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鎖完備な半順序ベクトル空間における凸計画法に対する鞍点について (SADDLE POINTS FOR CONVEX PROGRAMMING IN CHAIN COMPLETE PARTIALLY ORDERED VECTOR SPACES)

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ABSTRACT. In this paper, we give certain characterization of saddle points for a convex programming problem in some chain complete ordered vector space.

1. INTRODUCTION

In [6], Shizheng considered some characterization of saddle points for a certain optimization problems in some ordered vector space as follows. Let X be a vector space, Y a Dedekind complete ordered vector space, and Z be an ordered vector space. Consider the following convex programming problem:

$$(\mathrm{P}) \quad \min\{f(x) \in X \mid g(x) \leq 0, x \in D\}$$

where $f: D \to Y$ and $g: D \to Z$ are cone convex operators. He characterized the existence of saddle points of Lagrange function for optimization problem (P) based on the existence of (1) order automorphisms of vector space; (2) subgradients of the associated perturbing function; (3) directional derivative of the perturbing function, respectively. For a mapping from a vector space to an ordered vector lattice, Zowe also consider the characterization of saddle point for convex programming; see [8, 9, 10].

In ordered vector space, we consider chain completeness which is more weak one rather than Dedekind completeness. As examples of chain complete ordered vector spaces, the algebraic dual of ordered vector space and the set of bounded self-adjoint linear operators of Hilbert space are well-known; see [1, 2, 3, 7]. For analysis of chain complete ordered vector spaces, Borwein [1, 2] consider the subgradient of sublinear operator taking values on a partially ordered vector space. In [2], he consider as another example of chain complete ordered vector spaces, Daniel spaces and monotone complete ordered vector spaces.

In this paper, motivated by Borwein's work, we characterize the existence of optimization problem in chain complete ordered vector spaces. As an example of our method, we consider a mapping taking value in a line (totally ordered set) in R^2.

2. PRELIMINARIES

Firstly, we give preliminary terminology and notation used in this paper. Let X, Y and Z be real vector spaces, D a nonempty convex subset of X, and $x, y \in X$.

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We denote the algebraic dual space of X by X'; the vector space of all linear mapping from X into Y by $\mathcal{L}(X,Y)$; the origin of X by $\{0_X\}$; the linear hull, affine hull, and conical hull of A by L_A, ${}^{l}A$, and cone(A), respectively; the algebraic interior, relatively algebraic interior, and algebraic closure of A by cor(A), ${}^{i}A$, and lin(A), respectively; the algebraic sum and algebraic difference of A and B by $A+B := \{a+b | a \in A, b \in B\}$ and $A - B := \{a - b | a \in A, b \in B\}$, respectively; the line segment of x and y by $[x,y]$; the line segment of x and y without $\{y\}$ by $[x,y)$. Moreover, let f be a function from X into Y and $x' \in X'$. Then we denote the composite function of f and x' by $x' \circ f$; the dual pair of x and x' by $\langle x', x \rangle$. Also we denote the set of all real numbers by R; $R \cup \{\infty\}$ by $\overline{R}$. Let us consider a proper convex cone C_Y in Y (that is, $C_Y \neq \emptyset$, $C_Y \neq \{0_Y\}$, $C_Y \neq Y$, $\lambda C_Y \subset C_Y$ for all $\lambda \geq 0$, and $C_Y + C_Y \subset C_Y$). Furthermore, a partial ordering on Y with respect to C_Y is defined as follows:

$$x \leq_{C_Y} y \quad \text{if} \quad y - x \in C_Y \quad \text{for} \quad x, y \in Y.$$

It is well known that $\leq_{C_Y}$ is reflexive and transitive. In particular, if C_Y is pointed, that is, $C_Y \cap (-C_Y) = \{0_Y\}$, then C_Y is antisymmetric. Moreover, $\leq_{C_Y}$ has invariable properties to vector space structures as translation and scalar multiplication. In the sequel, we consider $(Y, \leq_{C_Y})$ and $(Z, \leq_{C_Z})$ as a partially ordered vector space where C_Y and C_Z are pointed proper convex cones, respectively. Let $L \subset Y$ be a totally ordered linear subspace of Y. Naturally L is non-empty. Next, we recall several definitions of order completeness.

Definition 1. Let Y be a vector space ordered by a proper convex cone C_Y. Then Y is said to be

(1) *Dedekind complete* if every nonempty subset of Y which is bounded from below has an infimum;
(2) *chain complete* if every nonempty chain of Y which is bounded from below has an infimum;

It is clear that if Y is Dedekind complete, then it is chain complete. However, the converse is not true in general. We say that a sequence $\{x_n\}$ of elements order converges (o-converges) to x if there are sequences $\{p_n\}$ and $\{q_n\}$ such that (α) $p_n \leq_{C_Y} x_n \leq_{C_Y} q_n$ and (β) $p_n \nearrow x$ and $q_n \searrow x$, where by $p_n \nearrow x$ we mean that $\sup_{n \in N} p_n = x$ and by $q_n \searrow x$ that $\inf_{n \in N} q_n = x$. A sequence $\{u_n\} \subset Y$ said to be order converges to u, if there exists $\{p_k\} \subset Y$ with $p_k \searrow 0$ and for any k there exists n_k such that $-p_k < u_n - u < p_k$ for any $n \geq n_k$ and we denote by $o - \lim_{n \to \infty} u_n = u$.

In this paper we consider convex programming problem:

$$\text{(MP)} \ \min\{f(x) \in X \mid g(x) \leq_{C_Z} 0, x \in D\}$$

where $f : D \to L$ and $g : D \to Z$ are cone convex operators. We assume that there exists $\bar{x}$ which is an optimal solution of (MP), that is, if $\bar{x} \in D$, $g(\bar{x}) \in -C_Z$, and $x \in D$, $g(x) \in -C_Z$, then $f(\bar{x}) \leq_{C_Y} f(x)$. We also defined the Lagrangian function $\varphi : D \times \mathcal{L}^+(Z, L) \to L$ of (MP) by

$$\varphi(x,T) = f(x) + T(g(x)).$$

We say $(\bar{x}, \bar{T}) \in D \times \mathcal{L}^+(Z, L)$ is a saddle point of φ, if

$$\varphi(\bar{x}, T) \leq_{C_Y} \varphi(\bar{x}, \bar{T}) \leq_{C_Y} \varphi(x, \bar{T}).$$

for all $x \in D$, $T \in \mathcal{L}^+(Z,L)$. We also assume that positive cones C_Y, C_L and C_Z, and its algebraic interiors C_Y^o, C_L^o and C_Z^o in Y, L and Z, respectively, are non-empty. Suppose that C_Z is linearly closed and its interior point C_Z^o is non-empty, then saddle points $(\bar{x}, \bar{T})$ of φ implies that $\bar{x}$ is a optimum solution of (MP). Conversely if $\bar{x}$ is a optimum solution and if there exists $\bar{T} \in \mathcal{L}(X,Y)$ such that $(\bar{x}, \bar{T})$ is a saddle points of φ, then we say that (MP) satisfies saddle point criterion at x. By [8], we can consider a sufficient condition for (MP) satisfying saddle point criterion:

$$N = \{\lambda g(x) + z \mid x \in D, \lambda \in R_+\} \cup \{0\}$$

is a linear subspace of Z. In [10], Zowe also give a sufficient condition which is similar to Slater's condition.

3. SADDLE POINT CRITERION

In $L \times Z$, we denote

$$A := \bigcup_{x \in D} \{(f(x), g(x)) + C_L \times C_Z\}.$$

Lemma 2. *A is a convex set and its algebraic interior A^o is non-empty. Moreover, $(f(\bar{x}), 0)$ is algebraic boundary of A where $\bar{x}$ is a optimal solution of (MP).*

Proof. Since f and g are cone convex and proper cones C_L in L and C_Z in Z are convex set, A is convex set. Let $y_0 \in C_L^o$ and $z_0 \in C_Z^o$. Then $\alpha := (f(x), g(x)) + (y_0, z_0) \in A^o$ for any $x \in D$. Let $y_1 \in C_L$ and $z_1 \in C_Z$. Let $\mid k \mid \leq k_1$. Then we have $y_0 + k \cdot y_1 \in C_L$ where $k_0 \in R$ with $k_0 > 0$ and $z_0 + k \cdot z_1 \in C_Z$ where $k_1 \in R$ with $k_1 > 0$. Thus if we take $|k| \leq \min(k_0, k_1)$, then we have

$$(y_0, z_0) + k(y_1, z_1) = (y_0 + ky_1, z_0 + kz_1) \in C_L \times C_Z.$$

Put $\beta := (y_1, z_1)$, then $\alpha + k\beta \in (f(x), g(x)) + C_L \times C_Z \subset A$. Thus $\alpha \in A^o$. Moreover, since $\bar{x}$ is optimum solution, we have

$$\bar{\alpha} := (f(\bar{x}), 0) = (f(\bar{x}), g(\bar{x})) + (0, -g(\bar{x})) \in \cup_{x \in D} (f(x), g(x)) + C_L \times C_Z,$$

and $\bar{\alpha} + k(-y_0, 0) \notin A$ for all $k > 0$. Thus we have $\bar{\alpha} \notin A^o$. Since $\bar{\alpha} \in A$ and $\bar{\alpha} \notin A^o$, we have $\bar{\alpha} \in \partial A$. Then $\bar{\alpha}$ is an algebraic boundary solution of A. □

Next we give an order automorphisms of vector space. First, we give a following proposition by Köthe [4, Section 17.2(1)].

Proposition 3. *Let X be vector spaces and Y a Dedekind complete vector space. Let A be a convex subset of Y and its algebraic interior A^o is non-empty. Let M be a linear manifold which contain no algebraic interior point of A. Then there exists a hyperplane H which contains M and contain no boundary point of A, thus $A \not\subseteq H$.*

Lemma 4. *There exists $S \in \mathcal{L}^+(L \times Z, L)$ such that*

$$S(\bar{\alpha}) = \inf S(A) \tag{3.1}$$

$$S(y,z) = S_1(y) + S_2(z), \tag{3.2}$$

where $\bar{\alpha} = (f(\bar{x}), 0)$, $S_1 \in \mathcal{L}^+(L,L)$, and $S_2 \in \mathcal{L}^+(Z,L)$.

Proof. By Lemma 2 and Proposition 3 (Köthe's theorem), there exists a hyperplane H containing $\bar{\alpha}$ which also contains no algebraic interior point of A, thus $A \not\subseteq H$. We put $H; = \{(y,z) \mid t(y,z) = u\}$, where $t : L \times Z \to R$ is a linear functional and $u = t(\bar{(\alpha)})$. Since $\bar{x}$ is a optimal solution and $\bar{\alpha} = (f(\bar{x}), 0)$ is a algebraic boundary point of A, we have $t(A) \geq t(\bar{\alpha})$. Since $\bar{\alpha} + (y,z) \in A$, for all $(y,z) \in C_Y \times C_Z$, and $t(\bar{\alpha} + (y,z)) \geq t(\bar{\alpha})$, $t(y,z) \geq 0$. So that t is a positive functional. Take $\beta \in L \setminus \{0\}$, define $S : L \times Z \to L$ by $S(y,z) = t(y,z)\beta$. It is easy to see that S is a linear operator and $S(A) = t(A)\beta \geq u\beta = S(\bar{\alpha})$. Since t is positive functional, S is a positive operator. Since $S(y,z) = S(y,0) + S(0,z)$, we define $S_1(y) = S(y,0)$ and $S_2(z) = S(0,z)$, then we have equation (3.2). □

Theorem 5. *(MP) satisfies saddle point criterion at $\bar{x}$ if and only if there exists linear mapping $S \in \mathcal{L}^+(L \times Z, L)$ as in (3.1) (3.2) such that S_1 is order homomorphism, that is, there exists S_1^{-1} which is positive linear operator.*

Proof. Since (MP) satisfies saddle point criterion at $\bar{x}$, there exists $\bar{T} \in \mathcal{L}^+(Z, L)$ such that $(\bar{x}, \bar{T})$ is a saddle point of φ. Putting $S(y,z) := y + \bar{T}(z)$, then we have $S \in \mathcal{L}^+(Z \times L, L)$. We assume that $(y,z) \in A$. Then there exists $x \in D$ such that $(y,z) \geq (f(x), g(x))$ and

$$\begin{aligned} S(y,z) \quad {}_{C_Y}\!\geq \;& y + \bar{T}(z) \,{}_{C_Y}\!\geq f(x) + \bar{T}(g(x)) \\ = \;& \varphi(x, \bar{T}) \,{}_{C_Y}\!\geq \varphi(x, T) \,{}_{C_Y}\!\geq \varphi(\bar{x}, 0) \\ = \;& f(\bar{x}) = S(\bar{\alpha}). \end{aligned}$$

Since Y is chain complete, then note that $S \in \mathcal{L}^+(Z \times L, L)$, we have $\inf S(A) = S(\bar{\alpha})$, where $\bar{\alpha} = (f(\bar{x}), 0)$ It is easy to see that S is positive. We take S_1 is identical operator of L and $S_2 = \bar{T}$, then we have $S_1 \in \mathcal{L}^+(L, L)$, $S_2 \in \mathcal{L}^+(Z, L)$ and $S(y,z) = S_1(y) + S_2(z)$.

Conversely, if there exists $S \in \mathcal{L}^+(L \times Z, L)$ such that for any $x \in D$, $(f(x), g(x)) \in A$, then we have

$$S_1(f(\bar{x})) = S_1(f(\bar{x})) + S_2(0) = S(f(\bar{x}), 0) \leq_{C_Y} S(f(x), g(x)) = S_1(f(x)) + S_2(g(x)),$$

because $\inf S(A) = S(\bar{\alpha}) = S(f(\bar{x}), 0)$. Let $x = \bar{x}$, then we have $0 \leq_{C_Y} S_2(g(\bar{x}))$. Moreover, since $\bar{x}$ is optimal solution, if $g(\bar{x}) \leq 0$, then $S_2(g(\bar{x})) \leq_{C_Y} 0$. Thus $S_2(g(\bar{x})) = 0$. Therefore

$$S_1(f(\bar{x})) + S_2(g(\bar{x})) \leq_{C_Y} S_1(f(x)) + S_2(g(x)). \tag{3.3}$$

Since S_1 is automorphism, we operate S_1^{-1} to (3.3). We put $T = S_1^{-1} \circ S_2$, then

$$f(\bar{x}) + \bar{T}(g(\bar{x})) \leq_{C_Y} f(x) + \bar{T}(g(x)).$$

On the other hand, since for any $T \in \mathcal{L}^+(L, L)$, $T(g(\bar{x})) \leq_{C_Y} 0$, thus we have

$$f(\bar{x}) + T(g(\bar{x})) \leq_{C_Y} f(x) + \bar{T}(g(x)).$$

Then we have

$$f(\bar{x}) + T(g(\bar{x})) \leq_{C_Y} f(\bar{x}) + \bar{T}(g(\bar{x})) \leq_{C_Y} f(x) + \bar{T}(g(x)).$$

Put $\varphi(x, T) = f(x) + T(g(x))$, then we have

$$\varphi(\bar{x}, T) \leq_{C_Y} \varphi(\bar{x}, \bar{T}) \leq_{C_Y} \varphi(x, \bar{T}).$$

Then $(\bar{x}, \bar{T})$ is a saddle point of $\varphi(x, T) = f(x) + T(g(x))$. □

Let $Z_1 := g(D) + C_Z$, $C(z) := \{x \in D \mid g(x) \leq z\}$, $z \in Z_1$, Then

Proposition 6. *Z_1 is convex set, $C_Z \subset Z_1$ and $C_Z^o \subset Z_1^o$.*

Proof. It is clear from the definition. □

Since Y is chain complete ordered vector space (Daniell space), if $f(C(Z))$ has lower bounded, then there exists $\inf f(C(Z))$. We define $F : D \to Y \cup \{\pm\infty\}$ by

$$F(z) = \begin{cases} \inf_{z \in Z_1} f(C(z)) & \text{if } z \in Z_1 \text{ and } f(C(z)) \text{ has lower bound,} \\ -\infty & \text{if } z \in Z_1 \text{ and } f(C(z)) \text{ does not have lower bound,} \\ \infty & \text{if } z \notin Z_1 \\ f(\bar{x}) & \text{if } z = 0 \end{cases}$$

Lemma 7. *If $f(C(z))$ has lower bound and $F(z) > -\infty$, then F is convex.*

Proof. Let $z_1, z_2 \in Z_1$, $k \in (0,1)$, x_1, $x_2 \in C(Z)$, $g(x_1) \leq_{C_Y} z_1$ and $g(x_2) \leq_{C_Y} z_2$. Then we have

$$\begin{aligned} g(kx_1 + (1-k)x_2)) &\leq_{C_Y} kg(x_1) + (1-k)g(x_2) \\ &\leq_{C_Y} kz_1 + (1-k)z_2. \end{aligned}$$

Thus

$$kx_1 + (1-k)x_2 \in C(kz_1 + (1-k)z_2).$$

Since f is convex, we have

$$f(C(kz_1 + (1-k)z_2)) \leq_{C_Y} kf(C(z_1)) + (1-k)f(C(z_2)).$$

Fix z_1 and z_2, and let x_1 and x_2 run through $C(z_1)$ and $C(z_2)$, then we have

$$\begin{aligned} &F(kz_1 + (1-k)z_2) \\ &\leq_{C_Y} k \inf\{f(x) \mid x \in C(z_1)\} + (1-k)\inf\{f(x) \mid x \in C(z_2)\} \\ &= kF(z_1) + (1-k)F(z_2). \end{aligned}$$

If $z_1 \notin Z_1$ or $z_2 \notin Z_1$, then $F(z_1) = \infty$ or $F(z_2) = \infty$, and if $kz_1 + (1-k)z_2 \notin Z_1$, then either $z_1 \notin Z_1$ or $z_2 \notin Z_1$ as above. Then always we have

$$F(kz_1 + (1-k)z_2) \leq_{C_Y} kF(z_1) + (1-k)F(z_2).$$

for all $k \in (0,1)$ and $z_1, z2 \in Z$. Thus F is convex. □

We assume that $F(z) > -\infty$. We recall the algebraic sub-differential of F at 0 to be the set

$$\partial^\alpha F(0) = \{T \in \mathcal{L}(Z,Y) \mid T(z) \leq_{C_Y} F(z) - F(0), z \in Z\}.$$

Theorem 8. *(MP) satisfies saddle point criterion at $\bar{x}$ if and only if*
(1) $F(z) > -\infty$ and $F(z) \in L$ for all $z \in Z$;
(2) $\partial^\alpha F(0) \neq 0$.

Proof. By Theorem 5, there exists linear mapping $S : L \times Z \to L$ such that

$$S(y,z) = S_1(y) + S_2(z), \text{ and } S_1(f(\bar{x})) = S(\bar{a}) \leq_{C_Y} S(A),$$

where S_1 order automorphism from L to L and $S_2 \in \mathcal{L}(Z,L)$. Since $S_2 \in \mathcal{L}(Z,L)$ and positive, then

$$S_1(f(x)) \leq_{C_Y} S_1(f(x)) + S_2(z), \tag{3.4}$$

for all $z \in Z_1$ and $x \in C(z)$. Let S_1^{-1} act on (3.4) and denote $T := S_1^{-1} \circ S_2$. We have $f(\bar{x}) - \bar{T}(z) \in L$ and $f(\bar{x}) - \bar{T}(z) \leq_{C_Y} f(x)$. Therefore, $f(x) - \bar{T}(z)$ is an order bounded of $f(C(z))$ in L. Since L is chain in Y and Y is chain complete, there exists infimum $\inf f(C(z))$ such that $-\infty < f(\bar{x}) - \bar{T}(z) \leq_{C_Y} \inf f(C(z))$ for all $z \in Z_1$. Put $F(z) := \inf f(C(z))$. Thus $-\bar{T}(z) \leq_{C_Y} F(z) - F(0)$, $-\bar{T} \in \partial^\alpha F(0) \neq \emptyset$. we have (1) and (2).

Next we assume that $T \in \partial^\alpha F(0) \neq 0$, then $T(z) \leq_{C_Y} F(z) - F(0)$ for all $z \in Z_1$. Since $(y, z) \in A$, there exists $x \in D$ such that $(y, z) \geq (f(x), g(x))$. Therefore

(3.5) $y - f(\bar{x}) \leq_{C_Y} f(x) - f(\bar{x}) \leq_{C_Y} F(z) - F(0) \leq_{C_Y} T(z), f(\bar{x}) \leq_{C_Y} y - T(z)$.

Since $(f(\bar{x}), z) \in A$ for all $z \in C_Z^+$, we have

$$f(\bar{x}) - T(z) \leq_{C_Y} f(\bar{x}), \quad -T(z) \leq_{C_Y} 0, \quad -T \in \mathcal{L}^+(Z, Y).$$

Let $\bar{T} := -T$, $S(y, z) := y + \bar{T}(z)$, so by (3.5), it implies that $S(\bar{\alpha}) = f(\bar{x}) \leq_{C_Y} S(A)$, where $\bar{\alpha} = (f(\bar{x}), 0)$. By Theorem 5, we have the conclusion. □

4. Directional derivative

In this section we consider the directional derivative.
An operator $f : D(f) \subset X \to Y$ is called a convex operator, if the domain of definition $D(f)$ of f is a non-empty convex subset of X and if for all $x_1, x_2 \in D(f)$ and all real λ, $0 \leq \lambda \leq l$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq_{C_Y} \lambda f(x_1) + (1 - \lambda) f(x_2)$$

Lemma 9. *Let Z be a real vector space, Y an ordered vector space and $L \subset Y$ a non-empty chain. Let $Z_0 \subset Z$ be a linear subspace. Let $G : D(G) \subset Z \to Y$ be a convex operator, where $D(G)$ is a domain of G, and $T_0 : Z_0 \to Y$ a linear operator such that*

$$T_0(z) \leq G(z), \quad \textit{for all} \quad z \in D(G) \cap Z_0.$$

If Y is chain complete, $D(G) \cap Z_0 \neq \emptyset$, $G(D(G)) \in L$ and $T_0(Z_0) \in L$, then there exists a linear operator T form X into Y such that

$$T(x) \in L \textit{ for all } x \in Z, T(x) = T_0(x) \textit{ for all } x \in M \textit{ and}$$
$$T(x) \leq_{C_Y} G(x) \textit{ for all } x \in D(G).$$

Proof. The proof is similar to that of [10, Theorem 2.1] □

Let $F(z) \geq -\infty$ for any $z \in Z$. For any z and $k_1 \geq k_2 > 0$, since F is convex, we have

$$F(k_2 z) = F(k_1^{-1} k_2 (k_1 z) + (1 - k_1^{-1} k_2) 0) \leq_{C_Y} k_1^{-1} k_2 F(k_1 z) + (1 - k_1^{-1} k_2) F(0),$$

and

$$k_2^{-1} F(k_2 z) - F(0)) \leq_{C_Y} k_1^{-1} F(k_1 z) - F(0)).$$

Let

$$M(z) := \{k^{-1}(F(kz) - F(0)) \mid k > 0\} \subset \bar{Y}$$

and

$$dF(0, z) := o - \lim_{k \downarrow 0} \frac{1}{k}(F(kz) - F(0)); \text{ see [5].}$$

If Y is Dedekind complete or Y is totally ordered and chain complete, it is easy to see that there exists $dF(0, z) \in \bar{Y}$ and $dF(0, z) = \inf M(z)$ for all $z \in Z$.

Lemma 10. *Suppose that $F(z) > -\infty$ for all $z \in Z$. Then $dF(0,z)$ is a sublinear operator from Z into $\bar{Y}$. Moreover we have $kdF(0,z) \leq_{C_Y} dF(0,kz)$ for $k < 0$ and $F(z) - F(0) \geq dF(0,z)$.*

Proof. Since F is convex operator, we have

$$F((1-k)u + kv) \leq_{C_Y} (1-k)F(u) + kF(v) \quad \text{for any} \quad u, v \in Z \quad \text{and} \quad k \in (0,1).$$

We take $u = 2tz_1$, $v = 2tz_2$ and $k = \frac{1}{2}$, then

$$F(tz_1 + tz_2) \leq_{C_Y} \frac{1}{2}F(tz_1) + \frac{1}{2}F(tz_2).$$

$$t^{-1}(F(t(z_1+z_2)) - F(0)) \leq_{C_Y} (2t)^{-1}(F(2tz_1) - F(0)) + (2t)^{-1}(F(2tz_2) - F(0)).$$

$$\begin{aligned} &o - \lim_{t \downarrow 0} t^{-1}(F(t(z_1+z_2)) - F(0)) \\ &\quad \leq_{C_Y} o - \lim_{t \downarrow 0}(2t)^{-1}(F(2tz_1) - F(0)) + o - \lim_{t \downarrow 0}(2t)^{-1}(F(2tz_2) - F(0)). \end{aligned}$$

Thus we have

$$dF(0, z_1 + z_2) \leq_{C_Y} dF(0, z_1) + dF(0, z_2).$$

Moreover if $k \geq 0$, then

$$\begin{aligned} dF(0,kz) &= o - \lim_{t \downarrow 0} t^{-1}(F((kz)) - F(0)) \\ &= o - \lim_{t \downarrow 0} k(kt)^{-1}(F(ktz) - F(0)) = kdF(0,z). \end{aligned}$$

On the other hand let $k < 0$. Since

$$0 = dF(0,0) \leq_{C_Y} dF(0,kz) + dF(0,-kz),$$

Thus

$$kdF(0,z) = -dF(0,-kz) \leq_{C_Y} dF(0,kz)$$

and

$$F(z) - F(0) = 1^{-1}(F(z) - F(0)) \in M(z).$$

□

Theorem 11. *(MP) satisfies the saddle point criterion at $\bar{x}$ if and only if*
(i) $F(z) > -\infty$ and $F(z) \in L$ for all $z \in Z$.
(ii) $dF(0,\bar{z}) \in L$ for some $\bar{z} \in C_Z^o$.

Proof. Assume that (MP) satisfies saddle point criterion. By Theorem 8, (MP) satisfies saddle point criterion at $\bar{x}$ if and only if (1) $F(z) > -\infty$ and $F(z) \in L$ for all $z \in Z$; (2) $\partial^\alpha F(0) \neq 0$. If $T \in \partial^\alpha F(0)$, then $T(z) \leq_{C_Y} F(z) - F(0)$ for all $z \in Z$. Take any $\bar{z} \in Z_+$, then $z \in Z_1$ and $C(z) \neq \emptyset$, so we have $F(\bar{z}) < \infty$ and $dF(0,\bar{z}) \leq F(z) - F(0) < \infty$. On the other hand, since $T(k\bar{z}) \leq F(kz) - F(0)$ for all $k > 0$, we have $T(\bar{z}) = k^{-1}T(k\bar{z}) \leq_{C_Y} \frac{1}{k}(F(kz) - F(0))$. $M(z)$ is bounded from below and $M(z) \in L$, there exists $\inf M(z)$, $dF(0,\bar{z}) = \inf M(\bar{z}) \in L$ and $-\infty < T(\bar{z}) \leq_{C_Y} \inf M(\bar{z})$. Thus we have (ii).

We prove the converse. We assume that $\bar{z} \in C_Z^o$ and $y_0 := dF(0,\bar{z}) \neq \infty$. Let $Z_0 := \{k\bar{z} \mid k \in R\}$ and $T_0 : Z_0 \to Y$ defined by $T_0(k\bar{z}) = ky_0$. Now we consider F as from Z_1 to $\bar{Y}$, F is a convex operator as in Lemma 7. By assumption $F(z) \in L$

for all $z \in Z_1$. Let $G(z) := F(z) - F(0)$, then G is also a convex operator from Z into $\bar{Y}$ with $G(z) \in L$ for all $z \in Z_1$. For any $k \geq 0$, $k\bar{z} \in Z_1$,

$$T_0(k\bar{z}) = ky_0 = kdF(0, \bar{z}) = dF(0, k\bar{z}) \leq_{C_Y} F(k\bar{z}) - F(0) = G(k\bar{z})$$

and for any $k < 0$ which satisfies $k\bar{z} \in Z_1 \cap Z_0$, from Lemma 10, we have $G(k\bar{z}) = F(k\bar{z}) - F(0)$ and $ky_0 = kdF(0, \bar{z}) \leq_{C_Y} dF(0, \bar{z}) \leq_{C_Y} F(k\bar{z}) - F(0)$. Hence we have $T_0(z) \leq_{C_Y} G(z)$ for all $z \in Z_1 \cap Z_0$. Since Y is chain complete, $\bar{z} \in Z_1 \cap Z_0 \neq \emptyset$, $G(Z_1) \in L$ and $T_0(Z_0) \in L$, by Lemma 9, there exists a linear operator T from Z into Y such that

$$T(z) = T_0(z) \text{ for all } z \in Z_0$$

and

$$T(z) \leq_{C_Y} G(z) = F(z) - F(0) \text{ for all } z \in Z_1.$$

On the other hand for $z \notin Z_1$, $F(z) = \infty$. Thus we have

$$T(z) \leq_{C_Y} F(z) - F(0) \text{ for all } z \in Z.$$

Thus $\partial^\alpha F(0) \neq \emptyset$. Then the assertion holds from Theorem 8. □

Example 12. Let $X = Y = D = R^2$, $Z = R$, $C_X = C_Y = \{(x,y) \in R^2 \mid x \geq 0, y \geq 0\}$, $C_Z = R_+ \cup \{0\}$, $L = \{(x,x) \in R^2\}$, $g(x,y) = x^2 + y^2$. We denote the mapping $f : R^2 \to R^2$ be defined by the following matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{4.1}$$

and $f(x,y) = (x+y, x+y)$, $\bar{x} = 0$, $Z_1 = g(D) + C_Z = R_+ \cup \{0\}$.

$$F(z) = \begin{cases} \inf_{z \in Z_1} f(C(z)) = \{(-\sqrt{2}z, -\sqrt{2}z)\} & \text{if } z \in Z_1, \\ \infty & \text{if } z \notin Z_1 \\ f(0,0) = (0,0) & \text{if } z = 0 \end{cases}$$

It is easy to see that $T_k : k \to kz (k \leq -\sqrt{2})$ satisfies $T_k(z) \leq F(z) - F(0)$ for all $z \in Z$. $T_k \in \partial^\alpha F(0) \neq \emptyset$, or $\bar{z} = 1$, $dF(0, \bar{z}) = \inf\{k^{-1}(F(k\bar{z}) - F(0)) \mid k > 0\} = -\sqrt{2}$. Therefore (MP) satisfies the saddle point criterion as in Theorem 8 or 11, but it does not satisfies Zowe's condition, because $N = \{\lambda g(x) + z \mid x \in D, \lambda \in R_+ \cup \{0\}, z \in C_Z\} = R_+ \cup \{0\}$ is not a subsequence of $Z = R$. Moreover, it is not satisfy Slater's type condition, because $\{x \in D \mid g(x) < 0\} = \emptyset$.

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