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Analysis on the Optimal Default Boundaries where a Firm's Cross-ownership of Debts and Equities is Present[1]

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1 Introduction

Blundel-Wingnall (2011) reported that foreign banks' cross-border exposure to sovereign debt of Italy, France and Spain is large and heavily concentrated within EU banks. They also reported that a similar observation can be seen with respect to the cross-border exposure between banks. Therefore we need to consider the cross-holding structures of debts when we evaluate the credit risk of EU countries and banks in EU.

There are several papers for the pricing model under cross-holding of securities. Suzuki (2002) showed a credit pricing model by extending Merton's structural model under firms' cross-holdings of debts and equities within n firms. Fischer (2014) generalized Suzuki(2002) with bond seniority structures and derivatives. However there has been no study that attempted to extend these to an endogenous default model such as Black and Cox (1976) and Leland (1996). In this study, we will extend the structural model in which a firm chooses default to maximize their equity value to that with cross-holdings of securities. The purpose of this paper is to analyze the default decision of firms and the optimal default boundaries when two firms establish cross-holdings of debts and equities.

2 Debt, Equity and Financial Asset

2.1 Basic Assumption

We consider EBIT (earnings before interest and tax) based model proposed by Goldstein, Ju and Leland (2001). We suppose two firms, namely firm i and firm j that produces payout flows specified by

$$\frac{\mathrm{d}D_k(t)}{D_k(t)} = \mu_k^P \mathrm{d}t + \sigma_k \mathrm{d}W_k^P(t), \quad k = i, j \tag{1}$$

under phicial measure $\mathcal{P}$ where μ_k^P, σ_k are constants and where $W_i^P(t)$ and $W_j^P(t)$ are strandard Brownian motion with an instantaneous covariance $\mathrm{Cov}(\mathrm{d}W_i^P, \mathrm{d}W_j^P) = \rho \mathrm{d}t$.

Throughout our analysis, we suppose the risk-free rate r is constant. As in Goldstein, Ju and Leland (2001), the value of claim to the entire payout cash flow is given by

$$Z_k(t) = \frac{D_k(t)}{r - \mu_k}, \quad k = i, j \tag{2}$$

under risk neutral measure $\mathcal{Q}$ assuming some economic conditions about an agent's risk aversion. Then we can show that $D_i(t), D_j(t)$ follow

$$\frac{\mathrm{d}D_k(t)}{D_k(t)} = \mu_k \mathrm{d}t + \sigma_k \mathrm{d}W_k(t), \quad k = i, j, \tag{3}$$

[1] This is an early draft of our paper "Endogenous Default Model with Firms' Cross-holdings of Debts and Equities."

where μ_k are risk adjusted drifts and where $W_i(t)$ and $W_j(t)$ are standard Brownian motion under Q with instantaneous covariance $\mathrm{Cov}(\mathrm{d}W_i, \mathrm{d}W_j) = \rho dt$. We call the claims to entire EBITs flows firms' *business assets* to distinguish the firms' *financial assets* because we will suppose firms can hold debts and equities as their financial assets.

Since $Z_k(t)$ follows the same process with $D_k(t)$ and risk adjusted drift μ_k satisfies (2), business assets follow

$$\frac{\mathrm{d}Z_k(t)}{Z_k(t)} = \left(r - \frac{D_k(t)}{Z_k(t)}\right)\mathrm{d}t + \sigma_k \mathrm{d}W_k(t), \quad k = i, j \tag{4}$$

under Q. Hereafter we suppose $Z_i(t)$ and $Z_j(t)$ are the state variables for the values of the securities issued by firm i and firm j.

2.2 Levered Firm Holding Consol Bond

We begin by considering firms that hold a riskless consol bond as their financial asset and issue a corporate consol bond. Throughout this paper, we ignore corporate tax to simplify the firm's capital structure. We denote the random time of default of firm $k = i, j$ by τ_k. Assume that firm k holds a consol bond with coupon c. Assume also that the stock holder must pay a constant coupon payment c_k to debt holders by issuing new stock and can get the EBIT flow as stock holders and coupon flow from the holding consol bond as long as the firm is solvent. Since the stationarity of the payoffs of the debts and equities implies that the values of securities issued by firm k will be time-independent, the value of equity issued by firm k can be given by

$$q_k^s(Z_k(0)) = E^Q\left[\int_0^{\tau_k} \mathrm{e}^{-rt}(c - c_k)dt + \int_0^{\tau_k} \mathrm{e}^{-rt}D_k(t)dt\right], \quad k = i, j. \tag{5}$$

We assume further that at the time of default, debt holders can get the remaining business assets reduced by proportional default cost $\delta_k, (k = i, j)$ while the remaining financial assets are assumed to be safe. Thus the value of debt is equal to

$$q_k^d(Z_k(0)) = E^Q\left[\int_0^{\tau_k} \mathrm{e}^{-rt}c_k dt + \mathrm{e}^{-r\tau_k}\left((1-\delta_k)Z_k(\tau_k) + \frac{c}{r}\right)\right], \quad k = i, j. \tag{6}$$

Here, (5) and (6) can be given by the closed form expression as follows.

Lemma 1 *Let $q_k^d(Z_k; c), q_k^s(Z_k; c), (k = i, j)$ denote the values of debts and equities when each firm k holds a consol bond with coupon amount c as their financial asset. Assume $c_k > c$ then q_k^d, q_k^s are equal to:*

$$\begin{aligned} q_k^d(Z_k; c) &= \frac{c_k'}{r}\left\{1 - \left(\frac{Z_k}{b_k}\right)^{\gamma_k}\right\} + (1-\delta_k)b_k\left(\frac{Z_k}{b_k}\right)^{\gamma_k}, & (7)\\ q_k^s(Z_k; c) &= Z_k - \frac{c_k'}{r} - \left(b_k - \frac{c_k'}{r}\right)\left(\frac{Z_k}{b_k}\right)^{\gamma_k} & (8)\end{aligned}$$

where

$$b_k = \frac{\gamma_k}{\gamma_k - 1}\frac{c_k'}{r}, \quad c_k' = c_k - c, \tag{9}$$

and where

$$\gamma_k = \frac{1}{2} - \frac{\mu_k}{\sigma_k^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu_k}{\sigma_k^2}\right)^2 + \frac{2r}{\sigma_k^2}} < 0, \quad k = i, j. \tag{10}$$

We omit the proof because it is straightforward.

Remark 1 *The debts and equities prices and the optimal default thresholds can be written by the value of "net financial debt", c'_k/r.*

Remark 2 *If $Z_k > b_k, (k = i, j)$ then the value of debts $q_k^d(Z_k; c)$ and equities $q_k^s(Z_k; c)$ are increasing function of the coupon amount of consol bond c. This can be assured from the value of $\partial q_k^s(Z_k; c)/\partial c$ and $\partial q_k^d(Z_k; c)/\partial c$.*

2.3 Cash flow under Cross-holding of debts and equities

Now suppose that both firm i and j owns debts and equities in each other. Let $\pi_{ij}^d, \pi_{ij}^s \in [0, 1]$ denote the proportions of debt and equity issued by firm j and held by firm i. Let $\pi_{ji}^d, \pi_{ji}^s \in [0, 1]$ denote the proportions of debt and equity issued by firm i and held by firm j. Here any securities issued by firm i and j will be time-independent because both issued debt and holding debt are consol bonds. Therefore, we denote $p_i^d(Z_i, Z_j), p_j^d(Z_i, Z_j)$ the value of debts issued by firm i, j. We denote also by $p_i^s(Z_i, Z_j), p_j^s(Z_i, Z_j)$ the value of equities issued by firms i, j. To understand the cross-holding structures, we state the values of securities from a cash flow standpoint.

First, the cash flow to the equity holder of firm i is given by the sum of following: (i) payment of coupon to their debt holders and receiving a dividend on the issued equity until the time of firm i's default. (ii) receiving a coupon from holding debt issued by firm j until either firm i defaults or firm j defaults and the remaining value of debt at the time of default of firm j. (iii) payment of coupon to debt holders of firm j as the equity holder of firm j's equity and receiving dividend from firm j's equity until either firm i defaults or firm j defaults. Hence, risk-neutral expectation of cash flow to equity holders of firm i at time 0 is given by

$$\begin{aligned} p_i^s(Z_i(0), Z_j(0)) &= E\Big[- \int_0^{\tau_i} \mathrm{e}^{-rt} c_i dt + \int_0^{\tau_i} \mathrm{e}^{-rt} D_i(t) dt \\ &\quad + \int_0^{\tau_i \wedge \tau_j} \mathrm{e}^{-rt} \pi_{ij}^d c_j dt + 1_{\tau_j < \tau_i} \left\{ \mathrm{e}^{-r\tau_j} \pi_{ij}^d p_j^d(Z_i(\tau_j), Z_j(\tau_j)) \right\} \\ &\quad - \int_0^{\tau_i \wedge \tau_j} \mathrm{e}^{-rt} \pi_{ij}^s c_j dt + \int_0^{\tau_i \wedge \tau_j} \mathrm{e}^{-rt} \pi_{ij}^s D_j(t) dt \Big], \end{aligned} \tag{11}$$

where $\tau_i \wedge \tau_j$ denotes the minimum of τ_i and τ_j, and where $D_k(t) = (r - \mu_k) Z_k(t), k = i, j$ from (2).

Second, a cash flow to the debt holder of firm i is given by the sum of followings: (i) receiving a coupon payment until the firm i's default and remaining business assets of firm i at the time of firm i's default, (ii) the value of firm j's debt and equity at the time of firm j's default. Thus the value of debt issued by firm i is expressed by

$$\begin{aligned} p_i^d(Z_i(0), Z_j(0)) &= E^Q \Big[\int_0^{\tau_i} \mathrm{e}^{-rt} c_i dt + (1 - \delta_i) \mathrm{e}^{-r\tau_i} Z_i(\tau_i) \\ &\quad + 1_{\tau_i < \tau_j} \left\{ \mathrm{e}^{-r\tau_i} \pi_{ij}^d p_j^d(Z_i(\tau_i), Z_j(\tau_i)) \right\} + 1_{\tau_i < \tau_j} \left\{ \mathrm{e}^{-r\tau_i} \pi_{ij}^s p_j^s(Z_i(\tau_i), Z_j(\tau_i)) \right\} \Big]. \end{aligned} \tag{12}$$

Note that firm j's equity value $p_j^s(Z_i(0), Z_j(0))$ and debt value $p_j^d(Z_i(0), Z_j(0))$ can be given by replacing j with i in (11) and (12).

2.4 Assumption of Default and Liquidation

We will set up how firms' defaults happen under the cross-holdings of securities and how to divide the share of securities at the time of a firm's default. We suppose that simultaneous defaults may happen between firm i and firm j. We assume the possible scenario of default and the distribution of remaining assets with default costs as follows.

Suppose both firm i and firm j are in solvent at time 0. Assume both firm i and firm j independently choose their default boundaries on (Z_i, Z_j) in the market that are frictionless and free of informational asymmetries. Assume that firm i and j shall default in the following possible five scenarios where $(x_{ij}^{20}, y_{ij}^{20})$, (x_i^{20}, y_i^{20}), (x_j^{20}, y_j^{20}), (x_i^{21}, y_i^{21}) and (x_j^{21}, y_j^{21}) denote corresponding default boundaries on $(Z_i, Z_j) \in \boldsymbol{R}_+^2$. These will be the only set of default sequences under the assumption that only the equity holders can choose their firms' default and the assumption that the equity holders' objectives are the maximization of their equity values.

First we assume the default sequence and liquidation scheme on simultaneous default.

Assumption 1 (Simultaneous Default) *We assume that if $\{\tau_i = \tau_j\}$ happens, debt issued by firm i and j are liquidated at the same time and holding debts issued by counter firm under cross-holdings of debts are valued as defaulted debts at liquidation. Further we assume three type of simultaneous defaults as follows:*

Case 1. *$\{t = \tau_i = \tau_j\}$ happens on $(x_{ij}^{20}, y_{ij}^{20})$. Both firm i and j go into default at the same time on $(x_{ij}^{20}, y_{ij}^{20})$. The $(x_{ij}^{20}, y_{ij}^{20})$ is the optimal boundaries for both firm i and j's equity holders.*

Case 2. *$\{t = \tau_i = \tau_j\}$ happens on (x_i^{20}, y_i^{20}). Firm i chooses going into default so as to maximize their equity value on (x_i^{20}, y_i^{20}) where firm j immediately chooses default because it is too late for the maximization of firm j's equity value.*

Case 3. *$\{t = \tau_j = \tau_i\}$ happens on (x_j^{20}, y_j^{20}). As well as above, firm j chooses going into default on (x_j^{20}, y_j^{20}) and it causes firm i's default. The default on (x_j^{20}, y_j^{20}) is not the optimal for firm i.*

Second we assume the default sequence and liquidation scheme on non-simultaneous default. We assume two types of liquidation schemes as follows.

Assumption 2 (Non-simultaneous Default) *We assume that if $\{\tau_j < \tau_i\}$ or $\{\tau_i < \tau_j\}$ happens, then the debt issued by the defaulting firm is liquidated and the solvent firm invests the cash into a risk-free consol bond. Further we assume two types of non-simultaneous default as follows:*

Case 4. *$\{t = \tau_i < \tau_j\}$ happens on (x_i^{21}, y_i^{21}). Firm i chooses going into default on a boundary where firm j chooses being solvent to maximize their equity value.*

Case 5. *$\{t = \tau_j < \tau_i\}$ happens on (x_j^{21}, y_j^{21}). In the same manner with Case 4, firm j chooses going into default on (x_j^{21}, y_j^{21}) while firm i is solvent.*

2.5 Debt Values on Default Boundaries

For the purpose of analysis on the default boundaries, we need to investigate how the value of securities are distributed to each security holder at the time of default. First, we show the case when simultaneous default happens that is Case 1, 2, or 3 happen.

Lemma 2 *Suppose that* $t = \tau_i = \tau_j$ *happens on* (x^{20}, y^{20}) *where we define*

$$(x^{20}, y^{20}) = (x_{ij}^{20}, y_{ij}^{20}) \cup (x_i^{20}, y_i^{20}) \cup (x_j^{20}, y_j^{20}). \tag{13}$$

Then security values issued by firm i *and* j *are given as*

$$p_i^s(x^{20}, y^{20}) = 0, \qquad p_i^d(x^{20}, y^{20}) = \frac{(1-\delta_i)x^{20} + \pi_{ij}^d(1-\delta_j)y^{20}}{1 - \pi_{ij}^d \pi_{ji}^d} \tag{14}$$

$$p_j^s(x^{20}, y^{20}) = 0, \qquad p_j^d(x^{20}, y^{20}) = \frac{(1-\delta_j)y^{20} + \pi_{ji}^d(1-\delta_i)x^{20}}{1 - \pi_{ji}^d \pi_{ij}^d}. \tag{15}$$

Proof: From Assumption 1, the values of securities issued by firm i and j must satisfy followings:

$$p_i^d(x^{20}, y^{20}) = (1-\delta_i)x^{20} + \pi_{ij}^s p_j^s(x^{20}, y^{20}) + \pi_{ij}^d p_j^d(x^{20}, y^{20}), \tag{16}$$

$$p_j^d(x^{20}, y^{20}) = (1-\delta_j)y^{20} + \pi_{ji}^s p_i^s(x^{20}, y^{20}) + \pi_{ji}^d p_i^d(x^{20}, y^{20}). \tag{17}$$

It follows that debt values can be given by solving the system of equation (16) and (17). □

Note that Suzuki (2002) showed the payoff functions under Merton (1976)'s model with cross-holdings of debts and equities. Lemma 2 can be recognized as a slight extension of Suzuki (2002) with positive default costs.

Second we show the method to derive the security values when non-simultaneous default happens. Suppose Case 4. So suppose $t = \tau_i < \tau_j$ happens on (x_i^{21}, y_i^{21}). Since firm j liquidate firm i's debt and invest the cash into a riskless consol bond with a coupon $r\pi_{ji}^d p_i^d(x_i^{21}, y_i^{21})$ from Assumption 2, the value of securities issued by firm i and j must satisfy the following equation:

$$p_i^d(x_i^{21}, y_i^{21}) = (1-\delta_i)x_i^{21} + \pi_{ij}^s p_j^s(x_i^{21}, y_i^{21}) + \pi_{ij}^d p_j^d(x_i^{21}, y_i^{21}), \tag{18}$$

$$p_i^s(x_i^{21}, y_i^{21}) = 0, \tag{19}$$

$$p_j^s(x_i^{21}, y_i^{21}) = q_j^s(y_i^{21}; r\pi_{ji}^d p_i^d(x_i^{21}, y_i^{21})), \tag{20}$$

$$p_j^d(x_i^{21}, y_i^{21}) = q_j^d(y_i^{21}; r\pi_{ji}^d p_i^d(x_i^{21}, y_i^{21})). \tag{21}$$

Here we are interested in the value $p_i^d(x_i^{21}, y_i^{21})$ for a given (x_i^{21}, y_i^{21}). So we can find that $p_i^d(x_i^{21}, y_i^{21})$ is given as a root of implicit function

$$p_i^d(x_i^{21}, y_i^{21}) = (1-\delta_i)x_j^{21} + \pi_{ij}^s q_j^s(y_i^{21}; r\pi_{ji}^d p_i^d(x_i^{21}, y_i^{21})) + \pi_{ij}^d q_j^d(y_i^{21}; r\pi_{ji}^d p_i^d(x_i^{21}, y_i^{21})). \tag{22}$$

Finally we can show that the non-linear equation (22) with respect to $p_i^d(x_i^{21}, y_i^{21})$ has a unique solution. We state the results including Case 5 due to the symmetry of i and j.

Proposition 1 *The debt value* $w_i = p_i^d(x_i^{21}, y_i^{21})$ *and* $w_j = p_j^d(x_j^{21}, y_j^{21})$ *are given by a fixed point of the functions defined respectively by*

$$f(w_i) = (1-\delta_i)x_i^{21} + \pi_{ij}^s q_j^s(y_i^{21}; r\pi_{ji}^d w_i) + \pi_{ij}^d q_j^d(y_i^{21}; r\pi_{ji}^d w_i) \tag{23}$$

$$f(w_j) = (1-\delta_i)y_i^{21} + \pi_{ji}^s q_i^s(x_j^{21}; r\pi_{ij}^d w_j) + \pi_{ji}^d q_i^d(x_j^{21}; r\pi_{ij}^d w_j). \tag{24}$$

And firm j*'s optimal default boundary after firm* i*'s default is given by*

$$b_j(x_i^{21}, y_i^{21}) = \frac{\gamma_j}{\gamma_j - 1}\left(\frac{c_j}{r} - \pi_{ji}^d w_i\right). \tag{25}$$

Also firm i*'s optimal default boundary after firm* j*'s default is given by*

$$b_i(x_j^{21}, y_j^{21}) = \frac{\gamma_i}{\gamma_i - 1}\left(\frac{c_i}{r} - \pi_{ij}^d w_j\right). \tag{26}$$

Corollary 1 *If default cannot happen before bond maturity as in Merton (1976), we can show that the debt values at maturity date are not unique if we suppose the default cost on a firm's assets (See Appendix). However if default can happen before the maturity so as to maximize firms' equity values as in Leland (1996), the debt values at default are unique even though with a positive default cost.*

3 Analysis on Optimal Default Boundaries

We will present analytical implicit solutions for optimal default boundaries under cross-holding of debts and equities between two firms. We also derive debt values on the boundaries. We follow the approach developed by Adkins and Paxon (2011) that solves an optimal replacement decision under a two-factor model.

3.1 Cross-holdings of Debts and Equities

Since the security values $p_i^s(Z_i, Z_j), p_j^s(Z_i, Z_j)$ are time-independent, their infinitesimal generator $\mathcal{A}$ is given as follows

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\sigma_i^2 Z_i^2 \frac{\partial^2}{\partial Z_i^2} + \frac{1}{2}\sigma_j^2 Z_j^2 \frac{\partial^2}{\partial Z_j^2} + \rho\sigma_i\sigma_j Z_i Z_j \frac{\partial^2}{\partial Z_i \partial Z_j} \\ &\quad + \mu_i Z_i \frac{\partial}{\partial Z_i} + \mu_j Z_j \frac{\partial}{\partial Z_j} - r.\end{aligned}$$

The cash flow of equity issued by firm i is given in (11). We can also obtain the cash flow of equity issued by firm j by replacing j with i in (11). It follows that values of equities issued by firms i and j must satisfy the following partial differential equations:

$$\mathcal{A}p_i^s(Z_i, Z_j) + (r - \mu_i)Z_i - c_i + \pi_{ij}^s\left((r - \mu_j)Z_j - c_j\right) + \pi_{ij}^d c_j = 0, \tag{27}$$
$$\mathcal{A}p_j^s(Z_i, Z_j) + (r - \mu_j)Z_j - c_j + \pi_{ji}^s\left((r - \mu_i)Z_i - c_i\right) + \pi_{ji}^d c_i = 0. \tag{28}$$

The homogeneous part of the generic function of each security is given by the from $A_k Z_i^{\beta_k} Z_j^{\eta_k}$ ($k = i, j$) where A_k is an undetermined coefficient. Moreover, β_k and η_k are given by roots of the following characteristic equation:

$$Q(\beta, \eta) = \frac{1}{2}\sigma_i^2\beta(\beta - 1) + \frac{1}{2}\sigma_j^2\eta(\eta - 1) + \rho\sigma_i\sigma_j\beta\eta + \mu_i\beta + \mu_j\eta - r = 0. \tag{29}$$

Details about such analysis can be seen in Adkins and Paxon (2011).

Since the equity holder doesn't have any option except the option to default, the homogeneous part of each security takes a single term. It follows that the values of securities take forms as follows:

$$p_i^s(Z_i, Z_j) = A_i Z_i^{\beta_i} Z_j^{\eta_i} + Z_i - \frac{c_i}{r} + \pi_{ij}^s\left(Z_j - \frac{c_j}{r}\right) + \pi_{ij}^d\frac{c_j}{r}, \tag{30}$$
$$p_j^s(Z_i, Z_j) = A_j Z_i^{\beta_j} Z_j^{\eta_j} + Z_j - \frac{c_j}{r} + \pi_{ji}^s\left(Z_i - \frac{c_i}{r}\right) + \pi_{ji}^d\frac{c_i}{r}, \tag{31}$$

We assumed that firms cannot have any other options except the option to default. Then it is natural to suppose $\beta_i, \beta_j, \eta_i, \eta_j \leq 0$. Note also that these 4 coefficients are not necessarily constants. Let us examine this point in more detail.

First, suppose (Z_i, Z_j) hit the boundary

$$(x_i, y_i) = (x_i^{20}, y_i^{20}) \cup (x_i^{21}, y_i^{21}).$$

Now (x_i, y_i) are given by the firm i's maximization of their equity value. Then the necessary conditions for the optimal default boundaries (x_i, y_i) are smooth-pasting conditions:

$$\begin{aligned} \frac{\partial p_i^s}{\partial Z_i}(x_i, y_i) &= \beta_i A_i x_i^{\beta_i - 1} y_i^{\eta_i} + 1 = 0, & (32) \\ \frac{\partial p_i^s}{\partial Z_j}(x_i, y_i) &= \eta_i A_i x_i^{\beta_i} y_i^{\eta_i - 1} + \pi_{ij}^s = 0, & (33) \end{aligned}$$

The corresponding value-matching condition comes from (19), so the equation

$$p_i^s(x_i, y_i) = A_i x_i^{\beta_i} y_i^{\eta_i} + x_i - \frac{c_i}{r} + \pi_{ij}^s \left(y_i - \frac{c_i}{r} \right) + \pi_{ij}^d \frac{c_j}{r} = 0 \tag{34}$$

must be satisfied.

Here we have five unknowns: $A_i, \beta_i, \eta_i, x_i, y_i$ but the model consists of four equations: (29), (32), (33) and (34). Since our aim is to determine the default boundary (x_i, y_i), we will derive the boundary by the function of y_i. Then four equations: (29), (32), (33) and (34) can be seen as a system of equation with respect to $(A_i, \beta_i, \eta_i, x_i)$. Here it is useful to derive the relation between x_i and y_i. So we have

$$x_i = \frac{\beta_i}{\beta_i - 1} \left(\frac{c_i}{r} - \pi_{ij}^s \left(y_i - \frac{c_j}{r} \right) - \pi_{ij}^d \frac{c_j}{r} \right) \tag{35}$$

from (32) and (34).

Remark 3 *We can see that the default threshold (35) has the same component with (9). However the values of β_i and η_i depends on y_i. This is different from (9) with constant γ_k in that firms hold a riskless consol bond.*

Proposition 2 *Under the non-arbitrage condition $\mu_i < r$ and $\mu_j < r$, the system of non-linear equations (29), (32), (33) and (34) has the unique solutions $\beta_i < 0$, $\eta_i < 0$, $x_i > 0$ and $A_i > 0$ for a given $y_i > 0$.*

Second, suppose (Z_i, Z_j) hit the boundary

$$(x_j, y_j) = (x_j^{20}, y_j^{20}) \cup (x_j^{21}, y_j^{21}).$$

This is a symmetrical case to the boundary (x_i, y_i). Therefore it is easy to derive the same relations. However we state all of that because we need corresponding equations to study the simultaneous default. So the smooth-pasting condition and value-matching condition are given as follows:

$$\begin{aligned} \frac{\partial p_j^s}{\partial Z_i}(x_j, y_j) &= \beta_j A_j x_j^{\beta_j - 1} y_j^{\eta_j} + \pi_{ji}^s = 0, & (36) \\ \frac{\partial p_j^s}{\partial Z_j}(x_j, y_j) &= \eta_j A_j x_j^{\beta_j} y_j^{\eta_j - 1} + 1 = 0, & (37) \end{aligned}$$

$$p_j^s(x_j, y_j) = A_j x_j^{\beta_j} y_j^{\eta_j} + y_j - \frac{c_j}{r} + \pi_{ji}^s \left(x_j - \frac{c_i}{r} \right) + \pi_{ji}^d \frac{c_i}{r} = 0. \tag{38}$$

From (37) and (38), we have

$$y_j = \frac{\eta_j}{\eta_j - 1} \left(\frac{c_j}{r} - \pi_{ji}^s \left(x_j - \frac{c_i}{r} \right) - \pi_{ji}^d \frac{c_i}{r} \right) \tag{39}$$

Note that from the similarity of firm i and j and Proposition 2, we can find the set of values $\beta_j < 0$, $\eta_j < 0$, $y_j > 0$ and $A_j > 0$ for a given x_j.

Third, suppose again (Z_i, Z_j) hit the boundary (x_i, y_i) before hitting (x_j, y_j). We assumed that firm j chooses to be solvent on (x_i^{21}, y_i^{21}) while firm j chooses default on (x_i^{20}, y_i^{20}). Since (x_i^{21}, y_i^{21}) and (x_i^{20}, y_i^{20}) are on the same line represented by (35) which is continuous function with respect to (x_i, y_i), there can exists a threshold value y_i^* which divides y_i^{21} and y_i^{20}. We can specify (x_i^{20}, y_i^{20}) and (x_i^{21}, y_i^{21}) by Proposition 2 and the following result.

Proposition 3 *The optimal default boundaries for Case 2 and Case 4 can be given by*

$$(x_i^{21}, y_i^{21}) = \{(x_i, y_i) \mid y_i > y_i^*\}, \tag{40}$$
$$(x_i^{20}, y_i^{20}) = \{(x_i, y_i) \mid y_i \leq y_i^*\}, \tag{41}$$

where x_i is given by Proposition 2 for given y_i and where

$$y_i^* = \frac{\gamma_j \left\{ \pi_{ji}^d (1-\delta_i) \dfrac{\beta_i}{\beta_i - 1} \dfrac{c_i}{r} - \left(1 + \pi_{ji}^d \pi_{ij}^d (1-\delta_i) \dfrac{\beta_i}{\beta_i - 1} \right) \dfrac{c_j}{r} \right\}}{1 - \gamma_j \left\{ 1 - \pi_{ji}^d \left(\pi_{ij}^s (1-\delta_i) \dfrac{\beta_i}{\beta_i - 1} - \pi_{ij}^d (1-\delta_j) \right) \right\}}. \tag{42}$$

Note that the optimal default boundaries for Case 3 and Case 5 are given in the Appendix.

Fourth, suppose (Z_i, Z_j) hit the boundary $(x_{ij}^{20}, y_{ij}^{20})$ before hitting (x_i, y_i) and (x_j, y_j). Then, the firm i maximizes their equity value, so (35) is satisfied. Further the firm j maximizes their equity value, so (39) is satisfied. Therefore the optimal boundary for Case 1 is given by a crossing point of (35) and (i39) as follows.

Proposition 4 *The optimal boundary for Case 1 is given by*

$$x_{ij}^{20} = -\beta_i \left\{ \frac{c_i}{r} \left(1 - \eta_j + \eta_j \pi_{ij}^s (\pi_{ji}^s - \pi_{ji}^d) \right) + \frac{c_j}{r} \left(\pi_{ij}^s - (1-\eta_j) \pi_{ij}^d \right) \right\} \Big/ \Gamma, \tag{43}$$
$$y_{ij}^{20} = -\eta_j \left\{ \frac{c_i}{r} \left(\pi_{ji}^s - (1-\beta_i) \pi_{ji}^d \right) + \frac{c_j}{r} \left(1 - \beta_i + \beta_i \pi_{ji}^s (\pi_{ij}^s - \pi_{ij}^d) \right) \right\} \Big/ \Gamma, \tag{44}$$

where

$$\Gamma = (\beta_i - 1)(\eta_j - 1) - \beta_i \eta_j \pi_{ji}^s \pi_{ij}^s$$

and where $(\beta_i < 0, \eta_i < 0)$ are the root of the system of equations (29), (33), (32) and (34) and where $(\beta_j < 0, \eta_j < 0)$ are the root of the system of equations (29), (36), (37) and (38).

Theorem 1 *If $\delta_i > 0$ then $y_i^* > y_{ij}^{20}$. Then $(x_i^{20}, y_i^{20}) \neq \phi$. It follows that $\mathbb{P}(\tau_i = \tau_j) > 0$.*

Remark 4 *For given default boundaries, we can get the debts values by the Lemma 2 and Proposition 1*

3.2 Cross Holdings of Debts Only

As a special case, we suppose $\pi_{ij}^s = \pi_{ji}^s = 0$. Then $\eta_i = 0$ from (33). It follows that $\beta_i = \gamma_i$ by substituting $\eta_i = 0$ to (29). Also, since $\beta_j = 0$ from (36), $\eta_j = \gamma_j$. Finally, we can derive a closed form expression of the optimal default boundaries for the case of cross-holdings of debts only, not equities, as follows:

Theorem 2 *If $\pi_{ij}^s = \pi_{ji}^s = 0$ then the optimal default boundaries for Case 2 and Case 4 are given by*

$$(x_i^{21}, y_i^{21}) = \{(x_i, y_i) \mid y_i > y_i^*\}, \tag{45}$$

$$(x_i^{20}, y_i^{20}) = \{(x_i, y_i) \mid y_i \le y_i^*\}, \tag{46}$$

where

$$x_i = \frac{\gamma_i}{\gamma_i - 1}\left(\frac{c_i}{r} - \pi_{ij}^d \frac{c_j}{r}\right), \quad y_i \ge 0 \tag{47}$$

and where

$$y_i^* = \frac{\gamma_j \left\{\pi_{ji}^d (1-\delta_i) \frac{\gamma_i}{\gamma_i - 1} \frac{c_i}{r} - \left(1 + \pi_{ji}^d \pi_{ij}^d (1-\delta_i) \frac{\gamma_i}{\gamma_i - 1}\right) \frac{c_j}{r}\right\}}{1 - \gamma_j \left\{1 + \pi_{ji}^d \pi_{ij}^d (1 - \delta_j)\right\}}. \tag{48}$$

Also, the optimal default boundaries for Case 3 and Case 5 are given by

$$(x_j^{21}, y_j^{21}) = \{(x_j, y_j) \mid x_j > x_j^*\}, \tag{49}$$

$$(x_j^{20}, y_j^{20}) = \{(x_j, y_j) \mid x_j \le x_j^*\}, \tag{50}$$

where

$$x_j \ge 0, \quad y_j = \frac{\gamma_j}{\gamma_j - 1}\left(\frac{c_j}{r} - \pi_{ji}^d \frac{c_i}{r}\right). \tag{51}$$

and where

$$x_j^* = \frac{\gamma_i \left\{\pi_{ij}^d (1-\delta_j) \frac{\gamma_j}{\gamma_j - 1} \frac{c_j}{r} - \left(1 + \pi_{ij}^d \pi_{ji}^d (1-\delta_j) \frac{\gamma_j}{\gamma_j - 1}\right) \frac{c_i}{r}\right\}}{1 - \gamma_i \left\{1 + \pi_{ij}^d \pi_{ji}^d (1 - \delta_i)\right\}}. \tag{52}$$

Furthermore, the optimal default boundaries for Case 1 is given by a point:

$$(x_{ij}^{20}, y_{ij}^{20}) = (x_i, y_j). \tag{53}$$

4 Conclusion

We study the optimal default boundaries when firms establish cross-holdings of debts and equities and when firms can chooses the time of default so as to maximize their equity values. We showed the optimal default boundaries with cross-holdings of debts and equities can be given by the unique solution of a system of equations. We also showed that the optimal default boundaries with cross-holdings of debts only, not equities, are given by closed form expressions. In addition to these, we showed that simultaneous defaults can happen with positive probability even though we suppose that firms' reference assets follow geometric Brownian motions. We also showed that firms' payouts cannot be unique when default can happen only at the bond's maturity with positive default costs. In contrast to this, we showed that firms' payouts on default can be unique even though with a positive default cost when firms choose the time of default so as maximize their equity values.

Appendix

The optimal default boundaries for Case 3 and Case 5

It can be given by

$$(x_j^{21}, y_j^{21}) = \{(x_j, y_j) \mid x_j > x_j^*\}, \tag{54}$$
$$(x_j^{20}, y_j^{20}) = \{(x_j, y_j) \mid x_j \leq x_j^*\}, \tag{55}$$

where

$$x_j^* = \frac{\gamma_i \left\{ \pi_{ij}^d (1-\delta_j) \frac{\eta_j}{\eta_j - 1} \frac{c_j}{r} - \left(1 + \pi_{ij}^d \pi_{ji}^d (1-\delta_j) \frac{\eta_j}{\eta_j - 1}\right) \frac{c_i}{r} \right\}}{1 - \gamma_i \left\{ 1 - \pi_{ij}^d \left(\pi_{ji}^s (1-\delta_j) \frac{\eta_j}{\eta_j - 1} - \pi_{ji}^d (1-\delta_i) \right) \right\}}. \tag{56}$$

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