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ON THE EXISTENCE OF THE MEAN VALUES FOR CERTAIN ORDER-PRESERVING OPERATORS IN L^1.

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space with σ-algebra $\mathcal{A}$ and measure μ. It is known that if T is a linear contraction on $L^1 = L^1(\Omega, \mathcal{A}, \mu)$ which does not increase L^∞-norm (so called a Dunford-Schwartz operator on L^1) and μ is finite, then T is weakly almost periodic, that is, for each $f \in L^1$, the orbit $\{T^n f : n = 0, 1, \ldots\}$ of f under T is a relatively weakly compact subset of L^1. This is, however, not the case when μ is infinite and σ-finite. Indeed, in this case, there exists a Dunford-Schwartz operator T on L^1 which is not weakly almost periodic, but for each $f \in L^1$, the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of f converge strongly to a fixed point of T. Then, assigning to each $f \in L^1$ the limit of the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of f, the linear operator on L^1 is a unique projection P of L^1 onto the subspace of L^1 consisting of fixed points of T such that $PT = P = TP$ and for each $f \in L^1$, Pf is contained in the closure of convex hull of the orbit of f under T. Such a projection P is said to be ergodic; see Takahashi [21] and also Hirano, Kido and Takahashi [8]. Therefore, it is natural to ask a question of whether every Dunford-Schwartz operator on L^1 has the mean values on L^1 (in the sense defined in the following section) if μ is σ-finite.

Recently, we [15] discussed a method of constructing a separated locally convex topology $\tilde{\tau}$ on L^1 such that the weak topology of L^1 associated with $\tilde{\tau}$ is coarser than the weak topology on L^1 generated by $L^\infty = L^\infty(\Omega, \mathcal{A}, \mu)$ without the assumption that μ is finite. A sufficient and necessary condition was shown for a bounded subset of L^1 relative to L^1-norm to be relatively weakly compact in $(L^1, \tilde{\tau})$. We applied it to show the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on L^1. This result also gives an identification of the limit function in almost everywhere convergence of the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of an $f \in L^1$ for such an operator T on L^1.

In this paper, we summarize those arguments presented in [15] about weak compactness in $(L^1, \tilde{\tau})$ and the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on L^1. We also apply them to show the existence of the mean values for certain order-preserving operators T in L^1, for which it seems to be still unknown whether for each $f \in L^1_+$, the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of f converge weakly in L^1 in the case when μ is infinite and σ-finite.

2. Preliminaries

Throughout the paper, let $\mathbb{N}_+$ and $\mathbb{R}$ denote the set of non-negative integers and the set of real numbers, respectively. Let $\langle E, F\rangle$ be the duality between vector spaces E and F over $\mathbb{R}$. If A is a subset of E, then $A^\circ = \{y \in F : \langle x, y\rangle \leq 1\ (x \in A)\}$ is a subset of F, called the polar of A. For each $y \in F$, we define a linear form f_y on E by $f_y(x) = \langle x, y\rangle$. Then, $\sigma(E, F)$ denotes the weak topology on E generated by the family $\{f_y : y \in F\}$. Let $\tau(E, F)$ and $\beta(E, F)$ denote the Mackey topology on E with respect to $\langle E, F\rangle$ and the strong topology on E with respect to $\langle E, F\rangle$, respectively. Let $(E, \mathfrak{T})$ is a locally convex space. Then, the topological dual of E is denoted by E'. The bilinear form $(x, f) \mapsto f(x)$ on $E \times E'$ defines a duality $\langle E, E'\rangle$. The weak topology $\sigma(E, E')$ on E generated by E' is called the weak topology of E (associated with $\mathfrak{T}$ if this distinction is necessary). The topological dual of E under the strong topology $\beta(E', E)$ with respect to $\langle E, E'\rangle$ is denoted by E'_β, called the strong dual of E.

Let S be a semigroup. We denote by $l^\infty(S)$ the vector space of real-valued bounded functions defined on S; under the norm $f \mapsto \|f\| = \sup_{s\in S} |f(s)|$, $l^\infty(S)$ is a Banach space. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by $(l(s)f)(t) = f(st)$ and $(r(s)f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$, respectively. Then, a linear functional m on $l^\infty(S)$ is said to be a mean on S if $\|m\| = m(e) = 1$, where $e(s) = 1$ for each $s \in S$. For each $s \in S$, we define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in l^\infty(S)$. A convex combination of point evaluations is called a finite mean on S. As is well known, a linear functional m on $l^\infty(S)$ is a mean on S if and only if $\inf_{s\in S} f(s) \leq m(f) \leq \sup_{s\in S} f(s)$ for each $f \in l^\infty(S)$. We often write $m_s(f(s))$ for the value $m(f)$ of a mean m on S at an $f \in l^\infty(S)$. A mean m on S is said to be left (or right) invariant if $m = l(s)'m$ (or $m = r(s)'m$) for each $s \in S$, where $l(s)'$ and $r(s)'$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. If a mean m on S is left and right invariant, then m is said to be invariant. In particular, an invariant mean on $\mathbb{N}_+$ is called a Banach limit. If there exists a left (or right) invariant mean on S, then S is said to be left (or right) amenable. If S is left and right amenable, then S is said to be amenable. It is known that if S is commutative, then S is amenable, due to the fixed point theorem of Kakutani and Markov; for more details, see Day [4].

We denote by $l_c^\infty(S, E)$ the vector space of vector-valued functions f defined on a semigroup S with values in a locally convex space E for which the closure of convex hull of $f(S)$ is weakly compact. For each $s \in S$, we define the operators $L(s)$ and $R(s)$ on $l_c^\infty(S, E)$

by $(L(s)f)(t) = f(st)$ and $(R(s)f)(t) = f(ts)$ for each $t \in S$ and $f \in l_c^\infty(S,E)$, respectively. Motivated by an original work of Takahashi [21], we introduce a notion of the mean values for vector-valued functions in $l_c^\infty(S,E)$. Let m be a mean on S. For each $f \in l_c^\infty(S,E)$, we define a linear functional $\tau(m)f$ on the strong dual E'_β of E by $\tau(m)f : x' \mapsto m_s\langle f(s), x'\rangle$ for each $x' \in E$. Then, it follows from the separation theorem that $\tau(m)f$ is an element of E, which is contained in the closure of convex hull of $f(S)$. We denote by $\tau(m)$ the linear operator of $l_c^\infty(S,E)$ into E that assigns to each $f \in l_c^\infty(S,E)$ a unique element $\tau(m)f$ of E such that $m_s\langle f(s), x'\rangle = \langle \tau(m)f, x'\rangle$ for each $x' \in E'$. The operator $\tau(m)$ is called the vector-valued mean on S (generated by m if explicit reference to the mean m is needed); for more details, see Kada and Takahashi [9]. Note that it is also a vector-valued mean in the sense of Goldberg and Irwin [7]. Whenever S is left amenable, an $f \in l_c^\infty(S,E)$ is said to have the mean value if there exists an element p of E such that $p = \tau(m)f$ for each left invariant mean m on S. The element p is called the mean value of f; see Lorentz [13], Day [4] and Miyake [14]. It is shown in [14] that an $f \in l_c^\infty(S,E)$ has the mean value if and only if the closure of convex hull of the right orbit $\mathcal{RO}(f) = \{R(s)f \in l_c^\infty(S,E) : s \in S\}$ of f contains exactly one constant function, where $l_c^\infty(S,E)$ is endowed with the topology of weakly pointwise convergence, for which the family of finite intersections of sets of the form $U(s;x';\epsilon) = \{f \in l_c^\infty(S,E) : |\langle f(s), x'\rangle| < \epsilon\}$ ($s \in S, x' \in E'$ and $\epsilon > 0$) is a neighborhood base of 0. It is also known that whenever S is an amenable semigroup with identity, if a vector-valued function f defined on S with values in a bounded subset of a complete locally convex space is weakly almost periodic in the sense of Eberlein, then f has the mean value in the sense herein defined; see also von Neumann [17], Bochner and von Neumann [2], Eberlein [6], Ruess and Summers [19] and Miyake and Takahashi [16].

The notion of the mean values for vector-valued functions is applied to semigroups of transformations in the following way. Let C be a closed convex subset of a locally convex space $(E,\mathfrak{T})$ and let S be a left amenable semigroup acting on C. We assume that for each $x \in C$, the closure of convex hull of the orbit $\mathcal{O}(x) = \{s(x) : s \in S\}$ of x under S is weakly compact. Let m be a mean on S. We define a mapping ϕ_S of C into $l_c^\infty(S,E)$ by $(\phi_S(x))(s) = s(x)$ for each $x \in C$ and $s \in S$. We simply write $S(m)x$ in place of $\tau(m)(\phi_S(x))$. We denote by $S(m)$ the mapping of C into itself that assigns to each $x \in C$ a unique element $S(m)x$ of C such that $m_s\langle s(x), x'\rangle = \langle S(m)x, x'\rangle$ for each $x' \in E'$. An element p of E is said to be the mean value of an $x \in C$ under S (with respect to $\mathfrak{T}$ if this distinction is necessary) if p is the mean value of $\phi_S(x)$, that is, $p = S(m)x$ for each left invariant mean m on S. If there exists the mean value of x under S for each $x \in C$, then S is said to have the mean values on C (with respect to $\mathfrak{T}$). If S is a semigroup generated by a single element $\sigma \in S$, then we often write $\sigma(m)x$ (or $\sigma(m)$) instead of $S(m)x$ (or $S(m)$). Accordingly, the mean value of an $x \in C$ under S is simply called the mean value of x under σ. Moreover, if S has the mean values on C, then σ is also said to have the mean values on C; see Ruess and Summers [19], Miyake and Takahashi [16] and Miyake [14].

3. On weak compactness in a separated locally convex topology on L^1

Throughout the paper, let $(\Omega, \mathcal{A}, \mu)$ denote a positive measure space with σ-algebra $\mathcal{A}$ and measure μ and let $\mathcal{F}$ denote the family of measurable subsets of Ω with finite measure. Then, $\mathcal{F}$ is ordered by set inclusion in the sense that for $E, F \in \mathcal{F}$, $E \leq F$ if and only if $E \subset F$, so that each finite subset of $\mathcal{F}$ has an upper bound. Let $E \in \mathcal{A}$. If $\mathcal{A}_E$ denotes the family of intersections of members of $\mathcal{A}$ with E and μ_E denotes the restriction of μ to $\mathcal{A}_E$, then the triple $(E, \mathcal{A}_E, \mu_E)$ is a positive measure space. For $1 \leq p < \infty$, let $\mathcal{L}^p(E)$ be the vector space of measurable functions f defined on E for which $\|f\|_{E,p} = (\int_E |f|^p \, d\mu)^{\frac{1}{p}} < \infty$ and let $\mathcal{L}^\infty(E)$ be the vector space of measurable functions f defined on E for which $\|f\|_{E,\infty} = \inf_N \sup_{w \in E \setminus N} |f(w)| < \infty$, where N ranges over the null subsets of E. If $\mathcal{N}_E$ denotes the set of null functions defined on E and $[f]$ denotes the equivalence class of an $f \in \mathcal{L}^p(E)$ mod $\mathcal{N}_E$ $(1 \leq p \leq \infty)$, then $[f] \mapsto \|f\|_{E,p}$ is a norm on the quotient space $\mathcal{L}^p(E)/\mathcal{N}_E$, which thus becomes a Banach space, usually denoted by $L^p(E)$. For an $f \in L^p(\Omega)$, $\|f\|_{\Omega,p}$ is called the L^p-norm of f, simply denoted by $\|f\|_p$. A measurable function f defined on Ω is called essentially-bounded if $\|f\|_\infty < \infty$. Every element of $L^p(E)$ is considered as a measurable function f defined on E with $\|f\|_{E,p} < \infty$, if no confusion will occur. We note that $L^p(\Omega)$ is ordered by defining $f \leq g$ $(f, g \in L^p(\Omega))$ to mean that $f(x) \leq g(x)$ almost everywhere on Ω, so that $L^p(\Omega)$ is a Banach lattice. We call a function $f \in L^p(\Omega)$ non-negative if $f \geq 0$. The set of non-negative functions in $L^p(\Omega)$ will be denoted by $L^p_+(\Omega)$. For each $E \in \mathcal{A}$, the bilinear form on $L^1(E) \times L^\infty(E)$ that is defined by $\langle f, h \rangle = \int_E fh \, d\mu$ for each $f \in L^1(E)$ and $h \in L^\infty(E)$ places $L^1(E)$ and $L^\infty(E)$ in duality. For $E, F \in \mathcal{F}$ with $E \leq F$, let i_{EF} denote the mapping of $L^1(F)$ onto $L^1(E)$ that assigns to each $f \in L^1(F)$ the restriction $f|_E \in L^1(E)$ of f to E. Then, the canonical imbedding of $L^\infty(E)$ into $L^\infty(F)$ is the adjoint operator of i_{EF}, denoted by j_{FE}.

Let $\mathcal{L}^1_{loc}(\Omega)$ be the vector space of measurable functions defined on Ω which are locally integrable, that is, integrable on each $E \in \mathcal{F}$ and let $\mathcal{N}_{loc}$ be the vector subspace of $\mathcal{L}^1_{loc}(\Omega)$ consisting of measurable functions f defined on Ω for which $\mu\{w \in E : f(w) \neq 0\} = 0$ for each $E \in \mathcal{F}$. If $[f]$ denotes the equivalence class of an $f \in \mathcal{L}^1_{loc}(\Omega)$

mod $\mathcal{N}_{loc}$, then $[f] = [g]$ $(f, g \in \mathcal{L}^1_{loc}(\Omega))$ means that for each $E \in \mathcal{F}$, $f|_E(x) = g|_E(x)$ almost everywhere on E, where $f|_E$ and $g|_E$ are the restrictions of f and g to E, respectively. In particular, if μ is σ-finite, then $\mathcal{N}_{loc}$ equals the set $\mathcal{N}_\Omega$ of null functions defined on Ω and hence for $f, g \in \mathcal{L}^1_{loc}(\Omega)$, $[f] = [g]$ if and only if $f(x) = g(x)$ almost everywhere on Ω. For each $E \in \mathcal{F}$, $[f] \mapsto \|f\|_{E,1}$ is a semi-norm on the quotient space $\mathcal{L}^1_{loc}(\Omega)/\mathcal{N}_{loc}$, which becomes a locally convex space, denoted by $L^1_{loc}(\Omega)$, under the separated locally convex topology τ generated by the semi-norms $[f] \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$. Every element of $L^1_{loc}(\Omega)$ is also considered as a measurable, locally integrable function defined on Ω, if no confusion will occur.

In the sequel, we shall assume that the measure space $(\Omega, \mathcal{A}, \mu)$ *is* σ*-finite.* The product space $\mathcal{L}$ of $(L^1(E), \|\cdot\|_{E,1})$, $E \in \mathcal{F}$ is the Cartesian product $L = \prod_{E\in\mathcal{F}} L^1(E)$ endowed with the product topology. Then, $L^1_{loc}(\Omega)$ is identified as a closed (and hence complete) subspace of $\mathcal{L}$ by the isomorphism of $L^1_{loc}(\Omega)$ into $\mathcal{L}$ that is defined by $f \mapsto (f|_E)_{E\in\mathcal{F}}$, where $f|_E$ is the restriction of an $f \in L^1_{loc}(\Omega)$ to E. Let $D = \bigoplus_{E\in\mathcal{F}} L^\infty(E)$ be the direct sum of $L^\infty(E)$, $E \in \mathcal{F}$. The vector spaces L and D are placed in duality by the bilinear form on $L \times D$ that is defined by $\langle f, g\rangle = \sum_E \langle f_E, g_E\rangle$ for each $f = (f_E) \in L$ and $g = (g_E) \in D$, where $f_E \in L^1(E)$ and $g_E \in L^\infty(E)$ for each $E \in \mathcal{F}$ and the sum is taken over at most a finite number of non-zero terms of g. Then, the topological dual of $\mathcal{L}$ is D and the topological dual of $L^1_{loc}(\Omega)$ is the quotient space $D/(L^1_{loc}(\Omega))^\circ$, which is algebraically isomorphic to the vector subspace $L^\infty_{loc}(\Omega)$ of $L^\infty(\Omega)$ consisting of measurable, essentially-bounded functions f defined on Ω for which $\mu\{w \in \Omega : f(w) \neq 0\} < \infty$. Note that $L^1_{loc}(\Omega)$ is identified as the reduced projective limit $\varprojlim i_{EF} L^1(F)$ of the family $\{(L^1(E), \|\cdot\|_{E,1}) : E \in \mathcal{F}\}$ with respect to the mappings i_{EF} $(E, F \in \mathcal{F}$ and $E \leq F)$. If $\mathcal{D} = \bigoplus_{E\in\mathcal{F}} L^\infty(E)$ is the locally convex direct sum of $(L^\infty(E), \tau(L^\infty(E), L^1(E)))$, $E \in \mathcal{F}$, then the quotient space $\mathcal{D}/(L^1_{loc}(\Omega))^\circ$ is the inductive limit $\varinjlim j_{FE} L^\infty(E)$ of the family $\{(L^\infty(E), \tau(L^\infty(E), L^1(E))) : E \in \mathcal{F}\}$ with respect to the mappings j_{FE} $(E, F \in \mathcal{F}$ and $E \leq F)$.

Proposition 1. *$L^1_{loc}(\Omega)$ is a complete locally convex space. The topological dual of $L^1_{loc}(\Omega)$ is algebraically isomorphic to $L^\infty_{loc}(\Omega)$.*

It is clear that if μ is finite, then $L^1_{loc}(\Omega)$ equals $L^1(\Omega)$ and hence, τ is just the topology on $L^1(\Omega)$ generated by the metric $(f, g) \mapsto \|f - g\|_1$. We note that if C is a bounded subset of $L^1(\Omega) \cap L^p(\Omega)$ relative to L^p-norm, i.e. $\sup_{f\in C} \|f\|_p < \infty$, then the weak topology on C generated by $L^q(\Omega)$ is the relative topology of the weak topology of $L^1_{loc}(\Omega)$ to C, where p and q are a pair of conjugate exponents, that is, $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$.

A subset A of $L^1_{loc}(\Omega)$ is said to be locally uniformly integrable if for each $E \in \mathcal{F}$, the set $\{f|_E \in L^1(E) : f \in A\}$ of the restrictions of the functions in A to E is uniformly integrable, that is, for each $E \in \mathcal{F}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for each $F \in \mathcal{A}$ with $F \subset E$ and $\mu(F) < \delta$, $\sup_{f\in A} \int_F |f|\, d\mu < \epsilon$. It follows from the theorem of Tychonoff that if A is a locally uniformly integrable, bounded subset of $L^1_{loc}(\Omega)$, then A is relatively weakly compact, since $L^1_{loc}(\Omega)$ is a complete subspace of $\mathcal{L}$. The converse holds.

Proposition 2. *Let C be a subset of $L^1_{loc}(\Omega)$. Then, C is relatively weakly compact if and only if C is bounded and locally uniformly integrable.*

We apply Cantor's diagonal argument to obtain a characterization of an adherent point of a subset C of $L^1_{loc}(\Omega)$ as the limit function in almost everywhere convergence of some sequence of functions in C.

Lemma 1. *Let C be a subset of $L^1_{loc}(\Omega)$ and let f be a function in the closure of C. Then, there exists a sequence $\{f_n\}$ of functions in C such that $f_n(x)$ converges to $f(x)$ almost everywhere on Ω.*

Let $\tilde{\tau}$ denote the relative topology of τ on $L^1_{loc}(\Omega)$ to $L^1(\Omega)$, which is the locally convex topology on $L^1(\Omega)$ generated by the semi-norms $f \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$. *In the sequel, $L^1(\Omega)$ will be considered as a locally convex space under this topology $\tilde{\tau}$*, if $L^1(\Omega)$ is not specified explicitly as a Banach space $(L^1(\Omega), \|\cdot\|_1)$ under the norm $f \mapsto \|f\|_1$. Then, the topological dual of $L^1(\Omega)$ is algebraically isomorphic to $L^\infty_{loc}(\Omega)$. It follows from Lemma 1 that if a subset C of $L^1(\Omega)$ is bounded relative to L^1-norm, i.e. $\sup_{f\in C} \|f\|_1 < \infty$, then the closure in $L^1_{loc}(\Omega)$ of C is contained in $L^1(\Omega)$.

Proposition 3. *If C is a bounded subset of $L^1(\Omega)$ relative to L^1-norm, then the closure in $L^1(\Omega)$ of C is complete.*

A sufficient and necessary condition is also given by Lemma 1 for a bounded subset of $L^1(\Omega)$ relative to L^1-norm to be relatively weakly compact.

Proposition 4. *Let C be a bounded subset of $L^1(\Omega)$ relative to L^1-norm. Then, C is relatively weakly compact if and only if C is locally uniformly integrable.*

Remark 1. Let $\Omega = \mathbb{R}$, let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and let μ be Lebesgue measure on $\mathbb{R}$. Then, for each $f \in L^1(\mathbb{R})$, the subset $\{f_x : x \in \mathbb{R}\}$ of $L^1(\mathbb{R})$ is relatively weakly compact (or relatively compact relative to the weak topology of $L^1(\mathbb{R})$ associated with $\tilde{\tau}$), where $f_x(y) = f(y - x)$ for each $x, y \in \mathbb{R}$. For example, let f be the real-valued function on $\mathbb{R}$ which is defined by $f(x) = e^{-|x|}$ $(x \in \mathbb{R})$. Then, the subset $\{f_x : x \in \mathbb{R}\}$ of $L^1(\mathbb{R})$ is not relatively weakly compact in $(L^1(\mathbb{R}), \|\cdot\|_1)$, but relatively weakly compact.

Remark 2. Let $\Omega = \mathbb{R}^n$, i.e. n-dimensional Euclidean space, let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$ and let μ be Lebesgue measure on $\mathbb{R}^n$. Then, by considering $\mathcal{F}$ as the family $\mathcal{K}$ of compact subsets of $\mathbb{R}^n$, we can apply those arguments presented in this section to obtain similar results to the propositions in it, which concern weak compactness in the separated locally convex topology $\tilde{\tau}_{\mathcal{K}}$ on $L^1(\mathbb{R}^n)$ generated by the semi-norms $f \mapsto \|f\|_{K,1}$ $(K \in \mathcal{K})$. The topological dual of $(L^1(\mathbb{R}^n), \tilde{\tau}_{\mathcal{K}})$ is algebraically isomorphic to the vector subspace of $L^\infty(\mathbb{R}^n)$ consisting of Lebesgue measurable, essentially-bounded functions defined on $\mathbb{R}^n$ with compact support. Note that, in this case, a Lebesgue measurable function f defined on $\mathbb{R}^n$ is called locally integrable if f is Lebesgue integrable on each $K \in \mathcal{K}$, and a subset A of $L^1(\mathbb{R}^n)$ is said to be locally uniformly integrable if for each $K \in \mathcal{K}$, the set $\{f|_K \in L^1(K) : f \in A\}$ of the restrictions of the functions in A to K is uniformly integrable.

4. On existence of the mean values for operators

We apply the result about weak compactness in the separated locally convex topology $\tilde{\tau}$ on $L^1(\Omega)$ in the previous section to show the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on $L^1(\Omega)$. Similar results are also obtained for (commutative semigroups of) certain order-preserving operators in $L^1(\Omega)$.

A linear operator T on $L^1(\Omega)$ is said to be a Dunford-Schwartz operator on $L^1(\Omega)$ if $\|T\|_1 \leq 1$ and $\|Tf\|_\infty \leq \|f\|_\infty$ for each $f \in L^1(\Omega) \cap L^\infty(\Omega)$. In this section, T will denote such an operator on $L^1(\Omega)$, if T is not specified explicitly. For each $f \in L^1(\Omega)$, the orbit $\{T^n f : n = 0, 1, \dots\}$ of f under T (denoted by $\mathcal{O}(f)$) is a uniformly integrable, bounded subset of $L^1(\Omega)$ relative to L^1-norm.

Lemma 2. *For each $f \in L^1(\Omega)$, the orbit $\mathcal{O}(f)$ of f under T is relatively weakly compact. Moreover, if μ is finite, then T is weakly almost periodic, that is, for each $f \in L^1(\Omega)$, the orbit $\mathcal{O}(f)$ of f under T is relatively weakly compact in $(L^1(\Omega), \|\cdot\|_1)$.*

Let m be a mean on $\mathbb{N}_+$. It follows from this lemma that for each $f \in L^1(\Omega)$, there exists a unique function $T(m)f$ in $L^1(\Omega)$ such that $m_k(\int_\Omega (T^k f)h\,d\mu) = \int_\Omega (T(m)f)h\,d\mu$ for each $h \in L^\infty_{loc}(\Omega)$. Then, $f \mapsto T(m)f$ is a linear operator on $L^1(\Omega)$, denoted by $T(m)$. For each $f \in L^1(\Omega)$, $T(m)f$ is contained in the closure of convex hull of the orbit $\mathcal{O}(f)$ of f under T.

Lemma 3. *For each mean m on $\mathbb{N}_+$, $T(m)$ is a Dunford-Schwartz operator on $L^1(\Omega)$.*

Recall that a function p in $L^1(\Omega)$ is the mean value of an $f \in L^1(\Omega)$ under T with respect to $\tilde{\tau}$ if and only if $\int_\Omega ph\, d\mu = m_k \left(\int_\Omega (T^k f)h\, d\mu\right) = \int_\Omega (T(m)f)h\, d\mu$ for each $h \in L^\infty_{loc}(\Omega)$ and Banach limit m. It is known that T can be regarded as a linear contraction on $L^p(\Omega)$ $(1 < p < \infty)$, that is, a linear operator on $L^p(\Omega)$ whose norm is less than or equal to 1, due to Riesz-Thorin convexity theorem. It follows from the ergodic theorem of Yosida and Kakutani that for each $f \in L^1(\Omega) \cap L^2(\Omega)$, $n^{-1}\sum_{k=0}^{n-1} T^{k+h} f$ converges strongly to a fixed point of T in $L^2(\Omega)$ uniformly in $h \in \mathbb{N}_+$. In other words, T has the mean values on $L^1(\Omega) \cap L^2(\Omega)$ with respect to $\tilde{\tau}$; see Lorentz [13].

Theorem 1. *Every Dunford-Schwartz operator on $L^1(\Omega)$ has the mean values on $L^1(\Omega)$ with respect to $\tilde{\tau}$.*

The notion of the mean values for T allows us to give an identification of the limit function in almost everywhere convergence of the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of an $f \in L^1(\Omega)$ by virtue of the convergence theorem of Vitali.

Proposition 5. *If the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of an $f \in L^1(\Omega)$ converge almost everywhere on Ω, then the limit function is the mean value of f under T with respect to $\tilde{\tau}$.*

By the work of Takahashi [21], we are allowed to extend Theorem 1 to commutative semigroups of Dunford-Schwartz operators on $L^1(\Omega)$. It follows from Riesz-Thorin convexity theorem that every semigroup S of Dunford-Schwartz operators on $L^1(\Omega)$ can be regarded as a semigroup of linear contractions on $L^p(\Omega)$ $(1 < p < \infty)$. Moreover, if S is commutative, then S has the mean values on $L^2(\Omega)$ and also has the mean values on $L^1(\Omega) \cap L^2(\Omega)$ with respect to $\tilde{\tau}$; see also Kido and Takahashi [11].

Theorem 2. *If S is a commutative semigroup of Dunford-Schwartz operators on $L^1(\Omega)$, then S has the mean values on $L^1(\Omega)$ with respect to $\tilde{\tau}$.*

An operator T on $L^1_+(\Omega)$ is said to be order-preserving if $f \le g$ $(f, g \in L^1_+(\Omega))$ implies $Tf \le Tg$. Similar results to the above proposition and theorems in this section can be obtained for order-preserving operators T on $L^1_+(\Omega)$ for which $T(0) = 0$ and T is nonexpansive with respect to L^1-norm and L^∞-norm, that is, $\|Tf - Tg\|_1 \le \|f - g\|_1$ for each $f, g \in L^1_+(\Omega)$ and $\|Tf - Tg\|_\infty \le \|f - g\|_\infty$ for each $f, g \in L^1_+(\Omega) \cap L^\infty(\Omega)$, by means of the nonlinear interpolation theorem of Browder, which implies that such an operator on $L^1_+(\Omega)$ can be regarded as an operator W on $L^p_+(\Omega)$ $(1 < p < \infty)$ such that $\|Wf - Wg\|_p \le \|f - g\|_p$ for each $f, g \in L^p_+(\Omega)$; see Krengel and Lin [12].

Theorem 3. *If T is an order-preserving operator on $L^1_+(\Omega)$ and $T(0) = 0$ and if T is nonexpansive with respect to L^1-norm and L^∞-norm, then T has the mean values on $L^1_+(\Omega)$ with respect to $\tilde{\tau}$.*

Finally, we note that the last theorem can be also generalized to commutative semigroups of such operators on $L^1_+(\Omega)$.

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