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General System of Split Monotonic Variational Inclusion Problem with Applications

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Abstract

In this paper, we apply the convergence theorem of the multiply sets split feasibility problem to study the convergence theorems of the following problems: The split feasibility problem; the general system of split monotonic variational inclusion problem; the general system of split equilibrium problem; the system of split equilibrium problem; the split multiply equilibrium problem; the split equilibrium problem; the general system of split variational inequality problem; the system of split variational inequality problem; the split variational inequality problem. We establish iteration processes and prove strong convergence theorems of these problems.

Keywords: the general system of split monotonic variational inclusion problem; the general system of split equilibrium problem; fixed point problem; the general system of split variational inequality problem; mathematical programming; quadratic function programming.

1 Introduction

The split feasibility problem (**SFP**) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the split feasibility problem (**SFP**) has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated

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radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For example, one can see [2, 3, 4, 5, 6, 7].

Variational inequality theory has been studied quite extensively and has emerged as an essential tool in the study of a wide class of obstacle, free moving, equilibrium problem. It also has many applications in the optimization theory. Recently, Cai and Bu [8] considered the following systems of variational inequalities in the smooth Banach space X, which involves finding

$$\begin{cases} \text{Find } \bar{x} \in C, \ \bar{y} \in C \text{ such that} \\ \langle r\Upsilon_2\bar{x} + \bar{y} - \bar{x}, J(x - \bar{y})\rangle \geq 0, \\ \langle \lambda\Upsilon_1\bar{y} + \bar{x} - \bar{y}, J(x - \bar{x})\rangle \geq 0 \end{cases} \quad (1.1)$$

for all $x \in C$, where μ_1 and μ_2 are two positive constants, C is a nonempty closed convex subset of X, $\Upsilon_1, \Upsilon_2 : C \to X$ are two nonlinear mappings, J is the normalized duality mappings. For the recent trends and developments as problem (1.1) and its special cases, one can see [9, 10, 11, 12].

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. For each $i = 1, 2$, let $\varepsilon_i > 0$, let Υ_i be a ε_i-inverse-strongly monotone mapping of C into H_1, let $\delta > 0, \delta' > 0$, let B be a $\delta-$inverse-strongly monotone mapping of Q into H_2, let B' be a $\delta'-$inverse-strongly monotone mapping of Q into H_2. For each $i = 1, 2$, let Φ_i be a maximal monotone mapping on H_1 such that the domain of Φ_i is included in C. Let G, G' be maximal monotone mapping on H_2 such that the domain of G, G' are included in Q. Throughout this paper, we use these notations and assumptions unless specify otherwise.

We know that the equilibrium problem is to find $z \in C$ such that

$$(\textbf{EP}) \quad g(z, y) \geq 0 \text{ for each } y \in C,$$

where $g : C \times C \to \mathbb{R}$ is a bifunction. This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, minimax inequalities, and saddle point problems as special cases. (For examples, one can see [13] and related literatures.)

To the best of our knowledge, there is no result on the systems of split variational inequalities problem.

Motive by the the above problems, in this paper, we apply the convergence theorem of the multiply sets split feasibility problem to study the convergence theorems of the following problems: The split feasibility problem; the general system of split monotonic variational inclusion problem; the general system of split equilibrium problem; the system of split equilibrium problem; the split multiply equilibrium problem; the split equilibrium problem; the general system of split variational inequality problem; the system of split variational inequality problem; the split variational inequality problem. We establish iteration processes and prove strong convergence theorems of these problems.

2 Preliminaries

Throughout this paper, let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$, respectively and C be a nonempty closed convex subset of H.

For $\alpha > 0$, a mapping $A : H \to H$ is called $\alpha-$inverse-strongly monotone (α-ism) if

$$\langle x - y, Ax - Ay \rangle \geq \alpha ||Ax - Ay||^2, \forall x, y \in H.$$

A mapping $T : C \to H$ is said to be a firmly nonexpansive mapping if

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2$$

for every $x, y \in C$. Let $T : C \to H$ be a mapping. Then $p \in C$ is called an asymptotic fixed point of T [14] if there exists $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup p$, and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote by $F(\hat{T})$ the set of asymptotic fixed points of T. A mapping $T : C \to H$ is said to be demiclosed if it satisfies $F(T) = F(\hat{T})$.

A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r, and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

Lemma 2.1. [15] Let C be a nonempty closed convex subset of a Hilbert space H and let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_g as follows:

$$\text{(L4.1)} \quad A_g x = \begin{cases} \{z \in H : g(x,y) \geq \langle y - x, z \rangle, \forall y \in C\}, \forall x \in C \\ \emptyset, \forall x \notin C. \end{cases}$$

Then, $EP(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, i.e., $T_r^g x = (I + rA_g)^{-1}x$.

3 Convergence Theorems of Hierarchical Problems

For each $i = 1, 2, 3$, let H_i be a real Hilbert space, G_i be a maximal monotone mapping on H_1 such that the domain of G_i is included in C. Let $J_\lambda^{G_i} = (I + \lambda G_i)^{-1}$ for each $\lambda > 0$. Let $\{\theta_n\} \subset H_1$ be a sequence. Let V be a $\bar{\gamma}-$ strongly monotone and $L-$ Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to H_1$ be a quasi-nonexpansive mapping with $Fix(T) = Fix(\hat{T})$. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let F_1 be a firmly nonexpansive mappings of H_2 into H_2 and F_2 be a firmly nonexpansive mappings of H_3 into H_3. Let $A_1 : H_1 \to H_2$ and $A_2 : H_1 \to H_3$ be bounded linear operators. Let A_1^* be the adjoint of A_1 and A_2^* be the adjoint of A_2. Let $I : H_1 \to H_1$ be a identity mapping, and let $I_i : H_{i+1} \to H_{i+1}$ be a identity mapping for $i = 1, 2$. Throughout this paper, we use these notations and assumptions unless specify otherwise.

Now, we recall the following multiple sets split feasibility problem (**MSSFP – firmily**):

$$\text{Find } \bar{x} \in H_1 \text{ such that } A_1\bar{x} \in Fix(F_1) \text{ and } A_2\bar{x} \in Fix(F_2).$$

Let Ω is a solution of (**MSSFP – firmily**).

With the same proof as Theorem 3.3 in [16], we have the following theorem which is slightly different from Theorem 3.3 in [16] is an important tool in this paper.

Theorem 3.1. [16] Suppose that $\Delta =: Fix(T)\bigcap \Omega \bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset$. Let $\{x_n\} \subset H$ be defined by

$$(3.1) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - F_1)A_1)J_{r_n}^{G_2}(I - r_n A_2^*(I_2 - F_2)A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \le \lambda_n \le b < \frac{2}{\|A_1\|^2+2}$, and $0 < a \le r_n \le b < \frac{2}{\|A_2\|^2+2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_\Delta(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Delta$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0 \text{ for all } q \in \Delta.$$

Remark 3.1. Theorem 3.3 [16] assumes that F_2 is an firmly nonexpansive on H_2 and $A_2 : H_1 \to H_2$ ia a bounded linear operator, but Theorem 3.1 assumes that F_2 is an firmly nonexpansive on H_3 and $A_2 : H_1 \to H_3$ ia a bounded linear operator.

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Now, we recall the following split fixed point problem (**SFP − nonexpansive**):

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(\Psi) \text{ and } A_1\bar{x} \in Fix(\Psi_1).$$

where Ψ_1 is a nonexpansive mapping of H_2 into H_2 and Ψ is a nonexpansive mapping of H_1 into H_1. Let Ω_1 be a set of (**SFP − nonexpansive**).

Theorem 3.2. Let Ψ_1 be a nonexpansive mapping of H_2 into H_2, let Ψ be a nonexpansive mapping of H_1 into H_1. Suppose that

$$\Delta_1 =: Fix(T)\bigcap\Omega_1\bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

Let $\{x_n\} \subset H$ be defined by

$$(3.2) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{1}{\|A_1\|^2+2}$, and $0 < a \leq r_n \leq b < \frac{1}{3}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Delta_1}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Delta_1$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \text{ for all } q \in \Delta_1.$$

4 Applications to General System of Split Monotonic Variational Inclusion Problems

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. For each $i = 1,2$, let $\varepsilon_i > 0$, let Υ_i be a ε_i−inverse-strongly monotone mapping of C into H_1, let $\delta > 0, \delta' > 0$, let B be a δ−inverse-strongly monotone mapping of Q into H_2, let B' be a δ'−inverse-strongly monotone mapping of Q into H_2. For each $i = 1,2$, let Φ_i be a maximal monotone mapping on H_1 such that the domain of Φ_i is included in C. Let G, G' be maximal monotone mapping on H_2 such that the domain of G, G' are included in Q. Throughout this paper, we use these notations and assumptions unless specify otherwise. In this paper, we consider the following common solution problem.

(i) We consider the general system of split monotonic variational inclusion problem (**GSSMVIP**):

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(J_\lambda^{\Phi_1}(I - \lambda\Upsilon_1)J_r^{\Phi_2}(I - r\Upsilon_2)),$$

and

$$\bar{u} = A_1\bar{x} \in H_2 \text{ such that } \bar{u} \in Fix(J_\sigma^{G}(I_1 - \sigma B)J_\rho^{G'}(I_1 - \rho B')).$$

Let $GSSMVI(\Phi_1, \Phi_2, G, G')$ be the solution set of general system of split monotonic variational inclusion problem (**GSSMVIP**).

Theorem 4.1. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Suppose that

$$\Pi =: Fix(T)\bigcap GSSMVI(\Phi_1, \Phi_2, G, G')\bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

Let $\{x_n\} \subset H$ be defined by

$$(4.1)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n) \end{cases}$$

where $\Psi_1 = J_\sigma^G(I_1 - \sigma B)J_\rho^{G'}(I_1 - \rho B')$, $\Psi = J_\lambda^{\Phi_1}(I - \lambda\Upsilon_1)J_r^{\Phi_2}(I - r\Upsilon_2)$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{1}{\|A_1\|^2+2}$, and $0 < a \leq r_n \leq b < \frac{1}{3}$;

(iv) $0 < \lambda < 2\varepsilon_1,\ 0 < r < 2\varepsilon_2,\ 0 < \sigma < 2\delta$, and $0 < \rho < 2\delta'$;

(v) $\lim_{n\to\infty}\theta_n = \theta$ for some $\theta \in H$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Pi$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0 \text{ for all } q \in \Pi.$$

(ii) For $i = 1, 2$, let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4) and let $g_i : Q \times Q \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). We study the general system of split equilibrium problem (**GSSEP**):

$$\begin{cases} \text{Find } \bar{x} \in H_1,\ \bar{y} \in H_1 \text{ such that} \\ f_2(\bar{y}, x) + \frac{1}{r}\langle \bar{y} - x, \bar{x} - \bar{y}\rangle - \langle \bar{y} - x, \Upsilon_2\bar{x}\rangle \geq 0, \\ f_1(\bar{x}, x) + \frac{1}{\lambda}\langle \bar{x} - x, \bar{y} - \bar{x}\rangle - \langle \bar{x} - x, \Upsilon_1\bar{y}\rangle \geq 0 \end{cases}$$

for all $x \in C$, and

$$\begin{cases} \bar{u} = A_1\bar{x} \in H_2,\ \bar{v} \in H_2 \text{ such that} \\ g_2(\bar{v}, u) + \frac{1}{\rho}\langle \bar{v} - u, \bar{u} - \bar{v}\rangle - \langle \bar{v} - u, B'\bar{u}\rangle \geq 0, \\ g_1(\bar{u}, u) + \frac{1}{\sigma}\langle \bar{u} - u, \bar{v} - \bar{u}\rangle - \langle \bar{u} - u, B\bar{v}\rangle \geq 0 \end{cases}$$

for all $u \in Q$

Let $GSSEP(f_1, f_2, \Upsilon_1, \Upsilon_2, g_1, g_2, B, B')$ be the solution set of general system of split equilibrium problem (**GSSEP**).

Theorem 4.2. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. For each $i = 1, 2$, let A_{f_i}, A_{g_i} defined as (L4.1) in Lemma 2.1. Suppose that

$$\Pi_2 =: Fix(T)\bigcap GSSEP(f_1, f_2, \Upsilon_1, \Upsilon_2, g_1, g_2, B, B')\bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

Let $\{x_n\} \subset H$ be defined by

$$(4.2)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n) \end{cases}$$

where $\Psi_1 = J_\sigma^{A_{g_1}}(I_1 - \sigma B)J_\rho^{A_{g_2}}(I_1 - \rho B')$, $\Psi = J_\lambda^{A_{f_1}}(I - \lambda\Upsilon_1)J_r^{A_{f_2}}(I - r\Upsilon_2)$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{1}{\|A_1\|^2+2}$, and $0 < a \leq r_n \leq b < \frac{1}{3}$;

(iv) $0 < \lambda < 2\varepsilon_1$, $0 < r < 2\varepsilon_2$, $0 < \sigma < 2\delta$, and $0 < \rho < 2\delta'$;

(v) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_2}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Pi_2$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0 \text{ for all } q \in \Pi_2.$$

(iii) In the following theorem, we study the split multiple equilibrium problem (**SMEP**):

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that} \\ f_1(\bar{x}, x) \geq 0, f_2(\bar{x}, x) \geq 0 \end{cases}$$

for all $x \in C$, and

$$\begin{cases} \bar{u} = A_1\bar{x} \in H_2 \text{ such that} \\ g_1(\bar{u}, u) \geq 0, g_2(\bar{u}, u) \geq 0 \end{cases}$$

for all $u \in Q$.

Let $SMEP(f_1, f_2, g_1, g_2)$ be the solution set of split multiple equilibrium problem (**SMEP**).

Theorem 4.3. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. For each $i = 1, 2$, let A_{f_i}, A_{g_i} defined as (L4.1) in Lemma 2.1. Suppose that

$$\Pi_5 =: Fix(T)\bigcap SMEP(f_1, f_2, g_1, g_2)\bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

Let $\{x_n\} \subset H$ be defined by

$$
(4.5)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J^{G_1}_{\lambda_n}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J^{G_2}_{r_n}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n) \end{cases}
$$

where $\Psi_1 = J^{A_{g_1}}_{\sigma} J^{A_{g_2}}_{\rho}$, $\Psi = J^{A_{f_1}}_{\lambda} J^{A_{f_2}}_{r}$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \le \lambda_n \le b < \frac{1}{\|A_1\|^2+2}$, and $0 < a \le r_n \le b < \frac{1}{3}$;

(v) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_5}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Pi_5$ such that

$$
\langle V\bar{x} - \theta, q - \bar{x} \rangle \ge 0 \text{ for all } q \in \Pi_5.
$$

(iv) In the following theorem, we study the general system of split variational inequality problem (**GSSVIP**) :

$$
\begin{cases} \text{Find } \bar{x} \in H_1, \ \bar{y} \in H_1 \text{ such that} \\ \langle r\Upsilon_2\bar{x} + \bar{y} - \bar{x}, x - \bar{y} \rangle \ge 0, \\ \langle \lambda\Upsilon_1\bar{y} + \bar{x} - \bar{y}, x - \bar{x} \rangle \ge 0 \end{cases}
$$

for all $x \in C$, and

$$
\begin{cases} \bar{u} = A_1\bar{x} \in H_2, \ \bar{v} \in H_2 \text{ such that} \\ \langle \rho B'\bar{u} + \bar{v} - \bar{u}, u - \bar{v} \rangle \ge 0, \\ \langle \sigma B\bar{v} + \bar{u} - \bar{v}, u - \bar{u} \rangle \ge 0 \end{cases}
$$

for all $u \in Q$. Let $GSSVI(\Upsilon_1, \Upsilon_2, B, B')$ be the solution set of system of split variational inequality problem (**GSSVIP**).

Theorem 4.4. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Suppose that

$$\Pi_7 =: Fix(T)\bigcap GSSVI(\Upsilon_1, \Upsilon_2, B, B')\bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

Let $\{x_n\} \subset H$ be defined by

$$(4.7)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n) \end{cases}$$

where $\Psi_1 = P_Q(I_1 - \sigma B)P_Q(I_1 - \rho B')$, $\Psi = P_C(I - \lambda\Upsilon_1)P_C(I - r\Upsilon_2)$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{1}{\|A_1\|^2+2}$, and $0 < a \leq r_n \leq b < \frac{1}{3}$;

(iv) $0 < \lambda < 2\varepsilon_1$, $0 < r < 2\varepsilon_2$, $0 < \sigma < 2\delta$, and $0 < \rho < 2\delta'$;

(v) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_7}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Pi_7$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0 \text{ for all } q \in \Pi_7.$$

(v) In the following theorem, we study the split multiple variational inequality problem (**SMVIP**) :

$$\begin{cases} \text{Find } \bar{x} \in H_1 \text{ such that} \\ \langle \Upsilon_2\bar{x}, x - \bar{x}\rangle \geq 0, \langle \Upsilon_1\bar{x}, x - \bar{x}\rangle \geq 0 \end{cases}$$

for all $x \in C$, and

$$\begin{cases} \bar{u} = A_1\bar{x} \in H_2 \text{ such that} \\ \langle B'\bar{u}, u - \bar{u}\rangle \geq 0, \langle B\bar{u}, u - \bar{u}\rangle \geq 0 \end{cases}$$

for all $u \in Q$.

Let $SMVI(\Upsilon_1, \Upsilon_2, B, B')$ be the solution set of split multiple variational inequality problem (**SMVIP**).

Theorem 4.5. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Suppose that

$$\Pi_9 =: Fix(T)\bigcap SMVI(\Upsilon_1, \Upsilon_2, B, B')\bigcap Fix(J_{\lambda_n}^{G_1})\bigcap Fix(J_{r_n}^{G_2}) \neq \emptyset.$$

Let $\{x_n\} \subset H$ be defined by

$$(4.9) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n A_1^*(I_1 - \Psi_1)A_1)J_{r_n}^{G_2}(I - r_n(I - \Psi))x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n) \end{cases}$$

where $\Psi_1 = P_Q(I_1 - \sigma B)P_Q(I_1 - \rho B')$, $\Psi = P_C(I - \lambda\Upsilon_1)P_C(I - r\Upsilon_2)$ for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{1}{\|A_1\|^2+2}$, and $0 < a \leq r_n \leq b < \frac{1}{3}$;

(iv) $0 < \lambda < 2\varepsilon_1$, $0 < r < 2\varepsilon_2$, $0 < \sigma < 2\delta$, and $0 < \rho < 2\delta'$;

(v) $\lim_{n\to\infty} \theta_n = \theta$ for some $\theta \in H$.

Then $\lim\limits_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_9}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \Pi_9$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0 \text{ for all } q \in \Pi_9.$$

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