set*

$$\{\chi(X^G) - 1 \colon X \text{ is a finite contractible } G\text{-}CW\text{-complex}\}$$

forms a subgroup of the group $\mathbb{Z}$ of integers.

Therefore, if X is a finite contractible G-CW-complex, then

$$\chi(X) \equiv 1 \pmod{n_G}$$

for a unique integer $n_G \geq 0$. We refer to n_G as to the *Oliver number* of G.

If $n_G = 1$, G is called an *Oliver group*. In algebraic terms this means that G contains no normal series of subgroups $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime power order and H/P is cyclic. Examples of Oliver groups include finite nonsolvable groups, as well as finite nilpotent groups with three or more noncyclic Sylow subgroups.

Theorem 4.2. *Let G be a finite group not of prime power order, and let F be a finite CW-complex. Then there exists a finite contractible G-CW-complex X such that the fixed point set X^G is homeomorphic to F if and only if $\chi(F) \equiv 1 \pmod{n_G}$.*

For an abelian group A and a prime p, let $\mathrm{Div}_p^\infty A$ denote the subgroup of A consisting of the infinitely p divisible elements of A. Moreover, let $A_{(p)}$ denote the localization of A at p.

Let G be a finite group not of prime power order. For a finite dimensional, countable CW-complex F, consider the abelian group

$$\widetilde{KO}_{\mathcal{P}(G)}(F) = \widetilde{KO}(F) \oplus \bigoplus_{P \neq \{e\}} \widetilde{KO}_P(F)_{(p)} \Big/ \mathrm{Div}_p^\infty \widetilde{KO}_P(F)_{(p)}$$

where P varies within the family $\mathcal{P}(G)$.

According to Theorem 4.2, the Euler characteristic is the only obstruction for a finite CW-complex F to occur as the fixed point set of a finite contractible G-CW-complex X. The possibility of stable extension of a G-vector bundle η over F to a G-vector bundle ξ over X is obstructed by the location of the class $[\eta]$ in $\widetilde{KO}_G(F)$, namely, the stable extension ξ of η exists if and only if $[\eta]$ lies in the kernel of the canonical map $\widetilde{KO}_G(F) \to \widetilde{KO}_{\mathcal{P}(G)}(F)$.

In the case where X is not finite, the stable extension ξ of η is obstructed in the same way, but there is no restriction on the Euler characteristic of F.

Theorem 4.3. *Let G be a finite group not of prime power order, and let ν be a real G-vector bundle over a smooth manifold F, such that $\dim \nu^G = 0$. Assume also that F is compact and the Euler characteristic $\chi(F) \equiv 1 \pmod{n_G}$. Then the following three statements are equivalent.*

1. *The class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$.*
2. *The class $[\tau_F \oplus \nu]$ lies in the kernel of the canonical map*
$$\widetilde{KO}_G(F) \to \widetilde{KO}_{\mathcal{P}(G)}(F).$$
3. *There exists a finite contractible G-CW-complex X such that $X^G = F$ and the class $[\tau_F \oplus \nu]$ lies in the image of the restriction map*
$$\widetilde{KO}_G(X) \to \widetilde{KO}_G(F).$$

Theorem 4.4. *Let G be a finite group not of prime power order, and let ν be a real G-vector bundle over a smooth manifold F, such that $\dim \nu^G = 0$. Then the following three statements are equivalent.*

1. *The class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$.*
2. *The class $[\tau_F \oplus \nu]$ lies in the kernel of the canonical map*
$$\widetilde{KO}_G(F) \to \widetilde{KO}_{\mathcal{P}(G)}(F).$$
3. *There exists a finite dimensional, countable, contractible G-CW-complex X such that $X^G = F$ and the class $[\tau_F \oplus \nu]$ lies in the image of the restriction map*
$$\widetilde{KO}_G(X) \to \widetilde{KO}_G(F).$$

5. Group actions on disks and Euclidean spaces

Some of the results of this section were obtained in [11, 12], and the complete classification theorems presented here go back to Oliver [10].

Theorem 5.1. *Let G be a group not of prime power order. Then there exists a smooth action of G on some disk such that the fixed point set is diffeomorphic to a smooth manifold F if and only if*

(i) *F is compact, $\chi(F) \equiv 1 \pmod{n_G}$, and*
(ii) *the class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$.*

Theorem 5.2. *Let G be a group not of prime power order. There exists a smooth action of G on some Euclidean space such that the fixed point set is diffeomorphic to a smooth manifold F if and only if*

(i) *the boundary of F is empty, and*
(ii) *the class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$.*

Theorems 5.1 and 5.2 follow from Theorems 4.3 and 4.4, respectively, and Theorem 2.1.

6. Group actions on spheres

The results of this section have been obtained in the series of papers [4]–[8]. If G is a finite non-trivial perfect group, then any Sylow 2-subgroup of G is not normal in G. Therefore, the union of the classes

$$\mathcal{G}_{\mathbb{R}},\ \ \mathcal{C}_{\mathbb{C}} \setminus \mathcal{G}_{\mathbb{R}},\ \ (\mathcal{G}_{\mathbb{C}} \setminus \mathcal{C}_{\mathbb{C}}) \cap \mathcal{G}_2^{\ntriangleleft},\ \ (\mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}) \cap \mathcal{G}_2^{\ntriangleleft}$$

(cf. Definition 3.3) contains all finite non-trivial perfect groups. Moreover, every of the four classes above contains an infinite family of perfect groups.

Theorem 6.1. *Let G be a finite perfect group, and let F be a smooth manifold. There exists a smooth action of G on a sphere S such that the fixed point set S^G is diffeomorphic to F and $S^P \neq S^G$ for every $P \in \mathcal{P}(G)$, if and only if*

(i) *F is closed and*
(ii) *the class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$.*

Let G be a finite group with an element not of prime power order. Assume that G has a normal Sylow 2-subgroup G_2. Then $G \in \mathcal{G}_{\mathbb{C}} \cap \mathcal{G}_2^{\triangleleft}$ by Lemma 3.2. Unravelling the notion of well-G-location, we see that $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$ if and only if $[\tau_F]$ lies in the image of the map

$$r_{\mathbb{C}} \colon \widetilde{KU}(F) \to \widetilde{KO}(F).$$

This amounts to F being a *stably complex* manifold, i.e., the stable normal bundle of F admits a complex structure. In particular, the dimensions of the connected components of F are of the same parity.

Theorem 6.2. *Let G be a finite Oliver group with a quotient isomorphic to the cyclic group of order pqr for three distinct primes p, q, and r. Moreover, suppose G_2 is normal in G. Then there exists a smooth action of G on a sphere S such that the fixed point set S^G is diffeomorphic to F and $S^P \neq S^G$ for every $P \in \mathcal{P}(G)$, if and only if*

(i) *F is closed and*
(ii) *F is stably complex.*

In particular, Theorem 6.2 holds for any finite abelian, more generally, finite nilpotent group with three or more noncyclic Sylow subgroups.

7. Group actions on complex projective spaces

The results of this section are obtained in the PhD Thesis of Marek Kaluba [2].

Theorem 7.1. *Let G be a finite perfect group and let F be a smooth manifold. Assume also that either* (1) *or* (2) *below holds.*

(1) *$G \in \mathcal{G}_{\mathbb{R}}$ and* (i) *F is closed and* (ii) *there is no restriction on $[\tau_F]$.*
(2) *$G \in \mathcal{G}_{\mathbb{C}}$ and* (i) *F is closed, the connected components of F all are even dimensional, and* (ii) *the class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$.*

Then there exists a smooth action of G on a complex projective space such that the fixed point set is diffeomorphic to F.

The idea of the proof. Consider a smooth action of G on the sphere S^{2n} of dimension $2n$ for some integer $n \geq 1$, with the given fixed point set F, obtained by Theorem 6.1. Next, modify the action so that the fixed point set consists of F and an isolated point x. Aside, create the complex projective space $\mathbb{C}P^n$ equipped with the linear action of G coming from the projectivisation $P_{\mathbb{C}}(V \oplus \mathbb{C})$ of $V = T_x(S^{2n})$, the tangent G-representation space at the point x. Finally, perform the G-equivariant connected sum

$$S^{2n} \# \mathbb{C}P^n = S^{2n} \# P_{\mathbb{C}}(V \oplus \mathbb{C})$$

around the points x in S^{2n} and $[0 : \ldots : 0 : 1]$ in $P_{\mathbb{C}}(V \oplus \mathbb{C})$, to obtain the required action of G on $\mathbb{C}P^n$. □

Theorem 7.1 does not include perfect groups in $\mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}$, such as A_5 and A_6, the alternating groups on five and six letters, respectively.

Theorem 7.2. *Let $G = A_5$. Let F be a smooth manifold. Assume also that*

(i) *F is closed, the components of F are of the same, even dimension, and*
(ii) *$[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$, i.e., $[\tau_F] \in \operatorname{Tor} \widetilde{KO}(F)$.*

Then there exists a smooth action of G on a complex projective space such that the fixed point set is diffeomorphic to F.

In this setting, we are not able to repeat the arguments from the proof of Theorem 7.1, because if $G \in \mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}$, the lack of the appropriate real G-modules (cf. Definition 3.1) means that there is no smooth action of G on a sphere S^{2n} with fixed point set $F \sqcup \{x\}$ for $\dim F > 0$.

The idea of the proof. Consider a smooth action of G on the sphere S^{2n} of dimension $2n$ for some integer $n \geq 1$, with the given fixed point set F, obtained by Theorem 6.1. Next, modify the action so that the fixed point set consists of F and the sphere S^{2d}, where $2d = \dim F$.

Following the construction above, perform the G-equivariant connected sum $S^{2n} \# \mathbb{C}P^n$ around two points, one chosen from $S^{2d} \subset S^{2n}$ and one chosen from $\mathbb{C}P^d \subset \mathbb{C}P^n$. This yields a smooth action of G on $\mathbb{C}P^n$ such that the fixed point set consists of F and a number of components diffeomorphic to complex projective spaces, possibly of distinct dimesions.

The new step of the construction is to use the G-equivariant surgery to modify the action of G on $\mathbb{C}P^n$ so that the fixed point set is just F, i.e., the extra components diffeomorphic to complex projective spaces are deleted. More specifically, construct an appropriate G-equivariant normal map of degree 1,

$$f \colon X \to \mathbb{C}P^n.$$

To convert f into a homotopy equivalence $M \to \mathbb{C}P^n$, the intermediate surgery obstructions for the maps

$$f^H \colon X^H \to (\mathbb{C}P^n)^H, \quad H < G,$$

are killed by means of the (geometric) reflection method due to Morimoto [3]. The final surgery obstruction vanishes (algebraically) by the Dress Induction. As a result, one obtains a smooth action of G on a closed smooth manifold M homotopy equivalent to $\mathbb{C}P^n$, with fixed point set diffeomorphic to F.[2] □

We expect that similar arguments are true and Theorem 7.2 holds for any finite perfect group $G \in \mathcal{G} \setminus \mathcal{G}_{\mathbb{C}}$. We wish to pose the following problem, where we assume that G is a finite group not of prime power order, such that $n_G = 1$ (i.e., G is an Oliver group) and G is not a perfect group.

Problem 1. Let F be a smooth manifold such that (i) F is closed, the connected components of F all are even dimensional, and (ii) the class $[\tau_F]$ is well-G-located in $\widetilde{KO}(F)$. *Is it true that there exists a smooth action of G on some complex projective space, such that the fixed point set is diffeomorphic to F?*

Answering the following question seems to be a challenging project.

Problem 2. Given a smooth action of a finite group G on $\mathbb{C}P^n$ with fixed point set F, *what are the closed smooth manifolds homotopy equivalent to $\mathbb{C}P^n$ which admit a smooth action of G with fixed point set diffeomorphic to F?*

[2] We are grateful to Masaharu Morimoto for bringing to our attention the fact that the resulting manifold M is also normally cobordant to $\mathbb{C}P^n$ and therefore, M is actually diffeomorphic to $\mathbb{C}P^n$.

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