数理解析研究所講究録
第 1932 巻 2015 年 60-66

Two ways of measuring chaos in locally compact totally disconnected groups

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Abstract

We discuss some simplifying formulas for the topological entropy of continuous endomorphisms of totally disconnected locally compact groups. Various applications are given, the major one is a connection of the topological entropy to Willis' scale function.

1 Topological entropy in locally compact groups - simplifying formulas

The topological entropy for continuous self-maps of compact spaces was defined by Adler, Konheim and McAndrew in [1]. Later on, this definition was extended by Bowen in [3] to uniformly continuous self-maps of metric spaces. His definition of entropy was especially efficient in the case of locally compact spaces provided with some Borel measure with good invariance properties, so in particular for continuous endomorphisms of locally compact groups provided with their Haar measure. Hood in [11] extended Bowen's definition to uniformly continuous self-maps of arbitrary uniform spaces and hence in particular to continuous endomorphisms of (not necessarily metrizable) locally compact groups. In the sequel we recall this definition as well as its simplified measure-free form from [5, 9].

Let G be a locally compact group and $\phi: G\to G$ a continuous endomorphism. Let $\mathcal{C}(G)$ be a local base at 1 of compact neighborhoods and let μ be a right Haar measure on G. For every $U\in\mathcal{C}(G)$ and every positive integer n, let $C_n(\phi,U)=U\cap\phi^{-1}(U)\cap\ldots\cap\phi^{-n+1}(U)$ be the *n-th ϕ-cotrajectory* of U. Let

$$H_{top}(\phi,U)=\limsup_{n\to\infty}-\frac{\log\mu(C_n(\phi,U))}{n}. \tag{1}$$

It is important to note that $H_{top}(\phi,U)$ does not depend on the choice of the Haar measure μ. The *topological entropy* of ϕ is

$$h_{top}(\phi)=\sup\{H_{top}(\phi,U):U\in\mathcal{C}(G)\}.$$

In case G is totally disconnected, one can obtain a measure-free formula in place of (1) (see (2)). Indeed, by a classical theorem of van Dantzig from [13], the filter base $\mathcal{C}(G)$ contains another much more convenient filter base, namely the family $\mathcal{B}(G)$ of all open compact *subgroups* of G. Moreover, for $U\in\mathcal{B}(G)$, the index

$$s(\phi,U):=[\phi(U):U\cap\phi(U)]$$

is finite, as $U\cap\phi(U)$ is open and $\phi(U)$ is compact. Analogously, $[U:C_n(\phi,U)]$ is finite for every positive integer n. As done in [5, 9], using the elementary properties of the measure μ, one can easily see that the limit in (1) exists and, more precisely, that

$$H_{top}(\phi,U)=\lim_{n\to\infty}\frac{\log[U:C_n(\phi,U)]}{n}. \tag{2}$$

Since $H_{top}(\phi,U)\leq H_{top}(\phi,V)$ whenever $V\subseteq U$ for $V,U\in\mathcal{C}(G)$ (i.e., $H_{top}(\phi,-)$ is monotone with respect to inclusion), the computation of the topological entropy can be simplified when G is a totally disconnected locally compact group, that is,

$$h_{top}(\phi)=\sup\{H_{top}(\phi,U):U\in\mathcal{B}(G)\}. \tag{3}$$

By (2) and (3), we obtain the following

Proposition 1.1. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a continuous endomorphism. Then*

$$h_{top}(\phi) = \sup\left\{ \lim_{n\to\infty} \frac{\log[U : C_n(\phi, U)]}{n} : U \in \mathcal{B}(G) \right\}.$$

For $U \in \mathcal{C}(G)$ the formulas (1) and (2) measure how rapidly the partial co-trajectory $C_n(\phi, U)$ approximates the *co-trajectory*

$$C(\phi, U) := \bigcap_{n=0}^{\infty} C_n(\phi, U) = \bigcap_{n=0}^{\infty} \phi^{-n}(U),$$

that we shall denote also by U_-, following [14]. For the sake of completeness, let $U_+ = \bigcap_{n=0}^{\infty} \phi^n(U)$. Both U_- and U_+ are compact, and U_- is the greatest ϕ-invariant (i.e., $\phi(U_-) \subseteq U_-$) subgroup of G contained in U.

In case the locally compact group G is totally disconnected, $\phi : G \to G$ is a topological automorphism and $U \in \mathcal{B}(G)$, it is possible to obtain a limit-free formula for the topological entropy $H_{top}(\phi, U)$ of ϕ with respect to U (see Theorem 1.2). In the sequel Δ detoňes the modular function $\Delta : \mathrm{Aut}(G) \to \mathbb{R}_+$ showing the extent to which an automorphism ϕ "expands" the right Haar measure μ of G (recall that it is independent of μ).

Theorem 1.2. [10] *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then*

$$H_{top}(\phi, U) = \log[\phi^{-1}(U_-) : U_-] + \log \Delta(\phi) = \log[\phi(U_+) : U_+].$$

The next theorem offers a more precise result, as far as "topological automorphism" is replaced by the milder condition "continuous endomorphism satisfying (4)", however the price to pay is the compactness of the group. Normality of the open subgroup U of K is not restrictive, since an open subgroup of a compact group K contains always an open normal subgroup of K.

Theorem 1.3. [6] *Let K be a totally disconnected compact group, $\psi : K \to K$ a continuous endomorphism and U an open normal subgroup of K such that*

$$|K/(\mathrm{Im}\psi \cdot C(\psi, U))| < \infty. \tag{4}$$

Then

$$H_{top}(\psi, U) = \log\left|\frac{\psi^{-1}(C(\psi, U))}{C(\psi, U)}\right| - \log\left|\frac{K}{\mathrm{Im}\psi \cdot C(\psi, U)}\right|.$$

If K is also abelian, then (4) is necessarily satisfied by every open subgroup U of K.

2 Basic properties of topological entropy in locally compact groups

As a first application of the results recalled in §1, we observe that the finite values of the topological entropy of topological automorphisms of totally disconnected locally compact groups belong to the discrete subset $\log \mathbb{N}_+ := \{\log n : n \in \mathbb{N}_+\}$ of $\mathbb{R}$. This should be compared with the still open problem about the values of the topological entropy of topological automorphisms of compact abelian groups, equivalent to the eighty years old Lehmer problem (see [12]). According to this problem, it is unknown whether one can find topological automorphisms of compact abelian groups of sufficiently small positive topological entropy. A positive answer would imply that every positive real number is eligible as the value of the topological entropy of some topological automorphism of some compact abelian group (see [5] for more details).

We list in the sequel some known properties of the topological entropy that can be easily obtained from the above limit-free formula given in Theorem 1.2 (see [10] for such a deduction).

Let us start with the invariance under conjugation.

Proposition 2.1. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Let H be another totally disconnected locally compact group and $\xi : G \to H$ a topological isomorphism. Then $h_{top}(\phi) = h_{top}(\xi\phi\xi^{-1})$.*

The next property is a weak form of the so-called addition theorem:

Proposition 2.2. *Let G and H be totally disconnected locally compact groups, $\phi : G \to G$ and $\psi : H \to H$ topological automorphisms. Then $h_{top}(\phi \times \psi) = h_{top}(\phi) + h_{top}(\psi)$.*

Next comes monotonicity with respect to taking restrictions to stable normal subgroups N or with respect to the topological automorphisms induced on the quotients G/N.

Proposition 2.3. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and H a closed normal subgroup of G such that $\phi(N) = N$, and let $\overline{\phi} : G/H \to G/H$ be the topological automorphism induced by ϕ. Then:*

(a) $h_{top}(\phi) \geq h_{top}(\phi\restriction_N)$;

(b) $h_{top}(\phi) \geq h_{top}(\overline{\phi})$.

The next is the so-called logarithmic law for the topological entropy.

Proposition 2.4. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $k > 0$ an integer. Then $h_{top}(\phi^k) = k \cdot h_{top}(\phi)$.*

We end with the "continuity" of the topological entropy with respect to inverse limits.

Proposition 2.5. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. If $\{N_i : i \in I\}$ is a directed system of closed normal subgroups of G with $\phi(N_i) = N_i$ and $\bigcap_{i\in I} N_i = \{1\}$, then $G \cong \varprojlim G/N_i$ and $h_{top}(\phi) = \sup_{i\in I} h_{top}(\overline{\phi}_i)$, where $\overline{\phi}_i : G/N_i \to G/N_i$ is the continuous endomorphism induced by ϕ.*

3 The scale function

Following [14, 15], the *scale* of a topological automorphism $\phi : G \to G$ of a totally disconnected locally compact group G is

$$s_G(\phi) = \min\{s(\phi, U) : U \in \mathcal{B}(G)\}$$

(note that [14] deals only with inner automorphisms). We use the notation $s(\phi)$ whenever the group G is clear from the context. Moreover, a subgroup $U \in \mathcal{B}(G)$ is called *minimizing* for ϕ if $s(\phi) = s(\phi, U)$.

As $s(\phi, U) = 1$ precisely when U is ϕ-invariant, one has $s(\phi) = 1$ if and only if G has a ϕ-invariant open compact subgroup. In the non-trivial cases, minimizing subgroups are not always easy to come by. With this motivation, the following approach was adopted in [15]. For $U \in \mathcal{B}(G)$ consider, beyond U_- and U_+, also the subgroups

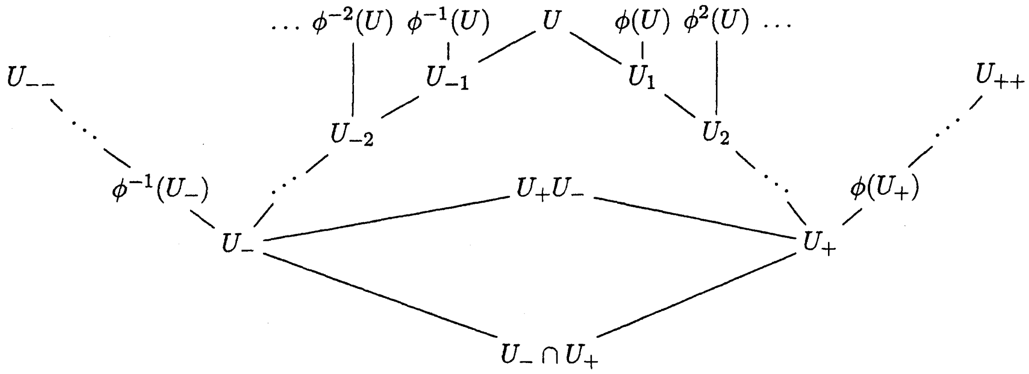
$$U_{++} = \bigcup_{n=0}^{\infty} \phi^n(U_+) \text{ and } U_{--} = \bigcup_{n=0}^{\infty} \phi^{-n}(U_-).$$

When ϕ is not clear from the context, these subgroups are denoted more rigorously by $U_{\phi,++}$ and $U_{\phi,--}$ respectively. Note that $U_{\phi,--} = U_{\phi^{-1},++}$. Following [14],

(a) U is *tidy above* for ϕ if $U = U_+U_-$;

(b) U is *tidy below* for ϕ if U_{++} is closed.

(c) U *tidy* for ϕ if it is tidy above and tidy below for ϕ.



The consequence of the so-called "tidying procedure" given in [15] is the following fundamental theorem showing that the minimizing subgroups are precisely the tidy subgroups.

Theorem 3.1. [15, Theorem 3.1] *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then U is minimizing for ϕ if and only if U is tidy for ϕ. In this case*

$$s(\phi) = [\phi(U_+) : (U_+)].$$

The following properties of the scale function, similar to some extent to those of the topological entropy recalled in §1, can be deduced from Theorem 3.1 (we refer to [2] for detailed proofs).

Proposition 3.2. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Let H be another totally disconnected locally compact group and $\xi : G \to H$ a topological isomorphism. Then $s(\phi) = s(\xi\phi\xi^{-1})$.*

Proposition 3.3. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and H a closed normal subgroup of G such that $\phi(N) = N$, and let $\overline{\phi} : G/H \to G/H$ be the topological automorphism induced by ϕ. Then:*

(a) $s(\phi) \geq s(\phi \restriction_N)$;

(b) $s(\phi) \geq s(\overline{\phi})$.

Proposition 3.4. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $k > 0$ an integer. Then $s(\phi^k) = s(\phi)^k$.*

As far as negative powers are concerned, one obtains the following corollary as a consequence of the "tidying procedure".

Corollary 3.5. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. If $U \in \mathcal{B}(G)$ is a tidy subgroup for ϕ, then it is tidy also for ϕ^{-1} and*

$$s(\phi) = s(\phi^{-1})\Delta(\phi).$$

The following "continuity" with respect to inverse limits was proved for inner automorphisms already in [14].

Proposition 3.6. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. If $\{N_i : i \in I\}$ is a directed system of closed normal subgroups of G with $\phi(N_i) = N_i$ and $\bigcap_{i\in I} N_i = \{1\}$, then $G \cong \varprojlim G/N_i$ and $s(\phi) = \sup_{i\in I} s(\overline{\phi}_i)$, where $\overline{\phi}_i : G/N_i \to G/N_i$ is the continuous endomorphism induced by ϕ.*

Applying Theorem 3.1 it is possible to prove also the following result, which is a weak addition theorem for the scale function.

Proposition 3.7. *Let G and H be two locally compact totally disconnected groups and ϕ, ψ two topological automorphisms of G and H respectively. Then $s(\phi \times \psi) = s(\phi) \cdot s(\psi)$.*

The following property is a p-adic version for the scale function of the celebrated Yuzvinski formula for the topological entropy from [17].

Proposition 3.8. *Let p be a prime and $\phi\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ a topological automorphism. Then $s(\phi) = \prod_{|\lambda|_p>1}|\lambda|_p$, where λ runs over the set of all eigenvalues of ϕ, taken eventually in some extension of $\mathbb{Q}_p$.*

4 The topological entropy compared with the scale function

For all the results in this section, the proofs can be found in [2].

According to (3) and Theorem 1.2, we have

$$h_{top}(\phi) = \sup\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\},$$

while the "tidying procedure" and Theorem 3.1 give

$$\log s(\phi) = \min\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

From these two equalities one obtains the following inequality. We give a more precise result below.

Proposition 4.1. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphisms. Then $h_{top}(\phi) \geq \log s(\phi)$.*

The above inequality can be deduced also from Proposition 1.1 and the formula from [4] showing that

$$\log s(\phi) = \lim_{n\to\infty} \frac{\log[\phi^n(U) : U \cap \phi^n(U)]}{n}, \tag{5}$$

for every $U \in \mathcal{B}(G)$. Indeed, $[\phi^n(U) : U \cap \phi^n(U)] \leq [\phi^n(U) : \phi^n(C_{n+1}(\phi, U))] = [U : C_{n+1}(\phi, U)]$ for every positive integer n.

The next example witnesses that the inequality in Proposition 4.1 can be strict. If K is topological group and $G = K^{\mathbb{Z}}$, the left Bernoulli shift $\sigma : G \to G$ of G is defined by

$$\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}} \tag{6}$$

for every $(x_n)_{n\in\mathbb{Z}} \in G$.

Example 4.2. Let p be a prime and $G = \mathbb{Z}(p^\infty)^{\mathbb{Z}}$. Imposing that $U = \mathbb{Z}(p)^{\mathbb{Z}}$ is open and compact in G, then G is given a locally compact (non-compact) topology. Consider the left shift $\sigma : G \to G$ defined as in (6). Since clearly $\sigma(U) = U$, it follows that

(a) $s(\sigma) = 1$, and

(b) $H_{top}(\sigma, U) = 0$.

On the other hand, if $V = \mathbb{Z}(p)^{-\mathbb{N}_+} \oplus \{0\} \oplus \mathbb{Z}(p)^{\mathbb{N}_+}$, then

(c) $H_{top}(\sigma, V) = \log p$, since $[\sigma(V_+) : V_+] = p$ and in view of Theorem 1.2.

This occurs since V is not tidy for σ. Indeed, $V_+ = \mathbb{Z}(p)^{\mathbb{N}_+}$ and $V_- = \mathbb{Z}(p)^{-\mathbb{N}_+}$, therefore V is tidy above for σ. On the other hand, $V_{++} = \mathbb{Z}(p)^{(-\mathbb{N}_+)} \oplus \{0\} \oplus \mathbb{Z}(p)^{\mathbb{N}_+}$, which is dense in U and so it is not closed, in other words V is not tidy below for ϕ.

Moreover, it is known that

(d) $h_{top}(\sigma \restriction_U) = \log p$.

This can be also computed by means of Theorem 1.2 as in item (c). In fact, every compact open subgroup of U contains one of the form $V_m = \bigoplus_{-\infty}^{-m} \mathbb{Z}(p) \oplus \bigoplus_{-m}^{m}\{0\} \oplus \bigoplus_{m}^{+\infty} \mathbb{Z}(p)$ for some $m \in \mathbb{N}$, and $[\sigma((V_m)_+) : (V_m)_+] = p$ for every $m \in \mathbb{N}$.

Definition 4.3. *Let $\mathfrak{W}$ be the class of locally compact totally disconnected groups G such that $h_{top}(\phi) = \log s(\phi)$ for every topological automorphism $\phi : G \to G$.*

Since $s(\phi) = 1$ for all compact totally disconnected groups G, the compact groups $G \in \mathfrak{W}$ are exactly those with $h_{top}(\phi) = 0$ for every topological automorphism $\phi : G \to G$ (some series of compact abelian groups $G \in \mathfrak{W}$ are built in [8]).

Example 4.4. Let p be a prime.

(a) Consider the left Bernoulli shift $\sigma : G \to G$ of $G = \mathbb{Z}(p)^{\mathbb{Z}}$, defined as in (6). Since G is compact and totally disconnected, $s(\sigma) = 1$; moreover, $h_{top}(\sigma) = \log p > 0$, as noted in Example 4.2(d), so $G \notin \mathfrak{W}$.

(b) The group G provided with the finer group topology having $U = \mathbb{Z}(p)^{\mathbb{N}}$ as un open (compact) subgroup is locally compact and non-compact. It coincides with the underlying additive group of the locally compact field $L = \mathbb{Z}/p\mathbb{Z}((X))$ of Laurent power series over the field $\mathbb{Z}/p\mathbb{Z}$. Now $\sigma : L \to L$ coincides with the multiplication by X^{-1} in the field L and now $\sigma : L \to L$ has $s(\sigma) = p$, so $h_{top}(\sigma) = \log s(\sigma)$.

Example 4.5. From Theorem 3.8, all groups $\mathbb{Q}_p^n \in \mathfrak{W}$. Hence, the underlying additive groups of the locally compact fields of characteristic 0 are in $\mathfrak{W}$.

Following [16], for a totally disconnected locally compact group G and $\phi : G \to G$ a topological automorphisms, we denote by $\mathrm{nub}(\phi)$ the intersection of all subgroups of G tidy for ϕ. The next proposition follows from the fact (due to the local compactness of G) that the tidy subgroups of ϕ form a local base at 1 in G whenever $\mathrm{nub}(\phi) = \{1\}$.

Theorem 4.6. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Then $h_{top}(\phi) = \log s(\phi)$ if and only if $\mathrm{nub}(\phi) = \{1\}$.*

In particular, $G \in \mathfrak{W}$ for every totally disconnected locally compact group G such that $\mathrm{nub}(\phi) = \{1\}$ for every topological automorphism ϕ of G (e.g., the p-adic Lie groups).

Our last theorem concerns the abelian case. Indeed, it uses Pontryagin duality to connect the scale of a topological automorphism ϕ with the scale of its dual $\widehat{\phi}$. We denote by $\widehat{G}$ the Pontryagin dual of a locally compact abelian group G.

Theorem 4.7. *Let $\phi\colon G \to G$ be a topological automorphism automorphism of a totally disconnected locally compact abelian group G, such that $\widehat{G}$ is totally disconnected too. Then $s(\widehat{\phi}) = s(\phi)$.*

This result is inspired by the so-called bridge theorem from [7] connecting, under the same assumptions, the topological entropy with the algebraic entropy by means of Pontryagin duality.

References

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965) 309–319.

[2] F. Berlai, D. Dikranjan, A. Giordano Bruno, *Scale function vs Topological entropy*, Topology Appl. **160** (2013) 2314–2334.

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971) 401–414.

[4] Z. Chatzidakis, E. Hrushovski, *An invariant for difference field extensions*, Ann. Fac. Sci. Toulouse Math. (6) **21** (2012) no. 2, 217–234.

[5] D. Dikranjan, A. Giordano Bruno, *Topological entropy and algebraic entropy for group endomorphisms*, Arhangel'skii A. V., Moiz ud Din Khan; Kocinac L., ed., Proceedings Islamabad ICTA 2011, Cambridge Scientific Publishers 133–214.

[6] D. Dikranjan, A. Giordano Bruno, *Limit free computation of entropy*, Rendiconti Istit. Mat. Univ. Trieste **44** (2012) 1–16.

[7] D. Dikranjan, A. Giordano Bruno, *The Bridge Theorem for totally disconnected LCA groups*, Topology Appl. **169** (2014) no.1, 21–32

[8] D. Dikranjan, M. Sanchis, *Infinitude of Bowen's entropy for groups endomorphisms*, in: Juan Carlos Ferrando and Manolo López Pellicer, eds, Proceeding from the first Meeting in Topology and Functional Analysis, in Elce, Spain (2013), dedicated to J. Kakol's 60-th birthday, Springer Verlag 2014, pp. 139–158.

[9] D. Dikranjan, M. Sanchis, S. Virili, *New and old facts about entropy on uniform spaces and topological groups*, Topology and its Applications **159** (2012) 1916–1942.

[10] A. Giordano Bruno, *Topological entropy for automorphisms of totally disconnected locally compact groups*, Topology Proc. **45** (2015) 175–187.

[11] B. M. Hood, *Topological entropy and uniform spaces*, J. London Math. Soc. **8** (2) (1974) 633–641.

[12] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. **34** (1933) 461–469.

[13] D. van Dantzig, *Studien over topologische Algebra*, Dissertation, Amsterdam 1931.

[14] G. A. Willis, *The structure of totally disconnected locally compact groups*, Math. Ann. **300** (2) (1994) 341–363.

[15] G. A. Willis, *Further properties of the scale function on a totally disconnected group*, J. Algebra **237** (1) (2001) 142–164.

[16] G. A. Willis, *The nub of an automorphism of a totally disconnected locally compact group*, Ergodic Theory Dynam. Systems **34** (2014) no. 4, 1365–1394.

[17] S. A. Yuzvinski, *Calculation of the entropy of a group-endomorphism*, Sibirsk. Mat. Ž. **8** (1967) 230–239.