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Group-Subgroup Pair Graphs

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1 Introduction

A Cayley graph is a graph defined from a finite group and a symmetric subset of its elements using the relations defined from the group operation. The resulting graphs are regular graphs, in other words, all of its vertices have a same fixed degree. Cayley graphs have been used as models for applications that require the use of regular graphs, as several of the graph properties can be determined by using the properties of the finite group and the subset. In particular we mention the construction of families of expander and Ramanujan graphs in (6) and (7).
In the article (8), the author defines a Cayley-type graph construction for group-subgroup pairs by analogy on the work on the wreath group determinant for group-subgroup pairs in (3). The main property of the resulting graphs is that the graphs are not regular, but all the vertices in the cosets of the subgroup have the same degree. Moreover, many of the graphs properties can be determined in a similar manner as that of Cayley graphs. The expectation is that these group-subgroup pair graphs may be useful for applications where r-partite graphs or graphs where vertices on a partition have the same degree are required. This articles summarizes and illustrates some of the results of the aforementioned article (8) that were presented on the RIMS Symposium on New Developments of Representation Theory and Harmonic Analysis on June, 2014 in Kyoto University.

2 Definition

Consider a finite group G, a subgroup H and a subset S such that $S\cap H$ is a symmetric subset.

Definition 2.1. *The* ***Group-Subgroup Pair Graph*** $\mathcal{G}(G,H,S)$ *is the graph with vertex set* G *and edges*

$$\begin{cases}(h,hs),(hs,h) & \forall h\in H,\ \ \forall s\in S-H,\\ (h,hs) & \forall h\in H\ \ \forall s\in S\cap H\end{cases}$$

Alternatively, the group-subgroup pair graph $\mathcal{G}(G,H,S)$ may be defined as

$$\mathcal{G}(G,H,S)=\bar{\mathcal{G}}(G,H,S_O)\oplus\mathcal{G}(H,S_H),$$

where $S_O=S-H$, $S_H=S\cap H$ and $\oplus$ is the generalized edge sum operator, as defined in (5). The graph $\mathcal{G}(H,S_H)$ is a Cayley graph and the graph $\bar{\mathcal{G}}(G,H,S_O)$ is the graph with vertices G and edges

$$\begin{cases}(h,hs) & \forall h\in H,\ \ \forall s\in S_O,\\ (x,xs^{-1}) & \forall x\in \bigcup_{s\in S_O} Hs,\ \ \forall s\in Hx\cap S_O.\end{cases}$$

This alternative definition expresses the group-subgroup pair graph $\mathcal{G}(G, H, S)$ as a union of two undirected graphs, a Cayley graph defined in the vertices of the subgroup H and an outer graph that connect the vertices of the subgroup H with the vertices of $G - H$. It is immediate from this definition that if $G = H$ the group-subgroup pair graph is a Cayley graph, that is, $\mathcal{G}(G, H, S) = \mathcal{G}(G, S)$.

Example 2.2. Consider the dihedral group $G = D_{12}$ of order 12 with generators r and s, and the subgroup $H = D_4$ of index 3. Note that the subgroup H is isomorphic to the Klein 4-group. Consider the following subsets of G.

$$
\begin{aligned}
s_1 &= \{s, s^2, rs, rs^2, rs^3, rs^4\}, \\
s_2 &= \{s, rs, rs^2, rs^5\}.
\end{aligned}
$$

The resulting pair-graphs shown in figures 1 and 2.

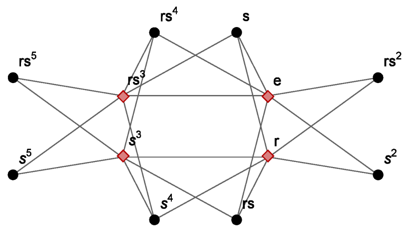

Figure 1: The pair-graph $\mathcal{G}(G, H, S_1)$.

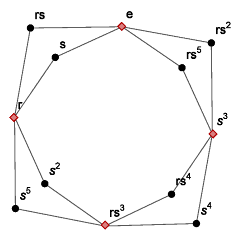

Figure 2: The pair-graph $\mathcal{G}(G, H, S_2)$.

The vertices with diamond shape correspond to elements of the subgroup H. Note that since the graphs are not regular they cannot be realized as Cayley graphs.

We will continue referring to these examples in order to illustrate the results of the following sections.

3 Basic Properties

3.1 Degree structure of the graph

The first result determines the existence of isolated vertices on a given group-subgroup pair graph and exhibits the relation of between the pair-graph and the coset structure of H in G.

Proposition 3.1. *i) The pair-graph $\mathcal{G}(G, H, S)$ contains no isolated vertices if and only if S contains a representative for each coset of H on G different from $He = H$.*

ii) The vertices H are isolated in $\mathcal{G}(G, H, S)$ if and only if S is the empty set.

This facts contrasts with the Cayley graph case, as a Cayley graph with non-empty generating set contains no isolated vertices. As mentioned before, Cayley graphs are regular graphs. In the case of the group-subgroup pair graph, the resulting graph is not regular but vertices on the same coset have the same degree.

Proposition 3.2. *In a pair-graph $\mathcal{G}(G,H,S)$, all the vertices in the same coset have the same degree. Namely, the vertices in H have degree $|S|$ and for $x \in H$ the degree of the vertices in the coset Hx is $|S \cap Hx|$.*

Returning to the examples from before, for the pair-graph $\mathcal{G}(G,H,S_1)$, since $|S_1| = 6$, the vertices of H have degree 6. The cosets of H in G are H, $Hs^2 = \{s^2, s^5, rs^2, rs^5\}$ and $Hs = \{s, s^4, rs, rs^4\}$, so the degree of the vertices in Hs^2 is 2 and the degree of the vertices of Hs is 3. For the graph $\mathcal{G}(G,H,S_1)$ the degree of the vertices in H is 4 and the degree of the vertices in the other two cosets is 2.

The following result establishes when a group-subgroup pair graph is a regular graph.

Proposition 3.3. *Let G be a group and H a subgroup of index $[G:H] = k+1$. For a subset $S \subset G$ with S_H symmetric, consider the pair-graph $\mathcal{G}(G,H,S)$. If $x_1,\dots,x_k$ is a set of representatives of the cosets of H incongruent to e, then for $h \in H$,*

$$\deg(h) = |S| \geqslant \sum_{i=1}^{k} |Hx_i \cap S_O| = |S_O| = \sum_{i=1}^{k} \deg(x_i),$$

with equality only when $S_H = \emptyset$. In particular, a nontrivial $\mathcal{G}(G,H,S)$ is regular if and only if $S_H = \emptyset$ and $[G:H] = 2$, or $[G:H] = 1$.

For the examples, since the index $[G:H] = 3$ the graphs are not regular. The case for $[G:H] = 2$ which results in regular pair-graphs is considered later in this article.

3.2 Connectedness

The connectedness of a Cayley graph $\mathcal{G}(G,S)$ is equivalent to the condition that the subgroup generated by S is equal to G. For the group-subgroup pair graph $\mathcal{G}(G,H,S)$ a similar condition is obtained, and as before, it involves the coset structure of the subgroup H of G.

Theorem 3.4. *A pair-graph $\mathcal{G}(G,H,S)$ is connected if and only if*

$$\langle H \cap (S_H \cup S_o S_o^{-1}) \rangle = H$$

and S contains representatives of all the cosets of H different from the coset H.

For the group-subgroup pair graphs of the example we have $S_{1_H} \cup S_{1_o} S_{1_o}^{-1} = \{e, rs^3, s^3, r\}$ and $S_{2_H} \cup S_{2_o} S_{2_o}^{-1} = \{e, s^3, r\}$. In both cases these subsetsi contain the generators r and s^3 of H and since S_1 and S_2 contain representatives of the cosets the corresponding pair-graphs are connected, in accordance to figures 1 and 2.

When a Cayley graph is not connected, the identity component consists of the vertices $\langle S \rangle$ and the number of connected components is $[G : \langle S \rangle]$. Since a group-subgroup pair graph may contain isolated vertices even in non trivial cases, the connected components may be different in structure to the identity component. The following two results completely characterize the connected components of the group-subgroup pair graphs.

Theorem 3.5. *With the same notation as before, let $U = \langle H \cap (S_H \cup S_O S_O^{-1}) \rangle$, then the identity component Γ_e of $\mathcal{G}(G,H,S)$ consists of the vertices $U \cup \left(\bigcup_{s \in S_O} Us \right)$. The remaining connected components of the pair-graph $\mathcal{G}(G,H,S)$ are either of the type $\Gamma_h = h\Gamma_e$ for $h \in H$ or the type $\{x\}$ for $x \in G - H$.*

Theorem 3.6. *The number of connected components of $\mathcal{G}(G, H, S)$ is*

$$[H : \langle H \cap (S_H \cup S_o S_o^{-1}) \rangle] + |G - H| - \left| \bigcup_{s \in S_O} Hs \right|.$$

3.3 Bipartite group-subgroup pair graphs.

A graph is bipartite when there is a partition V_+, V_- of the vertices such that any pair of vertices in the same subset are not adjacent. For a group G and symmetric subset S, if there is an homomorphism $\chi : G \to \{-1, 1\}$, such that $\chi(S) = \{-1\}$ then the Cayley graph $\mathcal{G}(G, S)$ is bipartite, this condition is also necessary when the graph is connected.

Proposition 3.7. *If a group homomorphism $\chi : G \to \{-1, 1\}$, such that $\chi(S) = \{-1\}$ exists, then the graph $\mathcal{G}(G, H, S)$ is bipartite. The converse is true if $\mathcal{G}(G, H, S)$ is connected and S is symmetric.*

The condition of symmetry on the generating set S cannot be removed, as shown in the following example.

Example 3.8. Let $G = A_4$, the alternating group of 4 letters, H, the Klein four group embedded as a subgroup of G, and $S = \{(1,2)(3,4), (1,4)(2,3), (1,2,3), (1,4,3), (2,3,4), (2,4,3)\}$, using cycle notation. Observe that $(1,3,2)$, the inverse of $(1,2,3)$, is not contained in S. The resulting $\mathcal{G}(G, H, S)$ is a bipartite graph, as shown in figure 3.

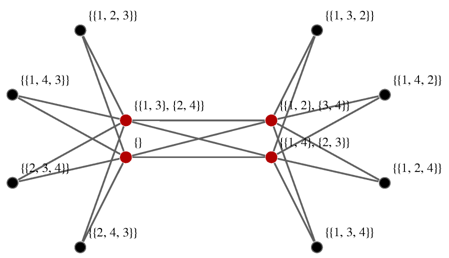

Figure 3: The bipartite pair-graph $\mathcal{G}(G, H, S)$.

Any homomorphism $\chi : G \to \{-1, 1\}$ with $\chi(S) = \{-1\}$, would have $\chi(S^{-1}) = \{-1\}$, but in this case $(1,3,2) = (1,2)(3,4)(1,4,3)$, therefore there are no homomorphisms that satisfy the conditions.

3.4 Vertex transitivity

A graph is vertex transitive when for any pair of different vertices x and y there is graph automorphism φ such that $\varphi(x) = y$. Cayley graphs are naturally vertex transitive by means of left translations L_g with $g \in G$. Any vertex transitive graph must be regular, therefore, in general group-subgroup pair graphs are not vertex transitive.

Proposition 3.9. *Nontrivial group subgroup pair graphs $\mathcal{G}(G, H, S)$ are not vertex transitive when $[G : H] \geqslant 3$.*

If we consider the action of the subgroup by left translations there is transitivity on the elements of the subgroup H.

Proposition 3.10. *The left action of H in $\mathcal{G}(G, H, S)$ is a graph automorphism. Moreover, for any $h_1, h_2 \in H$, the action is transitive. The same holds for any coset Hx, $x \in G$.*

4 The trivial eigenvalues of pair-graphs $\mathcal{G}(G, H, S)$

The eigenvalues of a graph are the eigenvalues of its associated adjacency operator, or matrix. By considering the constant function $f : x \to 1$ on the vertices it is immediate that a k-regular graph has the trivial eigenvalue $\mu = k$. This eigenvalue is the largest eigenvalue of the graph.

Theorem 4.1. *Let G be a group, H a subgroup of G of index $[G : H] = k + 1$ with $k \geqslant 1$, and $S \subset G$ a nonempty subset with S_H symmetric and $|S_O| \neq 0$. Consider $e = x_0, x_1, \ldots, x_k$ a set of representatives of the cosets of H in G and set $S_i = S \cap Hx_i$, for $i \in 1, 2, \ldots, k$. Then*

$$\mu^{\pm} = \frac{|S_H| \pm \sqrt{|S_H|^2 + 4\left(\sum_1^k |S_i|^2\right)}}{2}$$

are eigenvalues of the graph $\mathcal{G}(G, H, S)$.

Proposition 4.2. *With the same notation as before, the eigenvalue μ^+ is the largest eigenvalue of the graph $\mathcal{G}(G, H, S)$ with multiplicity $[H, \langle H \cap (S_H \cup S_O S_O^{-1}) \rangle]$.*

By considering the shape of the adjacency matrix of the group-subgroup pair graphs $\mathcal{G}(G, H, S)$ we also obtain the following result.

Proposition 4.3. *With the same notation as above, when $[G : H] \geq 3$ then $\mu_0 = 0$ is an eigenvalue of $\mathcal{G}(G, H, S)$ with multiplicity at least $|G| - 2|H|$.*

By analogy, we call the above eigenvalues, the **trivial eigenvalues** of the pair graph $\mathcal{G}(G, H, S)$.

Returning to the example, for the pair-graph $\mathcal{G}(G, H, S_1)$ we have $|S_H| = 1$, $|S_1| = 2$ and $|S_2| = 3$, therefore the trivial eigenvalues are

$$\mu^{\pm} = \frac{1 \pm \sqrt{53}}{2}.$$

For the pair-graph $\mathcal{G}(G, H, S_2)$ we have $|S_H| = 0$ and $|S_1| = |S_2| = 2$, then the trivial eigenvalues are

$$\lambda^{\pm} = \pm 2\sqrt{2}.$$

The spectrum of $\mathcal{G}(G, H, S_1)$ consists of $\frac{1}{2}\left(1 + \sqrt{53}\right)$,$\frac{1}{2}\left(1 + \sqrt{53}\right)$, $\frac{1}{2}\left(\sqrt{21} - 1\right)$, $\frac{1}{2}\left(1 + \sqrt{5}\right)$, $\frac{1}{2}\left(\sqrt{5} - 1\right)$, 0, 0, 0, 0, $\frac{1}{2}\left(1 - \sqrt{5}\right)$, $\frac{1}{2}\left(-1 - \sqrt{5}\right)$, $\frac{1}{2}\left(-1 - \sqrt{21}\right)$ and $\frac{1}{2}\left(1 - \sqrt{53}\right)$. On the other hand, the spectrum of $\mathcal{G}(G, H, S_2)$ consists of $2\sqrt{2}$, 2, 2, 0, 0, 0, 0, 0, 0, -2, -2, $-2\sqrt{2}$. Since $|G| - 2|H| = 4$, these computations confirms the above results.

5 Regular pair-graphs

In this section we consider the case $[G : H] = 2$ and $S \cap H = \emptyset$, since in this case H and $G - H$ is a bipartation of the vertices the resulting $\mathcal{G}(G, H, S)$ graph is a bipartite graph.

Proposition 5.1. *Let G, H and S be as described above. If S is a symmetric set, the resulting $\mathcal{G}(G, H, S)$ is a Cayley graph. Namely, $\mathcal{G}(G, H, S) = \mathcal{G}(G, S)$.*

As an example, for p, q prime numbers, when p is not a square modulo q, the $X^{p,q}$ Ramanujan graphs from Lubotzky, Phillips and Sarnak (6) can be identified with pair-graphs $\mathcal{G}(PGL_2(q), PSL_2(q), S_{p,q})$.

In the following we fix a finite group G and subgroup H of index 2 and consider the resulting pair graphs for distinct choices of $G - H$. The main result is that there is a symmetric relation for the nontrivial spectrum of certain choices of S. More precisely, for any subset S of $G - H$ there is a subset S' such that the set of eigenvalues of $\mathcal{G}(G, H, S)$ and $\mathcal{G}(G, H, S')$ differ only in the trivial eigenvalues.

Theorem 5.2. *For group G and subgroup H of index 2 and order n. Let \$$S \subset$ G-H \$ with $|S| = k$ and $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{2n}$ the spectrum of the pair-graph $\mathcal{G}(G, H, S)$. Then there is a $(n-k)$-regular pair-graph $\mathcal{G}(G, H, S')$ with spectrum $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{2n}$ such that*

$$\lambda_i = \mu_i$$

for $i \neq 1, 2n$.

One such corresponding subset S' is the relative complement of S in $G - H$, but in general, it is not unique. Recall that a Ramanujan graph is a connected k-regular graph such that for every eigenvalue μ different from $\pm k$, one has

$$|\mu| \leqslant 2\sqrt{k-1}.$$

As an application of the above result, we have the following bound on the size of the generating set that gives Ramanujan graphs.

Corollary 5.3. *With the same assumptions as before. If the pair-graph $\mathcal{G}(G, H, S)$ is connected and*

$$|S| \geqslant n + 2 - 2\sqrt{n},$$

then it is a bipartite Ramanujan graph.

Proof. Due to Theorem 5.2 any k-regular pair-graph $\mathcal{G}(G, H, S)$ has nontrivial eigenvalues μ satisfying $|\mu| \leqslant \min\{k, n-k\}$. Also, the pair-graph $\mathcal{G}(G, H, S)$ is a Ramanujan graph when said trivial eigenvalues satisfy $|\mu| \leqslant 2\sqrt{k-1}$. Considering the two inequalities, it follows that all k-regular pair-graphs $\mathcal{G}(G, H, S)$ with $k \leqslant 2$ or $k \geqslant n + 2 - 2\sqrt{n}$ are Ramanujan graphs. □

6 Summary

The diagram shown in figure 4 shows the type of graphs that may be obtained according to the choices of group G, subgroup H and subset S.

The table 1 shows a comparison of the main results of this article with comparable results on Cayley graphs. It can be seen that most of the results are similar but contain some complications related to the coset structure of the subgroup H on the group G.

7 Prospects

As mentioned in the summary, the similarity of the properties of the group-subgroup pair graphs with those of Cayley graphs suggest that we might continue working on this way to study the group-subgroup pair graphs.

- The eigenvalues of a Cayley graph for an abelian group are determined as character sums over the elements of the generating set. There may be a similar relation between the representation of the group, subgroup and generating set and the eigenvalues of the corresponding group-subgroup pair graph.

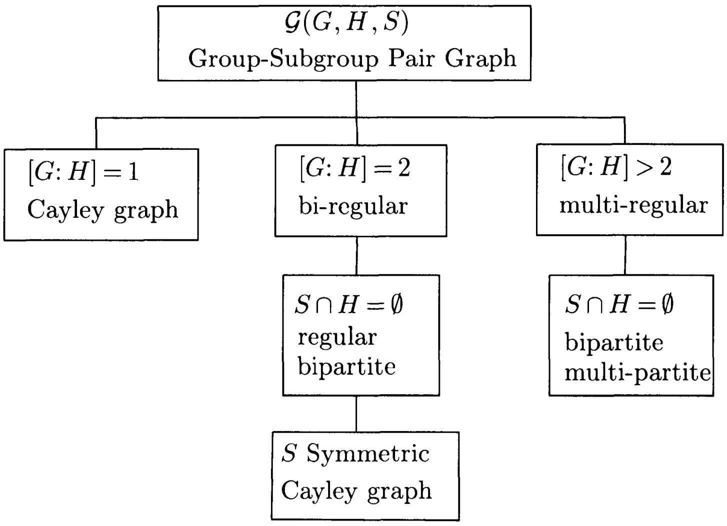

Figure 4: Characteristics of group-subgroup pair graphs.

	Cayley graph	**Group-Subgroup Pair Graph**
Notation	$\mathcal{G}(G,S)$	$\mathcal{G}(G,H,S)$
Isolated Vertices	None for $S \neq \emptyset$	May exist even for $S \neq \emptyset$
Degree Structure	S-regular	Vertices on a coset Hx have same degree
Connected graph	When $\langle S\rangle = G$	When $\langle H \cap (S_H \cup S_o S_o^{-1})\rangle = H$ and S contains a representative of every coset $Hx \neq H$
Connected components	$[G : \langle S\rangle]$	$[H : \langle H \cap (S_H \cup S_o S_o^{-1})\rangle] + \lvert G - H\rvert - \left\lvert \bigcup_{s\in S_O} Hs \right\rvert$
Vertex transitivity	Always	Not for $[G : H] \geq 3$
Trivial eigenvalues	$\mu = \lvert S\rvert$	$\mu^{\pm} = \frac{\lvert S_H\rvert \pm \sqrt{\lvert S_H\rvert^2 + 4(\sum_1^k \lvert S_i\rvert^2)}}{2}$ $\mu_0 = 0$ for $[G : H] >= 3$

Table 1: Comparison of Cayley graphs and group-subgroup pair-graphs

- Numerical computations show that the symmetric relation on the spectrum remains valid for non-regular group-subgroup pair-graphs, so the results of symmetry may be further generalized.
- Ramanujan graphs can be defined equivalently as the graphs that satisfy the graph-theoretical Riemann hypothesis for the Ihara Zeta function, see for example (9). By considering the Zeta functions of the group-subgroup pair graphs one might be able to define an analogue of Ramanujan graphs for r-partite graphs.
- The definition of the group-subgroup pair graph $\mathcal{G}(G,H,S)$ shows that all the information of the adjacency matrix is contained in the rows corresponding to the elements of H, which is a $|H| \times |G|$ rectangular matrix. As mentioned in the paper (8) one of the motivations for the introduction of the group-subgroup pair graphs was the extension of the group determinant to the wreath determinant for group-subgroup pairs, so the study of the properties of this matrix, including its wreath determinant may be useful to understand the properties of the resulting graph. A reference for the wreath determinant is (4).

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