数理解析研究所講究録
第 1932 巻 2015 年 30-35

行列の群の間の写像
MAPS BETWEEN GROUPS OF MATRICES

羽鳥 理
OSAMU HATORI

ABSTRACT. We give a structure theorem for isometries on the special unitary group. Applying a non-commutative Mazur-Ulam theorem we show that they are extended to a (complex or conjugate)-linear algebra isomorphism (or anti-isomorphism) between the full matrix-algebra followed by a multiplication by a unitary matrix whose determinant is 1. This is an anouncement of the forthcoming paper [6].

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The most prominent results on the study of isometries on normed spaces are the Banach-Stone theorem, its non-commutative generalization by Kadison [13, 14], and the celebrated Mazur-Ulam theorem. On the other hand, systematic studies of isometries of groups of operators and matrices have just begun and include the general linear groups [11] and unitary groups [8, 9, 5] of unital C^*-algebras, the (special) orthogonal groups [1]. Recent work of Molnár and Šemrl [17] and Molnár [15] describes the surjective isometries of the unitary group with the metrics induced by the unitarily invariant norms as well as other metrics. On the other hand the isometries on the special unitary group seem not to be described yet. The aim of this paper is to anounce the main result in the forthcoming paper on the isometries on special unitary group. We emphasise that in this paper an isometry merely means a distance preserving transformation, we do not assume that it respects any algebraic operation.

The celebrated Mazur-Ulam theorem states that a surjecitve isometry between real normed spaces preserves the algebraic midpoint; therefore it is a real linear isometry followed by a translation. The author, Hirasawa, Miura and Molnár [7] generalised it to a non-commutative version. It states that isometries between certain subsets of groups with metrics preserve the inverted Jordan triple product locally. It plays an important role in the study of isometries on groups. Applying it Molnár and the author [9] proved that isometries on the unitary group of a von Neumann algebra preserve the inverted Jordan triple product. Then they employed a one-parameter-group argument to replace the investigation on the unitary groups to that on the space of all self-adjoint elements. Applying Kadison's structure theorem for isometries on the space of all self-adjoint elements [14], the forms of the original isometries on the unitary groups are given in [9]. In this paper we also apply the non-commutative Mazur-Ulam theorem and the one-parameter-group argument to prove the main result.

For a positive integer n let $M_n(\mathbb{C})$ be the complex algebra of all $n \times n$ matrices of complex entries. In this paper the unit matrix is denoted by E. The eigenvalue of $X \in M_n(\mathbb{C})$ is denoted by $\sigma(X)$. For $X \in M_n(\mathbb{C})$ we denote the trace of X by $\mathrm{Tr}(X)$. The unitary group, which consists of all unitary matrices is denoted by $U(n)$. The special unitary group, which consists of all unitary matrices whose determinants are 1 is denoted by $SU(n)$. The space of all Hermitian matrices is denoted by $H(n)$. Note that $\sigma(X) \subset \mathbb{R}$ for every $X \in H(n)$, where

2000 *Mathematics Subject Classification.* Primary 15A86, 15B57; Secondary 47B49.
Key words and phrases. Isometry, special unitary group.

$\mathbb{R}$ is the set of all real numbers. The subspace $\{X : X \in H(n),\ \mathrm{Tr}(X) = 0\}$ of $H(n)$ which consists of Hermitian matrices whose traces are 0 is denoted by $H^0(n)$. For every $X \in H^0(n)$, denote

$$K_X = \{\pm\alpha : \alpha \in \sigma(X)\}, \quad K_X^0 = K_X \cup \{0\}$$

and

$$s(X) = \{|\alpha| : \alpha \in \sigma(X)\}.$$

Recall that the singular value of the Hermitian matrix coincides with the absolute value of the eigenvalue. Hence $s(X)$ is the set of all singular values of X for every $X \in H(n)$. It is well known that the Lie algebra of the Lie group $SU(n)$ is $iH^0(n)$, and $SU(n) = \exp(iH^0(n))$. In this paper the norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is the usual spectral norm; $\|X\| = \max\{\|Xv\| : v \in \mathbb{C}^n, \|v\| \leq 1\}$, hence $\|X\| = \max\{|\lambda| : \lambda \in \sigma(X)\}$ for every $X \in H(n)$. For $A \in M_n(\mathbb{C})$, A^* denotes the adjoint of A; A^{tr} denotes the transpose of A; $\bar{A}$ denotes the matrix whose (k,l)-entry is the complex-conjugate of the (k,l)-entry of A for every $1 \leq k,\ l \leq n$. The main result of the paper is the following.

Theorem 1.1. *Let ϕ be a map from $SU(n)$ into $SU(n)$. Then the following (i) and (ii) are equivalent.*

(i) ϕ is an isometry with respect to the metric induced by $\|\cdot\|$; $\|\phi(A) - \phi(B)\| = \|A - B\|$ for every pair $A, B \in SU(n)$.

(ii) There exists $U \in U(n)$ such that ϕ has of one of the following forms:

(a) $\phi(A) = \phi(E)UAU^*$ for every $A \in SU(n)$,

(b) $\phi(A) = \phi(E)UA^{tr}U^*$ for every $A \in SU(n)$,

(c) $\phi(A) = \phi(E)UA^*U^*$ for every $A \in SU(n)$,

(d) $\phi(A) = \phi(E)U\bar{A}U^*$ for every $A \in SU(n)$.

In these cases ϕ is automatically surjective.

If a map from $SU(n)$ into $SU(n)$ has one of the forms of (a), (b), (c) or (d) of (ii), then by a simple calculation ϕ is a surjective isometry from $SU(n)$ onto itself.

To prove the converse implication we employ so to say the CDA. The crucial point for the CDA to work with is that we need to prove that the given map admit propriate algebraic structure; T preserves the inverted Jordan product. Here we need a non-commutatvie Mazur-Ulam theorem.

2. THE COMMUTATIVE DIAGRAM ARGUMENT; CDA

We exhibit the commutative diagram argument in a general situation. Let L_j be a normed linear space for $j = 1, 2$ with which $\exp L_j$ is well defined. Suppose that $T : \exp L_1 \to \exp L_2$ is a surjective isometry. The picture is as follows. Given $T : \exp L_1 \to \exp L_2$, find $f : L_1 \to L_2$ such that the following diagram commute;

$$\begin{tikzcd} \exp L_1 \arrow[r, "T_0"] & \exp L_2 \\ L_1 \arrow[u, "\exp\cdot"] \arrow[r, "f"] & L_2 \arrow[u, "\exp\cdot"'] \end{tikzcd}$$

$$T_0(\exp x) = \exp f(x) \qquad x \in L_1,$$

where T_0 is the normalization of T, that is, $T_0(1) = 1$ by applying a suitable transformations on T. The one-parameter-group argument is not new; the argument is applied in several situations. The point here is that we do not assume any algebraic property on the given isometry. Prior to apply the one-parameter-group argument we need to prove that the given isometry does preserve a propriate algebraic structure.

- (1) Applying the non-commutative Mazur-Ulam theorem to ensure that T preserves the inverted Jordan product;

  $$T(\exp x \exp(-y) \exp x) = T(\exp x)(T(\exp y))^{-1}T(\exp x), \quad x, y \in L_1;$$

  This part is crucial for the following parts to work. Onece this part is establised the following arguments are usual ones.
- (2) Applying the above to prove that

  $$\mathbb{R} \ni r \mapsto T_0(\exp(rx))$$

  is a continuous one-parameter-group for every $x \in L_1$, where T_0 is the normalization of T;
- (3) By a representation theorem for the one-parameter-group to get the bijection $f : L_1 \to L_2$ with $T_0(\exp x) = \exp f(x)$ for every $x \in L_1$;
- (4) Prove that f is a surjective isometry and applying the celebrated Mazur-Ulam theorem to show that f is a surjective real linear isometry;
- (5) If the form of f is known, then applying it to describe the form of T_0 and T.

The crucial point for the CDA to work with is that we need to prove that the given map admits a propriate algebraic structure; T is expected to preserves the inverted Jordan product, at least locally. Here we need a non-commutatvie Mazur-Ulam theorem to prove it. By CDA we have described the form of isometries between the unitary groups in a von Neumann algebras in [9, Theorem 1] and a unital C^*-algebra [5] (cf.[15]). Note that the similar argument works for not only for unitary groups but for the space of all positive invertible elements in a unital C^*-algebras [9, Theorem 9] (cf. [12, 18, 16]), maps between the exponentials of Lipschitz algebras [10, Theorem 8], and maps between the exponentials of uniform algebras [19]. We also described by the CDA the forms of isometries on the special orthogonal group in [1].

To prove the converse implication of Theorem 1.1 we also apply the CDA; the non-commutative Mazur-Ulam theorem (cf. [8, Theorem 6]) and the one-parameter-group argument (see [9, 1, 5]) to infer that there exists a surjective real-linear isometry $f : H^0(n) \to H^0(n)$. Although the structure theorem for a surjective isometry from $H(n)$ onto $H(n)$ is already known by [14, Theorem 2], the author does not know the structure theorem for a surjective isometry on $H^0(n)$. Here comes a difficulty.

3. Preparation of the proof that (i) implies (ii)

To prove Theorem 1.1 by applying the CDA, put $L_j = iH^0(n)$ and $\exp iH^0(n) = SU(n)$. In the following Lemmas 3.1 to 3.10, $\phi : SU(n) \to SU(n)$ is an isometry and $\phi_0(\cdot) = \phi(E)^{-1}\phi(\cdot)$. We omit a proofs of Lemmas. Precise proofs are given in [6].

Lemma 3.1. *The map ϕ_0 is a surjective isometry from $SU(n)$ onto itself. There exists a real-linear isometry f from $H^0(n)$ onto itself such that*

$$\phi_0(\exp(itx)) = \exp(itf(x)), \quad t \in \mathbb{R}, \ x \in H^0(n).$$

Throughout this section f is the isometry given in Lemma 3.1. The structure of a surjective isometry (with respect to the spectral norm) between $H(n)$ is described by the theorem of Kadison [14, Theorem 2]. On the other hand the sturctue theorem for a surjective isometry from $H^0(n)$ onto itself seems to be missing. We will prove that either f or $-f$ preserves the spectrum (Lemma 3.10). Note that with respect to the following lemma a similar statements for

f^{-1} in the place of f are also satisfied since $f^{-1} : H^0(n) \to H^0(n)$ is a surjective isometry such that $\phi_0^{-1}(\exp(ity)) = \exp(itf^{-1}(y))$ for every $t \in \mathbb{R}$ and $y \in H^0(n)$.

Lemma 3.2. *For every $x \in H^0(n)$ $s(x) \setminus \{0\} = s(f(x)) \setminus \{0\}$.*

Recall that the Hausdorff distance $\Delta(F_1, F_2)$ between two non-empty compact sets F_1 and F_2 of $\mathbb{C}$ is

$$\Delta(F_1, F_2) = \max\{\sup_{z \in F_2} d(z, F_1), \sup_{w \in F_1} d(w, F_2)\},$$

where $d(v, F) = \inf_{w \in F} |v - w|$ for $v \in \mathbb{C}$ and a non-empty compact set F of $\mathbb{C}$. Due to Bhatia [2, Corollary VI.3.4] we have the inequality

$$d(\lambda, \sigma(y)) \leq \Delta(\sigma(x), \sigma(y)) \leq \|x - y\|, \quad x, y \in H(n) \tag{3.1}$$

for any $\lambda \in \sigma(x)$. Since $\sigma(y)$ is a finite set we have the following by (3.1).

Lemma 3.3. *Let $\varepsilon > 0$ and $x, y \in H^0(n)$. Suppose that $\|x - y\| \leq \varepsilon$. Then for every $\lambda \in \sigma(x)$, there exists $\lambda' \in \sigma(y)$ with $|\lambda - \lambda'| \leq \varepsilon$.*

Lemma 3.4. *For every $x \in H^0(n)$, $0 \in \sigma(x)$ if and only if $0 \in \sigma(f(x))$. Hence $s(x) = s(f(x))$, $K_x = K_{f(x)}$ and $K_x^0 = K_{f(x)}^0$ for every $x \in H^0(n)$.*

Lemma 3.5. *Let $x, y \in H^0(n)$ and let ε be such that*

$$0 < 3\varepsilon < \min\{|u - v| : u, v \in K_x^0, \ u \neq v\}.$$

Suppose that $\|x - y\| \leq \varepsilon$, $\lambda \in \sigma(x)$, and $-\lambda \notin \sigma(x)$. If $\mu \in \sigma(y)$ satisfies $|\lambda - \mu| \leq \varepsilon$ then $-\mu \notin \sigma(y)$.

Lemma 3.6. *Suppose that $x \in H^0(n)$ and $\sigma(x) = \{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_k\}$, where $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_k$ are all different. Suppose that*

$$\{\pm\alpha_1, \dots, \pm\alpha_l\} \cap \{\pm\beta_1, \dots, \pm\beta_k\} = \emptyset.$$

Let ε be a positive real number which satisfies that

$$3\varepsilon < \min\{|u - v| : u, v \in K_x^0, \ u \neq v\}.$$

Suppose that $y \in H^0(n)$ satisfies that $\|x - y\| \leq \varepsilon$ and

$$\{\beta_1, \dots, \beta_k\} \subset \sigma(y) \subset \{\alpha_1 \pm \varepsilon, \dots, \alpha_l \pm \varepsilon, \beta_1, \dots, \beta_k\}.$$

Then

$$\sigma(f(x)) \setminus \{\pm\alpha_1, \dots, \pm\alpha_l\} = \sigma(f(y)) \setminus \{\pm(\alpha_1 \pm \varepsilon), \dots, \pm(\alpha_l \pm \varepsilon)\}.$$

Lemma 3.7. *For every $x \in H^0(n)$, $\pm\lambda \in \sigma(x)$ if and only if $\pm\lambda \in \sigma(f(x))$.*

Lemma 3.8. *Let $x \in H^0(n)$. Suppose that there exists a $\lambda \in \sigma(x)$ which satisfies that $-\lambda \notin \sigma(x)$ and $\lambda \in \sigma(f(x))$ (resp. $-\lambda \in \sigma(f(x))$). Then $\mu \in \sigma(f(x))$ (resp. $-\mu \in \sigma(f(x))$) holds for every $\mu \in \sigma(x)$.*

Lemma 3.9. *For every $x \in H^0(n)$, $\sigma(f(x)) = \sigma(x)$ or $\sigma(f(x)) = -\sigma(x)$.*

Lemma 3.10. *The isometry f preserves the spectrum (i.e., $\sigma(f(x)) = \sigma(x)$ for every $x \in H^0(n)$) or $-f$ preserves the spectrum (i.e., $\sigma(f(x)) = -\sigma(x)$ for every $x \in H^0(n)$).*

4. Completion of the proof of Theorem 1.1

In this section we complete the proof that (i) of Theorem 1.1 implies (ii) of Theorem 1.1.

Suppose that (i) of Theorem 1.1 holds; $\phi : SU(n) \to SU(n)$ is an isometry. By Lemma 3.1 there exists a surjective real-linear isometry f from $H^0(n)$ onto itself such that

$$\phi_0(\exp(itx)) = \exp(itf(x)), \quad t \in \mathbb{R},\ x \in H^0(n),$$

where $\phi_0(\cdot) = \phi(E)^{-1}\phi(\cdot)$. Then by Lemma 3.10 the isometry f itself or $-f$ preserve the spectrum.

We consider in two cases: f preserves the spectrum; $-f$ preserves the spectrum. The argument is similar in both cases we only consider the case where f preserves the spectrum. Let $\tilde{f} : H(n) \to H(n)$ be defined by

$$x \mapsto f\left(x - \frac{\mathrm{Tr}(x)}{n}E\right) + \frac{\mathrm{Tr}(x)}{n}E$$

for $x \in H(n)$. It is easy to check that $\tilde{f}$ is a surjective real-linear map and preserves the spectrum. Since $\|x\| = \max\{|\lambda| : \lambda \in \sigma(x)\}$ for every $x \in H(n)$, $\tilde{f}$ is a real-linear isometry and by definition $\tilde{f}(E) = E$. According to the structure theorem of Kadison [14, Theorem 2] on surjective isometries on the real-linear space of all self-adjoint elements in a unital C^*-algebra there exists a Jordan $*$-isomorphism J from $M_n(\mathbb{C})$ onto itself such that $\tilde{f} = J$ on $H(n)$, hence $f = J$ on $H^0(n)$. The structure of J is already known that there is a unitary matrix U such that $J(X) = UXU^*$ for every $X \in M_n(\mathbb{C})$ or $J(X) = UX^{tr}U^*$ for every $X \in M_n(\mathbb{C})$, where X^{tr} denotes the transpose of X. Thus we have

$$\phi_0(\exp(ix)) = \exp(iUxU^*) = U\exp(ix)U^*, \quad x \in H^0(n)$$

or

$$\begin{aligned}\phi_0(\exp(ix)) = \exp(iUx^{tr}U) = U\exp(ix^{tr})U^* \\ = U(\exp(ix))^{tr}U^*, \quad x \in H^0(n).\end{aligned}$$

As $SU(n) = \exp(iH^0(n)))$ we get

$$\phi(A) = \phi(E)UAU^*, \quad A \in SU(n)$$

or

$$\phi(A) = \phi(E)UA^{tr}U^*, \quad A \in SU(n).$$

In the case when $-f$ preserves the spectrum, applying the same argument for $-f$ in the place of f we obtain a unitary matrix U such that

$$\begin{aligned}\phi_0(\exp(ix)) = \exp(-iUxU^*) = U\exp(-ix)U^* \\ = U(\exp(ix))^*U^*, \quad x \in H^0(n)\end{aligned}$$

or

$$\begin{aligned}\phi_0(\exp(ix)) = \exp(-iUx^{tr}U) = U\exp(-ix^{tr})U^* \\ = U\overline{\exp(ix)}U^*, \quad x \in H^0(n).\end{aligned}$$

Thus we have

$$\phi(A) = \phi(E)UA^*U^*, \quad A \in SU(n)$$

or

$$\phi(A) = \phi(E)U\bar{A}U^*, \quad A \in SU(n).$$

5. Problem

If the following problem is solved, the proof of Theorem 1.1 can be much simpler.

Problem 5.1. Describe the form of a surjective isometry from $H^0(n)$ onto itself.

References

[1] T. Abe, S. Akiyama and O. Hatori, *Isometries of the special orthogonal group,* Linear Algebra Appl., **439** (2013), 174–188

[2] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997

[3] M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge Univ. Press, 2002

[4] J. B. Conway, *A Course in Functional Analysis,* Springer-Verlag, 1990.

[5] O. Hatori, *Isometries of the unitary groups in C^*-algebras,* Studia Math., **221**(2014), 61–86

[6] O. Hatori, *Isometries on the special unitary group,* to appear, Contemp. Math., American Mathematical Society

[7] O. Hatori, G. Hirasawa, T. Miura and L. Molnár, *Isometries and maps compatible with inverted Jordan triple products on groups,* Tokyo J. Math., **35** (2012), 385–410.

[8] O. Hatori and L. Molnár, *Isometries of the unitary group,* Proc. Amer. Math. Soc., **140** (2012), 2141–2154.

[9] O. Hatori and L. Molnár, *Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^*-algebras,* J. Math. Anal. Appl., **409** (2014) 158–167.

[10] O. Hatori, A. Jiménex-Vargas and M. Villegas-Vallecillos, *Maps which preserve norms of non-symmetrical quotients between groups of exponentials of Lipschitz functions*, J. Math. Anal. Appl., **415** (2014), 825–845

[11] O. Hatori and K. Watanabe, *Isometries between groups of invertible elements in C^*-algebras,* Studia Math., **209** (2012), 103–106.

[12] S. Honma and T. Nogawa, *Isometries of the geodesic distances for the space of invertible positive operators and matrices*, Linear Algebra Appl., **444** (2014), 152–164

[13] R. V. Kadison, *Isometries of operator algebras,* Ann. Math. **54** (1951), 325–338.

[14] R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras,* Ann. Math., **56** (1952), 494–503.

[15] L. Molnár, *Jordan triple endomorphisms and isometries of unitary groups,* Linear Algebra Appl. **439** (2013), 3518–3531

[16] L. Molnár, *Jordan triple endomorphisms and isometries of spaces of positivve definite matrices*, Linear Multilinear Algebra (2014),
http://dx.doi.org/10.1080/03081087.2013.844231, in press

[17] L. Molnár and P. Šemrl, *Transformations of the unitary group on a Hilbert space,* J. Math. Anal. Appl., **388** (2012), 1205–1217.

[18] L. Molnár and P. Szokol, *Transformations n positive definite matrices preserving generalized distance measures*, J. Math. Anal. Appl., **466** (2015), 141–159

[19] T. Nogawa, *Maps which preserve a certain norm condition between the exponential groups of uniform algebras*, preprint

Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181 Japan
E-mail address: hatori@math.sc.niigata-u.ac.jp