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A new characterization of minimax identity problem in a two-person zero-sum dynamic game system

Hang-Chin Lai
Department of Mathematics
National Tsing Hua University, Taiwan

Jan-Tang Liu
Department of Applied Mathematics
Chung Yuan Christian University, Taiwan

Abstract

For any two stochastic spaces X and Y, we would like to search a real valued function $f: X\times Y \longrightarrow \mathbb{R}$ for $(x,y)\in X\times Y$ satisfying that whether the minimax identity (theorem) $\inf_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in X} f(x,y)$ holds. This problem established in a two-person zero-sum dynamic game under some conditions is solvable.

Keywords. *Minimax theorem, upper (lower) valued function, dynamic games, saddle value function.*

1 Preliminary

For any spaces X and Y, a real valued function f on $X\times Y$ is considered to search conditions in the function $f: X\times Y \longrightarrow \mathbb{R}$, and conditions in spaces X and Y satisfy the identity $\inf_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in X} f(x,y)$, namely minimax identity or minimax theorem. There are three types in minimax theorems described by Ky Fan (cf. [1], Fan). Thus the minimax theorems are multicriteria.

In this note, we assume the spaces X and Y are regarded as the strategy spaces of players I and II, respectively in a two person dynamic game, and we would assign a game function f in such a game system, and prove the minimax theorem holds.

Research means that one tries to find the conditions such that the objective result holds. The achieved research may be used the technique by

1. restriction,
2. extension, generalization
3. mix the conditions by restriction or extension, or create a new method and technique to explain the purpose result holds.

Our research work mostly obeys the above idea.

2 Performance of a two-person zero-sum dynamic game

We perform a two-person zero-sum dynamic game with a parameter θ by seven elements as following:

$$(DG_\theta)\ \ (S_n, A_n, B_n, t_{n+1}, u_n, v_n, \theta),\ n \in \mathbb{N}.$$

At first, we assume X and Y are metrizable separable spaces. A two-person zero-sum game means that, there are two players play a game in the state S_n by using their strategies $A_n \in X_n \subset X$ and $B_n \in Y_n \subset Y$ as the actions A_n and B_n, respectively. In the law of motion, they have the reward functions u_n and v_n at $n \in \mathbb{N}$ (the time space).

In order to evaluate process smoothly in mathematical analysis, we assume that all spaces are Borel measurability. Moreover, we assume the reward functions u_n and v_n are bounded.

After the step $S_nA_nB_n$, the game system is moving the state from S_n to S_{n+1} by transition probability t_{n+1}. This game system is continuously passing to infinity. For convenience, we use the stories of the game system by:

$$\begin{aligned}
&H_1 = S_1,\\
&H_2 = S_1 \times A_1 \times B_1 \times S_2 = H_1A_1B_1S_2,\\
&\quad\vdots\\
&H_n = S_1 \times A_1 \times B_1 \times S_2 \times A_2 \times B_2 \times \cdots \times S_{n-1} \times A_{n-1} \times B_{n-1} \times S_n\\
&\quad\ = H_{n-1}A_{n-1}B_{n-1}S_n, \quad n = 2, 3, \cdots
\end{aligned}$$

Assume that $u_n : H_nA_nB_n \longrightarrow \mathbb{R}$ and $v_n : H_nA_nB_n \longrightarrow \mathbb{R}_+$ at time $n \in \mathbb{N}$. By the bounded converging theorem, when the time n goes to infinity, they have limit functions

$$\lim_{n\to\infty} u_n = u : H_\infty \longrightarrow \mathbb{R} \ \text{ and } \ \lim_{n\to\infty} v_n = v : H_\infty \longrightarrow \mathbb{R}_+$$

where $h \in H_\infty$ is a stochastic variable for time n going to infinity. The function of u and v are density function on H_∞ with probability measure $P_{xy}(\cdot|\cdot)$. By the assumption, X and Y are separable, and so there exist sequences $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ dense in X and Y, respectively.

3 Conditional expectation in the game system (DG_θ)

Let $E_{x_n}, E_{y_n}, E_{t_{n-1}}$ denote the expectation operators with respect to $x_n \in X_n$, $y_n \in Y_n$ and the transition probability $\{t_{n+1}\}$. Thus the total conditional expectations of player I and player II are written as:

$$\begin{aligned} E(u_n, x, y)(s_1) &= \int_{H_\infty} u_n(h) P_{xy}(dh | s_1) = E_{xy} u_n(s_1) \\ &= E_{x_1} E_{y_1} E_{t_2} \cdots E_{x_{n-1}} E_{y_{n-1}} E_{t_n} E_{x_n} E_{y_n} u_n(s_1), \end{aligned}$$

and

$$\begin{aligned} E(v_n, x, y)(s_1) &= \int_{H_\infty} v_n(h) P_{xy}(dh | s_1) = E_{xy} v_n(s_1) \\ &= E_{x_1} E_{y_1} E_{t_2} \cdots E_{x_{n-1}} E_{y_{n-1}} E_{t_n} E_{x_n} E_{y_n} v_n(s_1), \end{aligned}$$

for $n \in \mathbb{N}$ by Fubini theorem. Hence the limits are given by bounded dominate (convergent) theorem as:

$$\lim_{n\to\infty} E(u_n; x, y)(s_1) = \int_{H_\infty} \lim_{n\to\infty} E(u; x, y) P_{xy}(dh | s_1) = U(x, y)(s_1) \in \mathbb{R},$$

and

$$\lim_{n\to\infty} E(v_n; x, y)(s_1) = \int_{H_\infty} \lim_{n\to\infty} E(v; x, y) P_{xy}(dh | s_1) = V(x, y)(s_1) \in \mathbb{R}_+,$$

respectively.

If a game function of the game system (DG_θ) is given by:

$$F_\theta^n = u_n - \theta v_n, \quad n \in \mathbb{N}, \tag{3.1}$$

it is regarded as the loss (gain) value function of player I, then player II has the gain (loss) value function denoted by

$$-F_\theta^n. \tag{3.2}$$

Consequently, the sum of (3.1) and (3.2) equals zero for any time $n \in \mathbb{N}$. By the bounded Lebesgue theorem,

$$\begin{aligned} F_\theta(x,y)(s_1) &= \lim_{n\to\infty} E_{xy} F_\theta^n(x,y)(s_1) \\ &= \lim_{n\to\infty} E_{xy} \left[u_n(x,y) - \theta v_n(x,y)\right](s_1) \\ &= U(x,y)(s_1) - \theta V(x,y)(s_1). \quad \text{(Since operator } \textstyle\int \text{ is linear.)} \end{aligned}$$

Hence it can be deduced to a minimax identity problem (cf. [5], Lai / Yu) to establish

$$\inf_{x\in X} \sup_{y\in Y} F_\theta(x,y)(s_1) = \sup_{y\in Y} \inf_{x\in X} F_\theta(x,y)(s_1)$$

holds.

4 Game function and lower (upper) value function

The upper value function is defined by

$$\overline{F}_\theta(s_1) = \inf_{x\in X} \sup_{y\in Y} F_\theta(x,y)(s_1).$$

Similarly, the lower value function of the game system is defined by:

$$\underline{F}_\theta(s_1) = \sup_{y\in Y} \inf_{x\in X} F_\theta(x,y)(s_1).$$

Like in a minimax programming problem, the value $\inf_{x\in X} \sup_{y\in Y} F(x,y)(s_1)$, needs $\sup_{y\in Y}$ must be attainable. Thus for a minimax theorem problem, it requires the same property which causes us to give the following two definitions.

Definition 4.1. A point $y^* \in Y$ is called a maximizer of $F_\theta(x,y)(s_1)$ over $y \in Y$ for each $x \in X$ in the system (DG_θ), if there exists a maximizer $y^* \in Y$ such that the following expression:

$$\sup_{y\in Y} F_\theta(x,y)(s_1) = F_\theta(x,y^*)(s_1) \text{ holds.}$$

Definition 4.2. We call $x^* \in X$ a minimizer of $F_\theta(x,y)(s_1)$ over $x \in X$ for each $y \in Y$ in the system (DG_θ), if there exists a minimizer $x^* \in X$ such that the following expression:

$$\inf_{x\in X} F_\theta(x,y)(s_1) = F_\theta(x^*,y)(s_1) \text{ holds.}$$

Since the game functions (loss / gain) of (DG_θ) performed by the form of player I

$$F^n_\theta(x, y)(s_1) = u_n(x, y)(s_1) - \theta(s_1)v_n(x, y)(s_1),$$

for any $(x, y) \in X \times Y$ at $n \in \mathbb{N}$ and $s_1 \in S_1$, the upper and lower values of players I and II are in the real interval: $\left[\underline{F}_\theta(s_1), \overline{F}_\theta(s_1)\right]$ which are not necessary positive value.

If $\overline{F}_\theta(s_1) \geq 0$, then player I has no lose and player II has no gain in the game system (DG_θ). Conversely, if $\underline{F}_\theta(s_1) \leq 0$, then player I has no gain and player II has no lose. Hence the following propositions are not hard to prove.

At first, we notice for upper function $\overline{F}_\theta$.

Proposition 4.3. *Let the parametric functions $\theta_1(s_1)$, $\theta_2(s_1)$ and $\theta(s_1)$ be given. Then we have*

(1) If $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then $\overline{F}_{\theta_1}(s_1) \leq \overline{F}_{\theta_2}(s_2)$,

(2) $\overline{F}_\theta(s_1) \geq 0 \Longleftrightarrow F_\theta(x, y)(s_1) \geq 0$,

(3) $\overline{F}_\theta(s_1) \leq 0 \Longleftrightarrow F_\theta(x, y)(s_1) \leq 0$.

Similarly, we state lower value function $\underline{F}_\theta(s_1)$.

Proposition 4.4. *Let $\theta_1(s_1)$, $\theta_2(s_1)$ and $\theta(s_1)$ be given. Then we have*

(1) If $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then $\underline{F}_{\theta_1}(s_1) \leq \underline{F}_{\theta_2}(s_2)$,

(2) $\underline{F}_\theta(s_1) \geq 0 \Longleftrightarrow F_\theta(x, y)(s_1) \geq 0$,

(3) $\underline{F}_\theta(s_1) \leq 0 \Longleftrightarrow F_\theta(x, y)(s_1) \leq 0$.

Consequently, we can establish several minimax theorems in the game function of the dynamic game of (DG_θ) defined on stochastic spaces X and Y as follows. For the existence of saddle valued function of (DG_θ), it is also not hard to prove these theorems.

5 Main Theorems

Theorem 5.1. *(1) Let $y^* \in Y$ be a maximizer of $F_\theta(x, y)(s_1)$ over $y \in Y$ for each $x \in X$. Then the minimax theorem holds:*

$$\overline{F}_\theta(s_1) = \underline{F}_\theta(s_1) \equiv F^*_\theta(s_1).$$

That is, $\sup\limits_{y \in Y} \inf\limits_{x \in X} F_\theta(x, y)(s_1) = \inf\limits_{x \in X} \sup\limits_{y \in Y} F_\theta(x, y)(s_1)$.

(2) If $\overline{F}_\theta(s_1)$ is not positive and there exists $\widetilde{y} \in Y$ such that $F_\theta(x,\tilde{y})(s_1) = 0$, then $\widetilde{y} \in Y$ is a maximizer of $\overline{F}_\theta(x,y)(s_1)$.

Question. In (1), we have known that if there is a maximizer, then the minimax theorem holds. The question arises that whether the maximizer exists? The answer is given in (2).

Proof. (1) If $y^* \in Y$ is a maximizer of $F_\theta(x,y)(s_1)$ over $y \in Y$, then for any $x \in X$,

$$\begin{aligned}\overline{F}_\theta(s_1) &= \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1) = \inf_{x\in X} F_\theta(x,y^*)(s_1)\\ &\le \sup_{y\in Y}\inf_{x\in X} F_\theta(x,y)(s_1) = \underline{F}_\theta(s_1).\end{aligned}$$

This shows that the saddle value function $F_\theta(x,y)(s_1)$ exists such that

$$\begin{aligned}&\overline{F}_\theta(s_1) \le \sup_{y\in Y}\inf_{x\in X} F_\theta(x,y)(s_1) = \underline{F}_\theta(s_1)\\ &\Longrightarrow \overline{F}_\theta(s_1) = F^*_\theta(s_1) = \underline{F}_\theta(s_1).\end{aligned}$$

That is, the minimax theorem of $F_\theta(x,y)(s_1)$ holds.

(2) Since $\overline{F}_\theta(s_1) \le 0$ and there exists a $\tilde{y} \in Y$ such that $F_\theta(x,\widetilde{y})(s_1) = 0$, it follows that

$$\begin{aligned}&\overline{F}_\theta(s_1) \le 0 \le F_\theta(x,\widetilde{y})(s_1) \le \sup_{y\in Y} F_\theta(x,y)(s_1), \text{ for all } x \in X\\ &\Longrightarrow 0 \le \inf_{x\in X} F_\theta(x,\widetilde{y})(s_1) \le \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1) = \overline{F}_\theta(s_1) \le 0,\ \forall x \in X.\end{aligned}$$

That is,

$$\inf_{x\in X} F_\theta(x,\widetilde{y})(s_1) = \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1).$$

Hence $\widetilde{y} \in Y$ is a maximizer of $F_\theta(x,y)(s_1)$. By (1), we see that the minimax theorem holds.

□

Theorem 5.2. *(1) Let $x^* \in X$ be a minimizer of $F_\theta(x,y)(s_1)$ over $x \in X$ for each $y \in Y$ such that*

$$\overline{F}_\theta(s_1) = \underline{F}_\theta(s_1) \equiv F^*_\theta(s_1).$$

That is, the minimax theorem holds for (DG_θ).

(2) If $\underline{F}_\theta(s_1)$ is not negative and there exists $\widetilde{x} \in X$ such that $F_\theta(\widetilde{x}, y)(s_1) = 0$, then $\widetilde{x} \in X$ is a minimizer of $\overline{F}_\theta(x, y)(s_1)$.

Question. In (1), we have known that if there is a minimizer, then the minimax theorem holds. The question arises that whether the minimizer exists? The answer is given in (2).

Proof. (1) If $x^* \in X$ is a minimizer of $F_\theta(x, y)(s_1)$ over $x \in X$, then for all $y \in Y$,

$$\begin{aligned}\underline{F}_\theta(s_1) &= \sup_{y\in Y}\inf_{x\in X} F_\theta(x, y)(s_1) = \sup_{y\in Y} F_\theta(x^*, y)(s_1)\\ &\geq \inf_{x\in X}\sup_{y\in Y} F_\theta(x, y)(s_1) = \overline{F}_\theta(s_1).\end{aligned}$$

Since $\underline{F}_\theta(s_1) \leq \overline{F}_\theta(s_1)$ is always true, we then get a saddle function $F^*_\theta(s_1)$ exists such that the above result implies:

$$\overline{F}_\theta(s_1) = F^*_\theta(s_1) = \underline{F}_\theta(s_1).$$

Thus the minimax theorem $\inf_{x\in X}\sup_{y\in Y} F_\theta(x, y)(s_1) = \sup_{y\in Y}\inf_{x\in X} F_\theta(x, y)(s_1)$ holds.

(2) Since $\underline{F}_\theta(s_1) \geq 0$ and $\exists \widetilde{x} \in X$ such that $F_\theta(\widetilde{x}, y)(s_1) = 0$, it follows that

$$\begin{aligned}&\overline{F}_\theta(s_1) \geq 0 \geq F_\theta(\widetilde{x}, y)(s_1) \geq \inf_{x\in X} F_\theta(x, y)(s_1), \text{ for all } y \in Y\\ \Longrightarrow\ & 0 \geq \sup_{y\in Y} F_\theta(\widetilde{x}, y)(s_1) \geq \sup_{y\in Y}\inf_{x\in X} F_\theta(x, y)(s_1) = \overline{F}_\theta(s_1) \geq 0, \text{ for all } y \in Y.\end{aligned}$$

That is,

$$\sup_{y\in Y} F_\theta(\widetilde{x}, y)(s_1) = \sup_{y\in Y}\inf_{x\in X} F_\theta(x, y)(s_1) = \underline{F}_\theta(x, y)(s_1) \geq 0.$$

Hence $\widetilde{x} \in X$ is a minimizer of $F_\theta(x, y)$ in the dynamic game system (DG_θ), and then by (1), we obtain that

$$\min_{x\in X}\max_{y\in Y} F_\theta(x, y)(s_1) = \max_{y\in Y}\min_{x\in X} F_\theta(s_1)$$

holds.

□

Consequently, from Theorem 5.1 and Theorem 5.2, we know that the existence of minimizer and maximizer to the function $F_\theta(x, y)$ if and only if the minimax identity problem is solved.

References

[1] Ky Fan. Minimax theorems. *Proceedings of the National Academic of Sciences of the United States of America*, 39:42–47, 1953.

[2] Y. Kimura, Y. Sawasaki, and K. Tanaka. A pertubation on two-person zero-sum game. *Annals of the International Society of Dynamic Games*, 5:279–288, 2000.

[3] H.C. Lai. On a dynamic fractional game. *Journal of Mathematical Analysis and Applications*, 294:644–654, 2004.

[4] H.C. Lai and K. Tanaka. Non-cooperative n-person game with a stopped set. *Journal of Mathematical Analysis and Applications*, 85:153–171, 1982.

[5] H.C. Lai and C.Y. Yu. Minimax theorem on a two-person zero-sum dynamic game. *Journal of Nonlinear and Convex Analysis*, 13(4):709–720, 2012.

[6] H.C. Lai and C.Y. Yu. Minimax theorem of the ratio of expectation for a two-person zero-sum dynamic game. *Journal of Nonlinear and Convex Analysis*, 14(1):89–101, 2013.

[LAI, Hang-Chin]
Department of Mathematics
National Tsing Hua University
Hsinchu 30013
TAIWAN
E-mail address: `laihc@math.nthu.edu.tw`

[LIU, Jen-Tang]
Department of Applied Mathematics
Chung Yuan Christian University
Taoyuan 32023
TAIWAN
E-mail address: `SeanLiu85@gmail.com`