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EQUIDISTRIBUTION OF HECKE OPERATORS ON SPECIAL CYCLES ON COMPACT SHIMURA VARIETIES

MASAO TSUZUKI (都築正男 ; 上智大学理工学部)

1. Introduction

Let F be a number field with adele ring $\mathbb{A}_F$. The set of places of F is denoted by Σ^F; it is a union of archimedean places Σ_∞^F and the non-archimedean ones Σ_{fin}^F. Let G be a connected reductive algebraic group over F and H a closed F-subgroup of G. We suppose the center of G is F-anisotropic for simplicity. We endow the adele groups $G(\mathbb{A}_F)$ and $H(\mathbb{A}_F)$ the Tamagawa measures. For an automorphic form φ on $G(F)\backslash G(\mathbb{A}_F)$, the H-period integral of φ is defined by

$$\mathcal{P}_H(\varphi)=\int_{H(F)\backslash H(\mathbb{A}_F)}\varphi(h)\,\mathrm{d}h$$

as long as the integral converges absolutely. An automorphic cuspidal representation $\pi\subset L^2(G(F)\backslash G(\mathbb{A}_F))$ of $G(\mathbb{A}_F)$ is said to be H-distinguished if it contains a vector $\varphi\in\pi$ such that $\mathcal{P}_H(\varphi)\neq 0$. The notion of H-distinction has a local counterpart. Suppose $v\in\Sigma_{\mathrm{fin}}^F$; an irreducible admissible representation π_v of the totally disconnected group $G(F_v)$ is called to be $H(F_v)$-distinguished if

$$\mathrm{Hom}_{H(F_v)}(\pi_v,\mathbf{1}_{H(F_v)})\neq 0.$$

Over an archimedean place $v\in\Sigma_\infty^F$, to define the corresponding notion, we should work on the category of smooth Frechet representations; an irreducible admissible representation π_v of the reductive Lie group $G(F_v)$ is defined to be $H(F_v)$-distinguished if its Casselmann-Wallach globalization admits a continuous non-zero $H(F_v)$-invariant distribution vector. The set of equivalence classes of irreducible unitary $H(F_v)$-distinguished representations of $G(F_v)$ is denoted by $\mathbb{X}_v$. It is not difficult to see that the global H-distinction of a cuspidal representation implies the $H(F_v)$-distinction of its local components at all places v. Precisely, if $\pi\cong\otimes_v\pi_v$ is an irreducible cuspidal representation, then for every place $v\in\Sigma^F$, the v-component π_v of π is $H(F_v)$-distinguished. The converse is more subtle and seems very difficult to establish in general if it is true. Here, we propose a weaker version of the converse statement in a slightly vague way.

Problem : Let S be a finite subset of Σ^F and $\{J_v\}_{v\in S}$ a collection of "good" subsets $J_v\subset\mathbb{X}_v$. Is there exists an irreducible cuspidal $\pi\cong\otimes_v\pi_v$ such that (i) π is H-distinguished (ii) (the class of) π_v belongs to J_v for all $v\in S$.

In this note, for unitary groups over CM-fields, we consider this problem in a more rigorous formulation and report an affirmative answer in a special case. Though our setting is rather restrictive, it includes an interesting case which yields a geometric consequence about certain equidistribution phenomenon for Satake parameters of automorphic representations contributing to the space of special cycles on compact unitary Shimura varieties. No proof is included.

2. Unitary groups and their representations

2.1. Let E be a CM-field and F the maximal totally real subfield in E. We suppose $[F:\mathbb{Q}] > 1$. The quadratic idele class character of $F^\times$ corresponding to the extension E/F by the class field theory is denoted by $\varepsilon_{E/F}$. The maximal order of E and F are denoted by $\mathfrak{o}_E$ and $\mathfrak{o}_F$, respectively. Let V be a finite m-dimensional E-vector space and $\mathbf{h}: V\times V \to E$ a non-degenerate hermitian form on V. For any $v \in \Sigma^F$, set $E_v = E\otimes_F F_v$ and $V_v = V\otimes_F F_v$. Let $\mathbf{h}_v$ denote the hermitian form induced on the E_v-module V_v by extension of scalars. We suppose that there exists an archimedean place v_1 such that $\mathbf{h}_{v_1}$ is of signature (n^+, n^-) with $n^+ \geqslant n^- \geqslant 2$ and $\mathbf{h}_v$ is positive definite at all $v \in \Sigma_\infty^F - \{v_1\}$. In particular, $m = n^+ + n^- \geqslant 4$. Let $G = U(\mathbf{h})$ be the unitary group of the hermitian space $(V, \mathbf{h})$, which we view as an F-algebraic group. From our assumption, $G(F_{v_1}) \cong U(n^+, n^-)$ and $G(F_v) = U(m)$ for $v \in \Sigma_\infty^F - \{v_1\}$. Since $\#\Sigma_F^\infty = [F:\mathbb{Q}] \geqslant 2$, this implies that G is F-anisotropic.

Let $\ell \in V$ be such that $\mathbf{h}_v[\ell] := \mathbf{h}_v(\ell,\ell)$ is a positive number of $F_v \cong \mathbb{R}$ for all $v \in \Sigma_\infty^F$; if this is the case, we say that ℓ is totally positive. Let H be the stabilizer of the subspace $E\ell$; we have $H \cong H_0 \times E^1$, where $H_0 = U(\mathbf{h}|\ell^\perp)$ is the unitary group of the hermitan space $\ell^\perp$, the orthogonal complement of ℓ in V, and E^1 is the torus of norm one elements in $E^\times$. From the assumptions, we have $H_0(F_{v_1}) \cong U(n^+ - 1, n^-)$ and $H_0(F_v) \cong U(m-1)$.

We fix an $\mathfrak{o}_E$-lattice $\mathfrak{L}$ in V (i.e., $\mathfrak{L}$ is a free $\mathfrak{o}_F$-submodule in V satisfying $\mathfrak{o}_E\,\mathfrak{L} \subset \mathfrak{L}$) such that $\ell \in \mathfrak{L}$ and it is maximal in the sense of [5].

2.2. $H(F_{v_1})$-distinguished representations. For a positive integer d such that $\sigma(d) := m - 1 - 2(n^- - d) > 0$, there corresponds an irreducible unitary representation δ_d of $G(F_{v_1}) \cong U(n^-, n^+)$ with the following properties.

(i) δ_d contains a $U(n^+)\times U(n^-)$-type τ_d with highest weight $[d, 0, \cdots, 0, -d; 0, \ldots, 0]$. (The $U(n^-)$-factor acts trivially.)
(ii) the Casimir operator of $G(F_{v_1})$ acts on δ_d with the scalar $\sigma(d)^2 - (m-1)^2$.
(iii) There exists a bounded $G(F_{v_1})$-intertwining operator from δ_d to $L^2(H(F_{v_1})\backslash G(F_{v_1}))$.

The representations δ_d are $H(F_{v_1})$-distinguished and comprise a family of unitary representations of $G(F_{v_1})$ called the $H(F_{v_1})$-relative discrete series representations of the symmetric space $H(F_{v_1})\backslash G(F_{v_1})$ ([1]).

2.3. $H(F_v)$-distinguished representation over a good place. We say that a finite place $v \in \Sigma_{\mathrm{fin}}^F$ is good if the following conditions are satisfied.

(a) $2 \in \mathfrak{o}_{F,v}^\times$.
(b) E_v is an unramified field extension of F_v, or E_v is isomorphic to $F_v \oplus F_v$.
(c) $\mathfrak{L}_v := \mathfrak{L}\otimes_{\mathfrak{o}_F} \mathfrak{o}_{F,v}$ is self-dual.
(d) $\mathbf{h}_v[\ell] \in \mathfrak{o}_{F,v}^\times$.

We note that almost all the finite places of F are good in this sense. Let v be a good place. Then $\mathbf{K}_v = \mathrm{GL}(\mathfrak{L}_v) \cap G(F_v)$ is a good maximal compact subgroup of $G(F_v)$. Let

q_v denote the cardinality of the residue field of F at v, ϖ_v a prime element of the integer ring $\mathfrak{o}_{F,v}$ of F_v and $|\,|_v$ the normalized valuation of F_v. Let $\mathcal{C}^0(F_v)$ be the light cone of V_v:

$$\mathcal{C}^0(F_v) = \{x \in V_v - \{0\} |\, \mathbf{h}_v[x] = 0\}.$$

For $s \in \mathbb{C}/2\pi(\log q_v)^{-1}\sqrt{-1}\mathbb{Z}$, letting the group $G(F_v)$ act on the $\mathbb{C}$-vector space of smooth functions $f : \mathcal{C}^0(F_v) \to \mathbb{C}$ such that

$$f(tx) = |\mathrm{N}_{E_v/F_v}(t)|_v^{s+(m-1)/2} f(x) \quad \text{for all } x \in \mathcal{C}^0(F_v) \text{ and } t \in E_v^\times,,$$

we define a smooth $G(F_v)$-module, denoted by $I_v(s)$. Now we are going to state several results, which does not seem an immediate consequence of works by Sakellaridis ([3],[4]) because the group $G(F_v)$ (when E_v is a field) is not split but should follow from an extension of his works to quasi-split groups. Anyway, we can prove what we need directly by a computational way due to the relatively simple structure of our symmetric space $H(F_v)\backslash G(F_v)$.

Proposition 1. *If s belongs to the set*

$$\mathbb{X}_v^{0+} := \sqrt{-1}[0, \pi(\log q_v)^{-1}] \cup (0, \nu_0/2)$$

where $\nu_0 \in \{0,1\}$ is the parity of $m-1$, then the $G(F_v)$-module $I_v(s)$ is irreducible, unitarizable, $\mathbf{K}_v$-spherical and $H(F_v)$-distinguished. Conversely, if π_v is an irreducible unitarizable $\mathbf{K}_v$-spherical and $H(F_v)$-distinguished representation of $G(F_v)$, then π_v is isomorphic to $I_v(s)$ with a unique $s \in \mathbb{X}_v^{0+}$.

Proposition 2. *For $s \in \mathbb{X}_v^0 := \sqrt{-1}[0, \pi(\log q_v)^{-1}]$, there exists a unique $H(F_v)$-invariant linear functional $\Lambda_v^0 : I_v(s) \to \mathbb{C}$ such that $\Lambda_v^0(f_v^0) = 1$, where $f_v^0 \in I_v(s)$ is the $\mathbf{K}_v$-invariant vector such that the restriction of f_v^0 to $\mathcal{C}^0(F_v) \cap \mathcal{L}_v$ is identically 1.*

Using the vector $f_v^0 \in I_v(s)^{\mathbf{K}_v}$ and the functional $\Lambda_v^0 \in \mathrm{Hom}_{H(F_v)}(I_v(s), \mathbb{C})$ in the previous proposition, we define the spherical function corresponding to $I_v(s)$ by setting

$$\Omega_v^{(s)}(g) = \langle \Lambda_v^0, I_v(s; g) f_v^0 \rangle, \quad g \in G(F_v).$$

Here $I_v(s;g)$ denotes the action of $g \in G(F_v)$ on $I_v(s)$. Obviously, the function $\Omega_v^{(s)}$ on $G(F_v)$ is left $H(F_v)$-invariant and right $\mathbf{K}_v$-invariant. From the structural theory of self-dual lattices, there exists a system of vectors $e_{v,j}$, $e_{v,j}$ $(1 \leqslant j \leqslant n_v)$ with $\mathbf{h}_v[e_{v,j}] = \mathbf{h}_v[e'_{v,j}] = 0$ and $\mathbf{h}_v(e_{v,j}, e'_{v,i}) = \delta_{ij}$ such that

$$\text{(2.1)} \quad \mathcal{L}_v = \mathfrak{o}_{E,v}e_{1,v} \oplus \mathfrak{o}_{E,v}e_{v,2} \oplus \cdots \mathfrak{o}_{E,v}e_{v,n_v} \oplus \mathcal{M}_v \oplus \mathfrak{o}_{E,v}e'_{v,n_v} \oplus \cdots \mathfrak{o}_{E,v}e'_{v,2} \oplus \mathfrak{o}_{E,v}e'_{v,1}$$

with $\mathcal{M}_v = \{0\}$ if m is even and $\mathcal{M}_v = \mathfrak{o}_{E,v}f_v$, $\mathbf{h}_v[f_v] \in \mathfrak{o}_{F,v}^\times$ if m is odd. Moreover, we may take (2.1) so that $\ell = a_v e_{1,v} + e'_{1,v}$ with some $a_v \in \mathfrak{o}_{F,v}$. By realizing G as a matrix group by (2.1), set $[\varpi_v^{-l}] = \mathrm{diag}(\bar{\varpi}_v^{-l}, \mathbf{1}_{m-2}, \varpi_v^l)$, where $\bar{\varpi}_v$ is the image of ϖ_v by the non trivial automorphism of E_v/F_v. Then we have the disjoint decomposition

$$G(F_v) = \bigcup_{l=0}^{\infty} H(F_v)\,[\varpi_v^{-l}]\mathbf{K}_v.$$

(*cf.* [2, Proposition 3.9] if E_v is a field.) Let $\tilde{v}$ be a place of E lying above v and $q_{\tilde{v}}$ the cardinality of the residue field of E at $\tilde{v}$. Let $\zeta_{E,v}(s)$ and $L_v(s, \varepsilon_{E/F})$ be the local v-factors of the Dedekind zeta function $\zeta_E(s)$ and the Hecke L-function $L(s, \varepsilon_{E/F})$ both viewed as Euler products over Σ^F, respectively. We define a smooth function $\Psi_v^{(s)}$ on $G(F_v)$ by requiring that it is left $H(F_v)$-invariant and right $\mathbf{K}_v$-invariant and satisfies

$$\Psi_v^{(s)}([\varpi_v^{-l}]) = q_v^{-s}\zeta_{E,v}(s+(m-1)/2)\, q_{\tilde{v}}^{-l(s+(m-1)/2)}, \quad l \in \mathbb{N}.$$

Theorem 3. *Let $s \in \mathbb{X}_v^0$.*

$$\Omega_v^{(s)}(g) = \frac{\zeta_{E,v}(-s+(m-1)/2)^{-1}\zeta_{E,v}(s+(m-1)/2)^{-1}}{Q_v(q_v^s - \varepsilon_{E/F}(\varpi_v)^m q_v^{-s})}\{-\Psi_v^{(s)}(g) + \Psi_v^{(-s)}(g)\}, \quad g \in G_v.$$

Here

$$Q_v = 1 - \varepsilon_{E/F}(\varpi_v)^{m-1}\, q_v^{-(m-1)}.$$

Proposition 4. *For any function $f : H(F_v)\backslash G(F_v)/\mathbf{K}_v \to \mathbb{C}$ with finite support, we define its spherical Fourier transform by setting*

$$\hat{f}(s) = \int_{H(F_v)\backslash G(F_v)} f(g)\, \Omega_v^{(s)}(g)\, \mathrm{d}g.$$

Then, we have the inversion formula

$$\int_{\mathbb{X}_v^0} \hat{f}(s)\, \mathrm{d}\mu_v^H(s) = f(1),$$

where

$$\mathrm{d}\mu_v^H(iy) = \frac{Q_v}{\pi}\left|\frac{L_v(2iy, \varepsilon_{E/F}^m)}{\zeta_{E,v}(iy+(m-1)/2)}\right|^2 \log q_v\, \mathrm{d}y.$$

2.4. H-distinguished automorphic representations. For any smooth $\mathbb{C}$-valued function φ on $G(F)\backslash G(\mathbb{A}_F)$, we define its H-period integral by

$$\mathcal{P}_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(h)\, |\omega_H|_{\mathbb{A}}(h),$$

where $|\omega_H|_{\mathbb{A}}$ is the Tamagawa measure on $H_{\mathbb{A}}$ (defined from an F-rational invariant gauge form ω_H). A subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F))$ is called to be an automorphic representation of $G(\mathbb{A}_F)$. An automorphic representation π, acting on an irreducible subspace $V_\pi \subset L^2(G(F)\backslash G(\mathbb{A}_F))$, is said to be H-distinguished if $\mathcal{P}_H(\varphi) \neq 0$ for some $\varphi \in V_\pi$. Since our H contains the center of G, an H-distinguished π has the trivial central character. For an integral ideal $\mathfrak{n}$ in E and for a positive integer d such that $\sigma(d) > 0$, let $\Pi^H(\mathfrak{n}, d)$ be the set of all the automorphic representations $\pi \cong \otimes_v \pi_v$ such that

(i) π is H-distinguished.
(ii) For each $v \in \Sigma_{\mathrm{fin}}^F$, π_v contains a non zero vector invariant by $\mathcal{U}_v(\mathfrak{n})$, the kernel of the reduction homomorphism $\mathbf{K}_v \to \mathrm{GL}(\mathcal{L}_v/\mathfrak{n}\mathcal{L}_v)$.
(iii) $\pi_{v_1} \cong \delta_d$.
(iv) $\pi_v \cong \mathbf{1}_{G(F_v)}$ for all $v \in \Sigma_\infty^F - \{v_1\}$.

Let $\pi \in \Pi^H(\mathfrak{n}, d)$. Let S be a finite set of good places relatively prime to $\mathfrak{n}$. Then for each $v \in S$, the v-component π_v is an $H(F_v)$-distinguished and $\mathbf{K}_v$-spherical irreducible unitary representation of $G(F_v)$. Thus, by Proposition 1, there exists a unique $\nu_v \in \mathbb{X}_v^0$ such that $\pi_v \cong I_v(\nu_v)$. Define the spectral parameter of π at S to be the point

$$\nu_S(\pi) = \{\nu_v\}_{v \in S}$$

in the product space

$$\mathbb{X}_S^0 = \prod_{v \in S} \mathbb{X}_v^0 = \prod_{v \in S} \sqrt{-1}[0, \pi(\log q_v)^{-1}].$$

We endow this space with the product topology of the Euclidean topology on the intervals. Let $\mu_S^H = \otimes_{v \in S} \mu_v^H$ be the product measure of μ_v^H's (defined in Proposition 4).

3. Main Results

Let E/F, $(V, \mathbf{h})$, G, ℓ, H, and $\mathfrak{L}$ be as in 2.1; we keep all the assumptions made there. Let $\mathfrak{n}$ be an integral ideal of E and d a positive integer such that $\sigma(d) > 0$. For $\pi \in \Pi^H(\mathfrak{n}, d)$, we set

$$\mathbb{P}^H(\mathfrak{n}, d; \pi) = \sum_{\varphi \in \mathcal{B}} |\mathcal{P}_H(\varphi)|^2$$

with $\mathcal{B}$ an orthonormal basis in $V_\pi[\tau_d]^{\mathcal{U}(\mathfrak{n})}$, the space of $\mathcal{U}(\mathfrak{n}) = \prod_{v \in \Sigma_{\mathrm{fin}}^F} \mathcal{U}_v(\mathfrak{n})$-fixed vectors in the τ_d-isotypic component of V_π. (By Harish-Chandra's finite dimensionality theorem on automoprphic forms, $\mathcal{B}$ is a finite set.)

Theorem 5. *Let S be a finite set of good places. Let $\{\mathfrak{n}_k\}$ be a sequence of integral ideals of E such that $\lim_{k \to \infty} \mathrm{N}_{E/\mathbb{Q}}(\mathfrak{n}_k) = \infty$ and any prime divisor of $\mathfrak{n}_k$ is away from S and is good. Then, for any Borel subset $\mathbb{J} \subset \mathbb{X}_S^0$ with $\mu_S^H(\partial \mathbb{J}) = 0$, we have*

$$\lim_{k \to \infty} \frac{\sum_{\substack{\pi \in \Pi^H(\mathfrak{n}_k, d) \\ \nu_S(\pi) \in \mathbb{J}}} \mathbb{P}^H(\mathfrak{n}_k, d; \pi)}{\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{n}_k)^m \mathrm{N}_{F/\mathbb{Q}}(\mathrm{tr}_{E/F}(\mathfrak{n}_k))^{-1}} = C \frac{\Gamma(\sigma(d) + m - 1)}{\Gamma(\sigma(d))} \mu_S^H(\mathbb{J}),$$

where C is an explicit positive constant which depends on E/F, $\mathfrak{L}$ and $\mathbf{h}$ but is independent of d and $\mathbb{J}$.

The next corollary partially answers the question raised in the introduction.

Corollary 6. *Let d be a positive integer such that $\sigma(d) > 0$. Let S be a finite set of good places. Then for a given Borel set $\mathbb{J} \subset \mathbb{X}_S^0$ such that $\mu_S^H(\partial \mathbb{J}) = 0$, we have an automorphic representation $\pi \cong \otimes_v \pi_v$ with the following properties:*

(i) *π is H-distinguished.*
(ii) *$\pi_{v_1} \cong \delta_d$, and $\pi_v \cong \mathbf{1}_{G(F_v)}$ for all $v \in \Sigma_\infty^F - \{v_1\}$.*
(iii) *There exists $\{\nu_v\}_{v \in S} \in \mathbb{J}$ such that $\pi_v \cong I_v(\nu_v)$ for all $v \in S$.*

3.1. Application to cycle geometry on a unitary Shimura variety. Let D be the set of all complex n^--dimensional subspaces $Z \subset V_{v_1}$ such that $\mathbf{h}_{v_1}$ is negative definite on Z. When viewed as a subset of the complex Grassmannian manifold of $V_{v_1} \cong \mathbb{C}^m$ on which $G(F_{v_1})$ acts naturally, D is an open $G(F_{v_1})$-orbit. For any open compact subgroup $\mathcal{U} \subset G(\mathbb{A}_{F,\mathrm{fin}})$, the group $G(F)$ acts on the product space $G(\mathbb{A}_{F,\mathrm{fin}})/\mathcal{U} \times D$ by the diagonal action. If $\mathcal{U}$ is neat, then, by passing to the quotient, we obtain a compact n^-n^+-dimensional complex manifold

$$X^{\mathcal{U}}(G, D) = G(F) \backslash [(G(\mathbb{A}_{F,\mathrm{fin}})/\mathcal{U}) \times D]$$

which is a finite disjoint union of locally symmetric manifolds $\Gamma_i \backslash D$ with cocompact arithmetic subgroups $\Gamma_i \subset G(F_{v_1})$. Let $\ell \in \mathfrak{L}$ and H be as above. Set

$$D_\ell = \{Z \in D |\, \mathbf{h}_{v_1}(Z, \ell) = \{0\}\}.$$

Then D_ℓ is an $H(F_{v_1})$-orbit and the inclusion $D_\ell \hookrightarrow D$ is a holomorphic embedding. For a neat open compact subgroup $\mathcal{U} \subset G(\mathbb{A}_{F,\text{fin}})$, consider the quotient space

$$X_\ell^{\mathcal{U}} = H(F)\backslash[(H(\mathbb{A}_{F,\text{fin}})/\mathcal{U} \cap H(\mathbb{A})) \times D_\ell]$$

together with the natural map

$$j : X_\ell^{\mathcal{U}} \longrightarrow X^{\mathcal{U}}(G, D). \tag{3.1}$$

The coset space $X_\ell^{\mathcal{U}}$ acquires a natural structure of complex manifold and the map j becomes a holomorphic map of complex manifolds with finite fibers. We have $\dim_{\mathbb{C}} X_\ell^{\mathcal{U}} = n^-(n^+ - 1)$, and thus (3.1) yields a chomomology class

$$\mathfrak{C}_\ell^{\mathcal{U}} \in H^{n^-,n^-}(X^{\mathcal{U}}(G, D), \mathbb{C})$$

such that

$$\mathfrak{C}_\ell^{\mathcal{U}} \cup [\alpha] = \int_{X_\ell^{\mathcal{U}}} j^*\alpha \quad \text{for all } [\alpha] \in H^{2n^-(n^+-1)}(X^{\mathcal{U}}(G, D), \mathbb{C}).$$

We fix a base point $Z_0 \in D_\ell$ and let K_{Z_0} denote the stabilizer of Z_0 in $G(F_{v_1})$. Let $\mathfrak{g}_{v_1}$ be the complexified Lie algebra of $G(F_{v_1})$. Then we have the Matsushima-Murakami decomposition

$$H^\bullet(X^{\mathcal{U}}(G, D), \mathbb{C}) = \bigoplus_\pi H^\bullet(\mathfrak{g}_{v_1}, K_{Z_0}; (\pi_{v_1})_{K_{Z_0}}) \otimes \pi_{\text{fin}}^{\mathcal{U}} \tag{3.2}$$

where π runs through all the automorphic representations of $G(\mathbb{A}_F)$ and $(\pi_{v_1})_{K_{Z_0}}$ denotes the K_{Z_0}-finite vectors. From now on, by choosing a $G(F_v)$-invariant Kaehler structure on D once and for all and putting the induced Kaehler form on D_ℓ, we make $X^{\mathcal{U}}(G, D)$ and $X_\ell^{\mathcal{U}}$ Kaehler manifolds. Thus we can speak about the primitive cohomology classes and the primitive decomposition of a general cohomology class of $X^{\mathcal{U}}(G, D)$ ([7]). Let $\mathfrak{n}$ be an integral ideal of E such that $\mathcal{U}(\mathfrak{n})$ is neat. By (3.2) and by invoking a result of [6], the primitive part of the class $\mathfrak{C}_\ell^{\mathcal{U}(\mathfrak{n})}$ has the decomposition

$$(\mathfrak{C}_\ell^{\mathcal{U}(\mathfrak{n})})_{\text{prim}} = \bigoplus_{\pi \in \Pi^H(\mathfrak{n}, n^-)} \mathfrak{C}_\ell^{\mathcal{U}}(\pi),$$

where only the representations in $\Pi^H(\mathfrak{n}, n^-)$ contributes to the sum. The integral $\int_{X^{\mathcal{U}}(G,D)} \alpha \wedge *\bar{\beta}$ for $\mathbb{C}$-valued differential forms induces a hermitian inner product $([\alpha]|[\beta])$ on the deRham cohomology group with trivial coefficients. As usual, the associated norm will be denoted by $||[\alpha]||$.

Theorem 7. *Let S be a finite set of good places. Let $\{\mathfrak{n}_k\}$ be a sequence of integral ideals of E as in the Theorem 5. Let $\mathbb{J} \subset \mathbb{X}_S^0$ be a Borel subset such that $\mu_S^H(\partial\mathbb{J}) = 0$. Then,*

$$\lim_{k \to \infty} \frac{\sum_{\substack{\pi \in \Pi^H(\mathfrak{n}_k, n^-) \\ \nu_S(\pi) \in \mathbb{J}}} ||\mathfrak{C}_\ell^{\mathcal{U}}(\pi)||^2}{\sum_{\pi \in \Pi^H(\mathfrak{n}_k, n^-)} ||\mathfrak{C}_\ell^{\mathcal{U}}(\pi)||^2} = \frac{\mu_S^0(\mathbb{J})}{\mu_S^0(\mathbb{X}_S^0)}.$$

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Masao TSUZUKI
Department of Science and Technology, Sophia University, Kioi-cho 7-1 Chiyoda-ku Tokyo, 102-8554, Japan
E-mail : m-tsuduk@sophia.ac.jp