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On a Characterization of the Bilinear Forms Graphs $Bil_q(d \times d)$

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1 Introduction

Much attention has been paid to a problem of classification of all Q-polynomial distance-regular graphs with large diameter [1] (for the definitions, we refer the reader to Section 2). One of the steps towards solution of this problem is a characterization of known distance-regular graphs by their intersection arrays. For the current status of the classification of the Q-polynomial distance-regular graphs, we refer the reader to the survey paper [3] by Van Dam, Koolen and Tanaka.

The bilinear forms graph denoted here by $Bil_q(d\times n)$ is a graph defined on the set of $d\times n$-matrices over $\mathbb{F}_q$ with two matrices being adjacent if and only if the rank of their difference is 1. We refer to [2, Chapter 9.5.A] for the detailed description of these graphs.

In 1999, K. Metsch [5] obtained the following result.

Result 1.1 *The bilinear forms graph $Bil_q(d \times n)$ is characterized by its intersection array if:*

- $q=2$ *and* $n\geq d+4$,
- $q\geq 3$ *and* $n\geq d+3$.

Thus, the open cases are:

- $q=2$ and $n\in\{d,d+1,d+2,d+3\}$,
- $q\geq 3$ and $n\in\{d,d+1,d+2\}$.

In this paper, we discuss a problem of characterization of the bilinear forms graphs $Bil_q(d,d)$, $d\geq 3$, by their intersection arrays.

This paper is based on a talk given at RIMS, and describes a sketch of the proof of our main result (see Section 3). The details of the proof will be given elsewhere.

2 Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple. Suppose that Γ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of adjacent vertices. The distance $d(x, y)$ between any two vertices x, y of Γ is the length of a shortest path connecting x and y in Γ.

For a subset X of the vertex set of Γ, we will also write X for the subgraph of Γ induced by X. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely i from x $(0 \leq i \leq D)$, where $D := \max\{d(x,y) \mid x, y \in V(\Gamma)\}$ is the *diameter* of Γ. In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset$. The subgraph induced by $\Gamma_1(x)$ is called the *neighborhood* or the *local graph* of a vertex x. The ball of radius 1 around x is denoted by $x^\perp$, i.e. $x^\perp = \{x\} \cup \Gamma_1(x)$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$ for short, and we denote $x \sim_\Gamma y$ or simply $x \sim y$ if two vertices x and y are adjacent in Γ. For a graph G, a graph Γ is called *locally* G if any local graph of Γ is isomorphic to G.

For a set of vertices $x_1, \ldots, x_n$, let $\Gamma(x_1, \ldots, x_n)$ denote $\cap_{i=1}^n \Gamma_1(x_i)$. Moreover, if x and y are at distance 2 in Γ, we call $\Gamma(x, y)$ the *μ-graph* of x, y.

The *eigenvalues* of a graph are the eigenvalues of its adjacency matrix (recall that they are algebraic integers). If, for some eigenvalue η of Γ, its eigenspace contains a vector orthogonal to the all ones vector, we say the eigenvalue η is *non-principal.* If Γ is regular with valency k then all its eigenvalues are non-principal unless the graph is connected and then the only eigenvalue that is principal is its valency k.

For a graph Γ and its vertex x, we say that η is a *local* eigenvalue *at* x, if η is an eigenvalue of $\Gamma_1(x)$.

A connected graph Γ with diameter D is called *distance-regular* if there exist integers b_{i-1}, c_i $(1 \leq i \leq D)$ such that, for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. In particular, any distance-regular graph is regular with valency $k := b_0$. We define $a_i := k - b_i - c_i$ for notational convenience and note that $a_i = |\Gamma(y) \cap \Gamma_i(x)|$ holds for any two vertices x, y with $d(x, y) = i$ $(1 \leq i \leq D)$. The array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of the distance-regular graph Γ.

A distance-regular graph with diameter 2 is called a *strongly regular* graph. We say that a strongly regular graph Γ has parameters (v, k, λ, μ), if $v = |V(\Gamma)|$, k is its valency, $\lambda := a_1$, and $\mu := c_2$.

If a graph Γ is distance-regular then, for all integers h, i, j $(0 \leq h, i, j \leq D)$, and all vertices $x, y \in V(\Gamma)$ with $d(x, y) = h$, the number

$$p_{ij}^h := |\{z \in V(\Gamma) \mid d(x, z) = i,\ d(y, z) = j\}|$$

does not depend on the choice of x, y. The numbers p^h_{ij} are called the *intersection numbers* of Γ. Note that $c_i = p^i_{1i-1}$, $a_i = p^i_{1i}$, and $b_i = p^i_{1i+1}$.

For each integer i $(0 \leq i \leq D)$, the ith *distance matrix* A_i of Γ has rows and columns indexed by the vertex of Γ, and, for any $x, y \in V(\Gamma)$,

$$(A_i)_{x,y} = \begin{cases} 1 \text{ if } d(x,y) = i, \\ 0 \text{ if } d(x,y) \neq i. \end{cases}$$

Then $A := A_1$ is just the *adjacency matrix* of Γ, $A_0 = I$, $A_i^\top = A_i$ $(0 \leq i \leq D)$, and

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \quad (0 \leq i, j \leq D),$$

in particular,

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leq i \leq D-1),$$
$$A_1 A_D = b_{D-1} A_{D-1} + a_D A_D,$$

and this implies that $A_i = p_i(A_1)$ for certain polynomial p_i of degree i.

The *Bose-Mesner* algebra $\mathcal{M}$ of Γ is a matrix algebra generated by A_1 over $\mathbb{C}$. It follows that $\mathcal{M}$ has dimension $D+1$, and it is spanned by the set of matrices $A_0 = I, A_1, \ldots, A_D$, which form a basis of $\mathcal{M}$.

Since the algebra $\mathcal{M}$ is semi-simple and commutative, $\mathcal{M}$ also has a basis of pairwise orthogonal idempotents $E_0 := \frac{1}{|V(\Gamma)|} J, E_1, \ldots, E_D$ (the so-called *primitive idempotents* of $\mathcal{M}$):

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq D), \quad E_i = E_i^\top \quad (0 \leq i, j \leq D),$$
$$E_0 + E_1 + \ldots + E_D = I,$$

where J is the all ones matrix.

In fact, E_j $(0 \leq j \leq D)$ is the matrix representing orthogonal projection onto the eigenspace of A_1 corresponding to some eigenvalue of Γ. In other words, one can write

$$A_1 = \sum_{j=0}^{D} \theta_j E_j,$$

where θ_j $(0 \leq j \leq D)$ are the real and pairwise distinct scalars, known as the *eigenvalues* of Γ. We say that the eigenvalues are in *natural* order if $b_0 = \theta_0 > \theta_1 > \ldots > \theta_D$. We denote $\hat{\theta}_i = -1 - \frac{b_1}{\theta_i + 1}$ for $i \in \{1, D\}$.

The Bose-Mesner algebra $\mathcal{M}$ is also closed under entrywise (Hadamard or Schur) matrix multiplication, denoted by $\circ$. Then the matrices $A_0, A_1, \ldots, A_D$ are the primitive idempotents of $\mathcal{M}$ with respect to $\circ$, i.e., $A_i \circ A_j = \delta_{ij} A_i$, and $\sum_{i=0}^{D} A_i = J$. This implies that

$$E_i \circ E_j = \sum_{h=0}^{D} q^h_{ij} E_h \quad (0 \leq i, j \leq D)$$

holds for some real numbers q_{ij}^h, known as the *Krein parameters* of Γ.

Let Γ be a distance-regular graph, and E be a primitive idempotent of its Bose-Mesner algebra. The graph Γ is called *Q-polynomial* (with respect to E) if there exist real numbers c_i^*, a_i^*, b_{i-1}^* $(1 \le i \le D)$ and an ordering of primitive idempotents such that $E_0 = \frac{1}{|V(\Gamma)|}J$ and $E_1 = E$, and

$$E_1 \circ E_i = b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1} \quad (1 \le i \le D-1),$$

$$E_1 \circ E_D = b_{D-1}^* E_{D-1} + a_D^* E_D.$$

Note that a Q-polynomial ordering of the eigenvalues/idempotents does not have to be the natural ordering.

Further, the *dual eigenvalues* of Γ *associated with* E are the real scalars θ_i^* $(0 \le i \le D)$ defined by

$$E = \frac{1}{|V(\Gamma)|} \sum_{i=0}^{D} \theta_i^* A_i.$$

We say that a distance-regular graph Γ has *classical parameters* (D, b, α, β) if the diameter of Γ is D, and the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right), \tag{1}$$

so that, in particular, $c_2 = (b+1)(\alpha+1)$,

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right), \tag{2}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \ldots + b^{j-1}.$$

The following important fact about Q-polynomial distance-regular graphs was proven in [7].

Result 2.1 *Let Γ be a Q-polynomial distance-regular graph with diameter $D \ge 3$. Then, for any $i = 2, \ldots, D-1$, there exists a polynomial T_i of degree 4 such that, for any vertex $x \in V(\Gamma)$ and any non-principal eigenvalue η of the local graph of x, $T_i(\eta) \ge 0$ holds. The polynomials T_i, $i = 2, \ldots, D-1$, differ only in a scalar multiple.*

We call these polynomials the *Terwilliger* polynomials of Γ. The existence of these polynomials was established in [7]. In [4], the polynomial T_2 was calculated explicitly.

Result 2.2 *Suppose that Γ has classical parameters (D, b, α, β). Then the Terwilliger polynomial $T_2(\lambda)$ of Γ is*

$$T_2(\lambda) = \frac{b_2}{\alpha+1}\Big(- \lambda^2 + \lambda\big(\alpha\begin{bmatrix} D \\ 1 \end{bmatrix} + \beta - \alpha - 1 - (\alpha+1)(b+1)\big) + \beta\begin{bmatrix} D \\ 1 \end{bmatrix} - (\alpha+1)(b+1)\Big) \times$$
$$\times \Big(\lambda^2 + \lambda(2 - \alpha b) - \alpha b + 1\Big) - b_2^2(\lambda+1)^2. \quad (3)$$

Furthermore, the roots of $T_2(\lambda)$ are

$$\beta - \alpha - 1, \ -1, \ -b-1, \ \alpha b\frac{b^{D-1}-1}{b-1} - 1.$$

Note that the bilinear forms graph $Bil_q(d \times n)$, $n \geq d$, has classical parameters $(D, b, \alpha, \beta) = (d, q, q-1, q^n - 1)$. In particular, if Γ is a distance-regular graph with the same intersection array as $Bil_q(d \times d)$, $d \geq 3$, then, for any vertex $x \in V(\Gamma)$ and any non-principal eigenvalue η of the local graph of x, one has:

$$\eta \in [-q-1, -1] \ \text{ or } \ \eta = q^n - q - 1, \quad (4)$$

3 Main result

In this section, we suppose that Γ is a distance-regular graph with the same intersection array as $Bil_2(d \times d)$, $d \geq 3$.

Proposition 3.1 *The local graph of any vertex x of Γ is the $(2^d - 3) \times (2^d - 3)$-grid.*

Proof: By (4), for $q = 2$, a local non-principal eigenvalue η at any vertex $x \in \Gamma$ satisfies:

$$\eta \in [-3, -1] \text{ or } \eta = 2^d - 3.$$

Claim 3.2 *$\Gamma_1(x)$ has only integral eigenvalues, i.e., -3, -2, -1, or $2^d - 3$.*

Proof: Recall that the eigenvalues of a graph are algebraic integers, and their product is an integer. Let $\eta_1, \ldots, \eta_s$ be all *irrational* eigenvalues of $\Gamma_1(x)$. Then $\eta_i \in (-3, -1)$ and $\Pi_{i=1}^s \eta_i$ is an integer, and thus $\Pi_{i=1}^s(\eta_i + 2)$ is an integer. Now $\eta_i \in (-3, -1) \Rightarrow |\eta_i + 2| < 1 \Rightarrow \Pi_{i=1}^s(\eta_i + 2) = 0$. The claim is proved.

Claim 3.3 *$\Gamma_1(x)$ has spectrum* $2(2^n - 2)^1$, $(2^n - 3)^{2(2^n-2)}$, $(-2)^{(2^n-1)^2}$.

Proof: Recall the following basic fact from algebraic graph theory. Let $\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_s^{m_s}$ be the spectrum of a regular (with valency k) graph on v vertices, and A be its adjacency matrix. Then:

$$\sum_{i=0}^{s} m_i = v, \quad tr(A) = \sum_{i=0}^{s} m_i \theta_i = 0, \quad tr(A^2) = \sum_{i=0}^{s} m_i \theta_i^2 = vk, \tag{5}$$

where we may put $\theta_0 = k$ and, moreover, $m_0 = 1$ if the graph is connected.

Apply this fact to $\Gamma_1(x)$. In our notation:

$$b_0 = v = (2^n - 1)^2, \quad \theta_0 = k = a_1 = 2(2^n - 2),$$

$$\theta_1 = 2^n - 3, \quad \theta_2 = -1, \quad \theta_3 = -2, \quad \theta_4 = -3,$$

and m_1, m_2, m_3, m_4 are unknown multiplicities of $\theta_1, \theta_2, \theta_3, \theta_4$, respectively, while $m_0 = 1$ (as $\Gamma_1(x)$ is connected).

Then (5) gives a system of (three) linear equations with respect to (four) unknowns $m_1, \ldots, m_4$. One can show that this system has the only non-negative integral solution:

$$m_1 = 2(2^n - 2), \quad m_2 = 0, \quad m_3 = (2^n - 1)^2, \quad m_4 = 0,$$

which shows the claim.

We now see that $\Gamma_1(x)$ is a regular graph with exactly 3 distinct eigenvalues. This yields that $\Gamma_1(x)$ is a strongly regular graph with smallest eigenvalue -2. It now easily follows from Seidel's classification of strongly regular graphs with smallest eigenvalue -2, see [9], that $\Gamma_1(x)$ is a $(2^d - 3) \times (2^d - 3)$-grid. ∎

Lemma 3.4 *For every pair of vertices $x, y \in \Gamma$ with $d(x, y) = 2$, the induced subgraph $\Gamma(x) \cap \Gamma(y)$ is a 6-gon.*

Proof: The lemma easily follows from Proposition 3.1 and the fact that $c_2 = 6$. ∎

We now see that Γ has the same local graphs as $Bil_2(d \times d)$.

Let $\mathcal{H}$ denote the bilinear forms graph $Bil_2(d \times d)$. For vertices $\mathbf{x} \in \mathcal{H}, x \in \Gamma$, an isomorphism $\varphi : \mathbf{x}^{\perp} \to x^{\perp}$ is called *extendable* if there is a bijection $\varphi' : \mathbf{x}^{\perp} \cup \mathcal{H}_2(\mathbf{x}) \to x^{\perp} \cup \Gamma_2(x)$, mapping edges to edges, such that $\varphi'|_{\mathbf{x}^{\perp}} = \varphi$ (in this case φ' is called the extension of φ). We say that Γ has distinct μ-graphs if $\Gamma(x, y) = \Gamma(x, z)$ for $y, z \in \Gamma_2(x)$ implies $y = z$. This property yields that the extension φ' above is unique.

A graph Δ is called *triangulable* if every cycle in it can be decomposed into a product of triangles (see [6, Section 6]).

For the following result, see [6, Theorem 7.1].

Result 3.5 *Assume:*

(1) Γ *has distinct* μ*-graphs.*

(2) *There exist a vertex* $\mathbf{x}$ *of* $\mathcal{H}$ *and a vertex* x *of* Γ*, and an extendable isomorphism* $\varphi:\ \mathbf{x}^{\perp} \to x^{\perp}$.

(3) *If* $\mathbf{x}, x$ *are vertices of* $\mathcal{H}$*,* Γ*, respectively,* $\varphi:\ \mathbf{x}^{\perp} \to x^{\perp}$ *is an extendable isomorphism,* φ' *is its extension, and* $\mathbf{w} \in \mathcal{H}(\mathbf{x})$*, then* $\varphi'|_{\mathbf{w}^{\perp}}:\ \mathbf{w}^{\perp} \to \varphi(\mathbf{w})^{\perp}$ *is extendable.*

(4) $\mathcal{H}$ *is triangulable.*

Then Γ *is covered by* $\mathcal{H}$.

Indeed, since Γ and $\mathcal{H}$ have the same intersection arrays, Result 3.5 implies that $\Gamma \cong \mathcal{H}$.

It is not difficult to see that Γ satisfies Conditions (1) and (4) of Result 3.5.

Let $\Gamma(x) := \{w_{ij}\}_{i,j}$, and, as usually, for distinct pairs (i,j) and (i',j'), $w_{ij} \sim w_{i'j'}$ holds if and only if $i = i'$ or $j = j'$. Denote by L_i the maximal clique of $\Gamma(x)$ that contains the vertices w_{ij} for all j, and by $L_j^{\top}$ the maximal clique of $\Gamma(x)$ that contains the vertices w_{ij} for all i. For a vertex $x \in \Gamma$, $x^{\perp}$ denotes $\{x\} \cup \Gamma(x)$.

Without loss of generality, we may assume that there is a vertex $z \in \Gamma_2(x)$ such that $\Gamma(x,z) \subset L_1 \cup L_2 \cup L_3$. Define a subgraph Σ induced in Γ by the vertex subset

$$\{x\} \cup L_1 \cup L_2 \cup L_3 \cup \{z' \in \Gamma_2(x) \mid \Gamma(x,z') \subset L_1 \cup L_2 \cup L_3\},$$

so that $\Sigma(x) = L_1 \cup L_2 \cup L_3$.

In order to show that Γ satisfies Conditions (2) and (3) of Result 3.5, one has to show the following.

Lemma 3.6 Σ *is isomorphic to* $Bil_2(2,d)$.

The main result of this work is the following theorem.

Theorem 3.7 *The bilinear forms graphs* $Bil_2(d,d)$*,* $d \geq 3$*, are uniquely determined by their intersection arrays.*

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