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Spherical functions on the space of p-adic unitary hermitian matrices, the case of odd size

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§0 Introduction

We have been interested in the harmonic analysis on p-adic homogeneous spaces based on spherical functions. We have considered the space of p-adic unitary hermitian matrices of even size in [HK], and in the present article we will report that of odd size. For details, please see the full version [HK2]. The results can be formulated in parallel, though the groups acting the spaces have different root structures, C_n for even and BC_n for odd. The spaces have a natural close relation to the theory of automorphic functions and classical theory of sesquilinear forms (e.g. [H3], [HS]).

We fix an unramified quadratic extension k'/k of $\mathfrak{p}$-adic field k such that $2 \notin \mathfrak{p}$, and consider hermitian and unitary matrices with respect to k'/k, and denote by a^* the conjugate transpose of $a \in M_{mn}(k')$. Let π be a prime element of k and q the cardinality of the residue class field $\mathcal{O}_k/(\pi)$ and we normalize the absolute value on k by $|\pi| = q^{-1}$.

Denote by $j_m \in GL_m(k)$ the matrix whose all anti-diagonal entries are 1 and others are 0. Set

$$\begin{aligned} &G = U(j_m) = \{g \in GL_m(k') \mid gj_mg^* = j_m\}, \quad K = G(\mathcal{O}_{k'}) \\ &X = \{x \in X \mid x^* = x,\ \Phi_{xj_m}(t) = \Phi_{j_m}(t)\}, \\ &g \cdot x = gxg^*, \quad (g \in G,\ x \in X), \end{aligned}$$

where $\Phi_y(t)$ is the characteristic polynomial of matrix y. We note that X is a single $G(\overline{k})$-orbit over the algebraic closure $\overline{k}$ of k. Set

$$n = \left[\frac{m}{2}\right].$$

According to the parity of m, G has the root structure of type C_n for even m and type BC_n for odd m. It is known in general that the spherical functions on various p-adic groups Γ can be expressed in terms of the specialization of Hall-Littlewood polynomials of the corresponding root structure of Γ (cf. [M2, §10], also [Car, Theorem 4.4]). For the present space X, the main term of spherical functions can be written by using Hall-Littlewood polynomials of type C_n with different specialization according to the parity of m (cf. Theorem 3 below).

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Let us note the results of even size in [HK] and odd size simultaneously, so that one can compare the results.

Theorem 1 (1) *A set of complete representatives of $K\backslash X$ can be taken as*

$$\left\{ x_\lambda \mid \lambda \in \Lambda_n^+ \right\}, \tag{0.1}$$

where

$$x_\lambda = \begin{cases} \mathrm{Diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \ldots, \pi^{-\lambda_1}) & \text{if } m = 2n \\ \mathrm{Diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}, 1, \pi^{-\lambda_n}, \ldots, \pi^{-\lambda_1}) & \text{if } m = 2n+1, \end{cases}$$
$$\Lambda_n^+ = \left\{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \right\},$$

(2) *There are precisely two G-orbits in X represented by $x_0 = x_{\mathbf{0}} = 1_m$ and $x_1 = x_{(1,0,\ldots,0)}$.*

The proof of Cartan decomposition for odd size needs a more delicate calculation than that for even size. If k has even residual characteristic, there are some K-orbits without any diagonal matrix besides the above types, independent of the parity of the size.

A spherical function on X is a K-invariant function on X which is a common eigenfunction with respect to the convolutive action of the Hecke algebra $\mathcal{H}(G,K)$, and a typical one is constructed by Poisson transform from relative invariants of a parabolic subgroup. We take the Borel subgroup B consisting of upper triangular matrices in G. For $x \in X$ and $s \in \mathbb{C}^n$, we consider the following integral

$$\omega(x;s) = \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i}\, dk, \tag{0.2}$$

where $d_i(y)$ is the determinant of the lower right i by i block of y, $1 \leq i \leq n$. Then the right hand side of (0.2) is absolutely convergent for $\mathrm{Re}(s_i) \geq 0$, $1 \leq i \leq n$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$. Since $d_i(x)$'s are relative B-invariants on X such that

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)) \quad (p \in B,\ x \in X,\ 1 \leq i \leq n),$$

we see $\omega(x;s)$ is a spherical function on X which satisfies

$$f * \omega(x;s) = \lambda_s(f)\omega(x;s), \quad f \in \mathcal{H}(G,K)$$
$$\lambda_s(f) = \int_B f(p) \prod_{i=1}^n |\psi_i(p)|^{-s_i}\, \delta(p) dp,$$

where dp is the left invariant measure on B with modulus character δ. We introduce the new variable $z \in \mathbb{C}^n$ related to s by

$$s_i = -z_i + z_{i+1} - 1 + \frac{\pi\sqrt{-1}}{\log q}, \quad 1 \leq i \leq n-1$$
$$s_n = \begin{cases} -z_n - \frac{1}{2} + \frac{\pi\sqrt{-1}}{\log q} & \text{if } m = 2n \\ -z_n - 1 + \frac{\pi\sqrt{-1}}{2\log q} & \text{if } m = 2n+1, \end{cases} \tag{0.3}$$

and denote $\omega(x;z)=\omega(x;s)$ and $\lambda_s=\lambda_z$.

The Weyl group W of G relative to B acts on rational characters of B, hence on z and s also. The group W is generated by S_n which acts on z by permutation of indices and by τ such that $\tau(z_1,\dots,z_n)=(z_1,\dots,z_{n-1},-z_n)$. We will give the functional equation of $\omega(x;z)$ with respect to W. To describe the results we prepare some notation. We set

$$\Sigma^+=\Sigma_s^+\sqcup\Sigma_\ell^+,$$
$$\Sigma_s^+=\{\,e_i+e_j,\ e_i-e_j\mid 1\le i<j\le n\,\},\quad \Sigma_\ell^+=\{\,2e_i\mid 1\le i\le n\,\},$$

where $e_i\in\mathbb{Z}^n$ is the i-th unit vector, and define a pairing

$$\langle\ ,\ \rangle:\mathbb{Z}^n\times\mathbb{C}^n\longrightarrow\mathbb{C},\ \langle\alpha,\,z\rangle=\sum_{i=1}^n\alpha_i z_i.$$

Theorem 2 (1) *For any $\sigma\in W$, one has*

$$\omega(x;z)=\prod_\alpha\frac{1-q^{\langle\alpha,\,z\rangle-1}}{q^{\langle\alpha,\,z\rangle}-q^{-1}}\cdot\omega(x;\sigma(z)),$$

where α runs over the set $\{\,\alpha\in\Sigma_s^+\mid-\sigma(\alpha)\in\Sigma^+\}$ for $m=2n$ and $\{\,\alpha\in\Sigma^+\mid-\sigma(\alpha)\in\Sigma^+\}$ for $m=2n+1$.
(2) *The function $G(z)\cdot\omega(x;z)$ is holomorphic and W-invariant, hence belongs to $\mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^W$, where*

$$G(z)=\prod_\alpha\frac{1+q^{\langle\alpha,\,z\rangle}}{1-q^{\langle\alpha,\,z\rangle-1}},$$

and α runs over the set Σ_s^+ for $m=2n$ and Σ^+ for $m=2n+1$.

As for the explicit formula of $\omega(x;z)$ it suffices to give for each x_λ by Theorem 1 (1).

Theorem 3 *(Explicit formula) For each $\lambda\in\Lambda_n^+$, one has*

$$\omega(x_\lambda;z)=\frac{c_n}{G(z)}\cdot q^{\langle\lambda,\,z_0\rangle}\cdot Q_\lambda(z;t),$$

where $G(z)$ is given in Theorem 2, z_0 is the value in z-variable corresponding to $s=\mathbf{0}$,

$$c_n=\begin{cases}\dfrac{(1-q^{-2})^n}{w_m(-q^{-1})} & \text{if } m=2n\\[2mm] \dfrac{(1+q^{-1})(1-q^{-2})^n}{w_m(-q^{-1})} & \text{if } m=2n+1,\end{cases}\qquad w_m(t)=\prod_{i=1}^m(1-t^i),$$

$$Q_\lambda(z;t)=\sum_{\sigma\in W}\sigma\left(q^{-\langle\lambda,\,z\rangle}c(z;t)\right),\quad c(z;t)=\prod_{\alpha\in\Sigma^+}\frac{1-t_\alpha q^{\langle\alpha,\,z\rangle}}{1-q^{\langle\alpha,\,z\rangle}}$$

$$t_\alpha=\begin{cases}t_s & \text{if } \alpha\in\Sigma_s^+\\ t_\ell & \text{if } \alpha\in\Sigma_\ell^+,\end{cases}\quad t_s=-q^{-1},\quad t_\ell=\begin{cases}q^{-1} & \text{if } m=2n\\ -q^{-2} & \text{if } m=2n+1.\end{cases}$$

We see $Q_\lambda(z;t)$ belongs to $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$ by Theorem 2. It is known that $Q_\lambda(z;t) = W_\lambda(t)P_\lambda(z;t)$ with Hall-Littlewood polynomial $P_\lambda(z;t)$ and Poincaré polynomial $W_\lambda(t)$ and the set $\{P_\lambda(z;t) \mid \lambda \in \Lambda_n^+\}$ forms an orthogonal $\mathbb{C}$-basis for $\mathcal{R}$ for each $t_\alpha \in \mathbb{R}, |t_\alpha| < 1$ (cf. [HK, Appendix C]). As is written as above, we need the different specialization for $Q_\lambda(z;t)$ according to the parity of m, and $G(z)$ is different also.

In particular, we have

$$\omega(x_0;z) = \frac{(1-q^{-1})^n w_n(-q^{-1}) w_{m'}(-q^{-1})}{w_m(-q^{-1})} \cdot G(z)^{-1}, \qquad m' = \left[\tfrac{m+1}{2}\right] \tag{0.4}$$

and we may modify the spherical function $\omega(x;z)$ as

$$\Psi(x;z) = \frac{\omega(x;z)}{\omega(x_0;z)} \in \mathcal{R}. \tag{0.5}$$

We define the spherical Fourier transform on the Schwartz space $\mathcal{S}(K\backslash X)$ as follows

$$\widehat{\ }: \mathcal{S}(K\backslash X) \longrightarrow \mathcal{R}, \ \varphi \longmapsto \widehat{\varphi}(z) = \int_X \varphi(x)\Psi(x;z)dx$$

where dx is a G-invariant measure on X.

Theorem 4 (1) *The above spherical transform is an $\mathcal{H}(G,K)$-module isomorphism and $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 2^n.*
(2) *All the spherical Fourier functions on X are parametrized by $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\right)^n/W$ through λ_z, and the set $\left\{\Psi(x;z+u) \;\middle|\; u \in \{0, \frac{\pi\sqrt{-1}}{\log q}\}^n\right\}$ forms a $\mathbb{C}$-basis of spherical functions corresponding to z.*
(3) *(Plancherel formula) Set a measure $d\mu(z)$ on $\mathfrak{a}^* = \left\{\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q}\mathbb{Z}\right)\right\}^n$ by*

$$d\mu(z) = \frac{1}{2^n n!} \cdot \frac{w_n(-q^{-1})w_{m'}(-q^{-1})}{(1+q^{-1})^{m'}} \cdot \frac{1}{|c(z;t)|^2}dz, \qquad m' = \left[\tfrac{m+1}{2}\right]$$

where dz is the Haar measure on $\mathfrak{a}^*$. By an explicitly given normalization of dx depending on the parity of m, one has

$$\int_X \varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*} \widehat{\varphi}(z)\overline{\widehat{\psi}(z)}d\mu(z) \quad (\varphi, \psi \in \mathcal{S}(K\backslash X)).$$

(4) *(Inversion formula) For any $\varphi \in \mathcal{S}(K\backslash X)$, one has*

$$\varphi(x) = \int_{\mathfrak{a}^*} \widehat{\varphi}(z)\Psi(x;z)d\mu(z), \quad x \in X.$$

§1 The Space X of unitary hermitian matrices

Let k' be an unramified quadratic extension of a $\mathfrak{p}$-adic field k of odd residual characteristic and consider hermitian and unitary matrices with respect to k'/k, and denote by a^* the conjugate transpose of $a \in M_{mn}(k')$. Let π be a prime element of k and q the cardinality of the residue class field $\mathcal{O}_k/(\pi)$ and we normalize the absolute value on k by $|\pi| = q^{-1}$ and denote by $v_\pi(\ )$ the additive value on k. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which $k' = k(\sqrt{\epsilon})$.

We consider the unitary group

$$G = G_n = \left\{ g \in GL_{2n+1}(k') \mid g^* j_{2n+1} g = j_{2n+1} \right\}, \qquad j_{2n+1} = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \in M_{2n+1},$$

and the space X of unitary hermitian matrices in G

$$X = X_n = \left\{ x \in G \mid x^* = x,\ \Phi_{xj_{2n+1}}(t) = (t^2-1)^n(t-1) \right\}, \tag{1.1}$$

where $\Phi_y(t)$ is the characteristic polynomial of the matrix y. It should be noted that (1.1) implies $\det x = 1$. We note that X is a single $G(\overline{k})$-orbit containing 1_{2n+1} over the algebraic closure $\overline{k}$ of k ([HK, Appendix A]). The group G acts on X by

$$g \cdot x = gxg^* = gxj_{2n+1}g^{-1}j_{2n+1}, \quad g \in G,\ x \in X.$$

We fix a maximal compact subgroup K of G by

$$K = K_n = G \cap M_{2n+1}(\mathcal{O}_{k'}),$$

(cf. [Sa, §9]), and take a Borel subgroup B of G, which consists of all the upper triangular matrices in G and is given by

$$B = \left\{ \begin{pmatrix} A & & \\ & u & \\ & & j_nA^{*-1}j_n \end{pmatrix} \begin{pmatrix} 1_n & \beta & C \\ & 1 & -\beta^* j_n \\ & & 1_n \end{pmatrix} \;\middle|\; \begin{array}{l} A \in B_n, \beta \in (k')^n, u \in \mathcal{O}_{k'}^1, C \in M_n(k') \\ \beta\beta^* + Cj_n + j_nC^* = 0_n \end{array} \right\}, \tag{1.2}$$

where B_n is the set of all the upper triangular matrices in $GL_n(k')$, $\mathcal{O}_{k'}^1 = \left\{ u \in \mathcal{O}_{k'}^\times \mid N(u) = 1 \right\}$. Here and hereafter empty entries in matrices should be understood as 0 and N as the norm map $N_{k'/k}$. The group G satisfies the Iwasawa decomposition $G = BK = KB$.

In this section, we give the K-orbit decomposition and the G-orbit decomposition of the space X.

Theorem 1.1 *The K-orbit decomposition of X_n is given as follows:*

$$X_n = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda, \tag{1.3}$$

where

$$\Lambda_n^+ = \left\{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \right\},$$
$$x_\lambda = \mathrm{Diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}, 1, \pi^{-\lambda_n}, \cdots, \pi^{-\lambda_1}).$$

We recall the case of unramified hermitian matrices. The group $GL_m(k')$ acts on the set $\mathcal{H}_m(k') = \{x \in GL_m(k') \mid x^* = x\}$ by $g \cdot x = gxg^*$, and it is known (cf. [Ja])

$$\mathcal{H}_m(k') = \bigsqcup_{\lambda \in \Lambda_m} GL_m(\mathcal{O}_{k'}) \cdot \pi^{\mu} \tag{1.4}$$

where

$$\Lambda_m = \{\mu \in \mathbb{Z}^m \mid \mu_1 \geq \cdots \geq \mu_m\}, \quad \pi^{\mu} = \mathrm{Diag}(\pi^{\mu_1}, \ldots, \pi^{\mu_m}).$$

Hence, we see that $K \cdot x_\lambda \cap K \cdot x_\mu = \emptyset$ if $\lambda \neq \mu$ in Λ_n^+. Moreover, since $N(\mathcal{O}_{k'}^\times) = \mathcal{O}_k^\times$, we see that any diagonal $x \in X$ is reduced to some x_λ, $\lambda \in \Lambda_n^+$ by the action of K.

The strategy of the proof of Theorem 1.1 is the same as in the even case [HK]; general n cases are reduced to the case $n = 1$. However the case $n = 1$, the size of matrices is 3, is more complicated and technical than the even case.

Remark 1.2 *If k has even residual characteristic, then there are some K-orbits without any diagonal matrix. In fact, the following matrices are contained in X_n and can not be diagonalized by the action of K, for $n = r + s$, $s > 0$:*

$$\begin{pmatrix} 1_r & & & & \\ & & & -j_{\frac{s}{2}} & \\ & & j_{s+1} & & \\ & -j_{\frac{s}{2}} & & & \\ & & & & 1_r \end{pmatrix} \ (\textit{if } 2 \mid s), \qquad \begin{pmatrix} 1_r & & & & \\ & & & j_{\frac{s+1}{2}} & \\ & & -j_s & & \\ & j_{\frac{s+1}{2}} & & & \\ & & & & 1_r \end{pmatrix} \ (\textit{if } 2 \not| \, s).$$

Before giving the G-orbit decomposition of X_n, we recall the case of unramified hermitian matrices. It is known that there are precisely two $GL_m(k')$-orbits in $\mathcal{H}_m(k')$ for $m \geq 1$:

$$\begin{aligned} \mathcal{H}_m(k') &= GL_m(k') \cdot 1_m \sqcup GL_m(k') \cdot \pi^{(1,0,\ldots,0)}, \\ &= \Bigg(\bigsqcup_{\substack{\mu \in \Lambda_m \\ |\mu| \text{ is even}}} GL_m(\mathcal{O}_{k'}) \cdot \pi^{\mu} \Bigg) \sqcup \Bigg(\bigsqcup_{\substack{\mu \in \Lambda_m \\ |\mu| \text{ is odd}}} GL_m(\mathcal{O}_{k'}) \cdot \pi^{\mu} \Bigg), \end{aligned} \tag{1.5}$$

where $|\mu| = \sum_{i=1}^m \mu_i$.

Theorem 1.3 *There are precisely two G-orbits in X_n :*

$$G \cdot x_0 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| \text{ is even}}} K \cdot x_\lambda, \quad G \cdot x_1 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| \text{ is odd}}} K \cdot x_\lambda.$$

where $|\lambda| = \sum_{i=1}^n \lambda_i$, $x_0 = 1_{2n+1}$ and $x_1 = \mathrm{Diag}(\pi, 1, \ldots, 1, \pi^{-1})$.

§2 Spherical function $\omega(x;s)$ on X

We introduce a spherical function $\omega(x;s)$ on X by Poisson transform from relative B-invariants. For a matrix $g \in G$, denote by $d_i(g)$ the determinant of lower right i by i block of g. Then $d_i(x)$, $1 \le i \le n$ are relative B-invariants on X associated with rational characters ψ_i of B, where

$$d_i(p \cdot x) = \psi_i(p)d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)), \quad (x \in X,\ p \in B). \tag{2.1}$$

We set

$$X^{op} = \{x \in X \mid d_i(x) \neq 0,\ 1 \le i \le n\}. \tag{2.2}$$

then $X^{op}(\overline{k})$ is a Zariski open $B(\overline{k})$-orbit, where $\overline{k}$ is the algebraic closure of k. For $x \in X$ and $s \in \mathbb{C}^n$, we consider the integral

$$\omega(x;s) = \int_K |\mathbf{d}(k \cdot x)|^s\, dk, \quad |\mathbf{d}(y)|^s = \prod_{i=1}^{n} |d_i(y)|^{s_i}, \tag{2.3}$$

where dk is the normalized Haar measure on K, and k runs over the set $\{k \in K \mid k \cdot x \in X^{op}\}$. The right hand side of (2.3) is absolutely convergent if $\mathrm{Re}(s_i) \ge 0$, $1 \le i \le n$, and continued to a rational function of $q^{s_1}, \dots, q^{s_n}$, and we use the notation $\omega(x;s)$ in such sense. We call $\omega(x;s)$ a spherical function on X, since it becomes an $\mathcal{H}(G,K)$-common eigenfunction on X (cf. [H1, §1], or [H2, §1]). Indeed, $\mathcal{H}(G,K)$ is a commutative $\mathbb{C}$-algebra spanned by all the characteristic functions of double cosets $KgK, g \in G$ by definition, and we see

$$\begin{aligned}(f * \omega(\ ;s))(x) \quad &\left(= \int_G f(g)\omega(g^{-1}\cdot x;s)dg\right) \\ &= \lambda_s(f)\omega(x;s), \quad (f \in \mathcal{H}(G,K)),\end{aligned} \tag{2.4}$$

where dg is the Haar measure on G normalized by $\int_K dg = 1$, and λ_s is the $\mathbb{C}$-algebra homomorphism defined by

$$\begin{aligned}\lambda_s : \mathcal{H}(G,K) &\longrightarrow \mathbb{C}(q^{s_1}, \dots, q^{s_n}), \\ f &\longmapsto \int_B f(p)\,|\psi(p)|^{-s}\,\delta(p)dp.\end{aligned}$$

Here $|\psi(p)|^{-s} = \prod_{i=1}^n |\psi_i(p)|^{-s_i}$, dp is the left-invariant measure on P such that $\int_{P\cap K} dp = 1$ and $\delta(p)$ is the modulus character of dp ($d(pq) = \delta(q)^{-1}dp$).

We introduce a new variable z which is related to s by

$$s_i = -z_i + z_{i+1} - 1 + \frac{\pi\sqrt{-1}}{\log q} \quad (1 \le i \le n-1), \quad s_n = -z_n - 1 + \frac{\pi\sqrt{-1}}{2\log q} \tag{2.5}$$

and write $\omega(x;z) = \omega(x;s)$. We see

$$|\psi(p)|^s = (-1)^{v_\pi(p_1\cdots p_n)} \prod_{i=1}^{n} |N_{k'/k}(p_i)|^{z_i}\, \delta^{\frac{1}{2}}(p), \quad p \in B, \tag{2.6}$$

where p_i is the i-the diagonal entry of p, $1 \leq i \leq n$. The Weyl group W of G with respect to the maximal k-split torus in B acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1} b n_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega(x; s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \ltimes C_2^n$, S_n acts on z by permutation of indices, and W is generated by S_n and $\tau : (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_{n-1}, -z_n)$. Keeping the relation (2.5), we also write $\lambda_z(f) = \lambda_s(f)$. Then the $\mathbb{C}$-algebra map λ_z is an isomorphism (the Satake isomorphism)

$$\lambda_z \ : \ \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}]^W, \tag{2.7}$$

where the ring of the right hand side is the invariant subring of the Laurent polynomial ring $\mathbb{C}[q^{2z_1}, q^{-2z_1}, \ldots, q^{2z_n}, q^{-2z_n}]$ by W.

By the embedding

$$K_0 = GL_n(\mathcal{O}_{k'}) \longrightarrow K, \quad h \longmapsto \widetilde{h} = \begin{pmatrix} jh^{*-1}j & & \\ & 1 & \\ & & h \end{pmatrix},$$

we obtain

$$\omega(x; s) \ = \ \int_K \zeta_*^{(h)}(D(k \cdot x); s) dk,$$

where $D(y)$ is a spherical function on the space $\mathcal{H}_n(k')$ defined by

$$\zeta_*^{(h)}(y; s) = \int_{K_0} |\mathbf{d}(h \cdot y)|^s \, dh, \quad (h \cdot y = hyh^*),$$

and we see $G_1(z) \cdot \zeta_*^{(h)}(y; s)$ is holomorphic for $z \in \mathbb{C}^n$ and S_n-invariant(cf. [H1, §2]). Thus we obtain the following.

Theorem 2.1 *The function $G_1(z) \cdot \omega(x; s)$ is invariant under the action of S_n on z, where*

$$G_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j - 1}}. \tag{2.8}$$

As for $\tau \in W$, first we calculate explicitly the spherical function $\omega^{(1)}(x; s)$ of $n = 1$ and read

$$\frac{1 + q^{2z}}{1 - q^{-1+2z}} \cdot \omega^{(1)}(x; z) \in \mathbb{C}[q^z + q^{-z}], \ \omega^{(1)}(x; z) = \frac{1 - q^{-1+2z}}{q^{2z} - q^{-1}} \cdot \omega^{(1)}(x; \tau(z)). \tag{2.9}$$

Then, for $n \geq 2$, by using the embedding

$$K_1 = U(j_3)(\mathcal{O}_{k'}) \hookrightarrow K = K_n, \ h \longmapsto \widetilde{h} = \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix},$$

we show the identity

$$\omega(x;s) \;=\; \int_K \prod_{i=1}^{n-2} |d_i(k\cdot x)|^{s_i} \cdot |d_{n-1}(k\cdot x)|^{s_{n-1}+s_n} \cdot \omega^{(1)}(D_1(k\cdot x); s_n)dk, \quad (2.10)$$

where, $D_1(y) \in X_1$ for $y \in X^{op}$ and $D_1(y) = \mathrm{Diag}(y_n, y_0, y_n^{-1})$ if y is diagonal with diagonal n-th entry y_n and $(n+1)$-entry y_0. By (2.9) and (2.10), we obtain the following.

Theorem 2.2 *For general size n, the spherical function satisfies the functional equation*

$$\omega(x;z) = \frac{1-q^{-1+2z_n}}{q^{2z_n}-q^{-1}}\omega(x;\tau(z)),$$

where $\tau(z) = (z_1,\ldots,z_{n-1},-z_n)$.

We prepare some notation. Set

$$\Sigma = \{\pm e_i \pm e_j,\ \pm 2e_i \mid 1 \le i,j \le n,\ i \ne j\}, \quad \Sigma^+ = \Sigma_s^+ \cup \Sigma_\ell^+,$$
$$\Sigma_s^+ = \{e_i + e_j,\ e_i - e_j \mid 1 \le i < j \le n\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \le i \le n\},$$

where e_i is the i-th unit vector in $\mathbb{Z}^n$, $1 \le i \le n$. We note here that $\Sigma \cup \{\pm e_i \mid 1 \le i \le n\}$ is the set of roots of G. We consider the pairing

$$\langle t,\, z\rangle = \sum_{i=1}^{n} t_i z_i, \quad (t,z) \in \mathbb{Z}^n \times \mathbb{C}^n,$$

which satisfies

$$\langle \alpha,\, z\rangle = \langle \sigma(\alpha),\, \sigma(z)\rangle, \quad (\alpha \in \Sigma,\ z \in \mathbb{C}^n,\ \sigma \in W).$$

Then the following two theorems are proved in the same way as in the even case, based on Theorem 2.1 and Theorem 2.2.

Theorem 2.3 *The spherical function $\omega(x;z)$ satisfies the following functional equation*

$$\omega(x;z) = \Gamma_\sigma(z)\cdot\omega(x;\sigma(z)), \quad (2.11)$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha\in\Sigma^+(\sigma)} \frac{1-q^{\langle\alpha,\,z\rangle-1}}{q^{\langle\alpha,\,z\rangle}-q^{-1}}, \quad \Sigma^+(\sigma) = \{\alpha \in \Sigma^+ \mid -\sigma(\alpha) \in \Sigma^+\}.$$

Theorem 2.4 *The function $G(z)\cdot\omega(x;z)$ is holomorphic on $\mathbb{C}^n$ and W-invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1},\ldots,q^{\pm z_n}]^W$, where*

$$G(z) = \prod_{\alpha\in\Sigma^+} \frac{1+q^{\langle\alpha,\,z\rangle}}{1-q^{\langle\alpha,\,z\rangle-1}}.$$

§3 The explicit formula for $\omega(x; z)$

In order to determine the explicit formula of $\omega(x;z)$, it suffices to give the explicit formula for each representative of K-orbits, hence for $x_\lambda, \lambda \in \Lambda_n^+$ by Theorem 1.1.

Theorem 3.1 *For $\lambda \in \Lambda_n^+$, one has the explicit formula*

$$\omega(x_\lambda; z) \quad = \quad \frac{(1+q^{-1})(1-q^{-2})^n}{w_{2n+1}(-q^{-1})} \cdot \frac{1}{G(z)} \cdot q^{\langle \lambda, z_0 \rangle} \cdot Q_\lambda(z),$$

where $G(z)$ is given in Theorem 2.4, $z_0 \in \mathbb{C}^n$ is the value in z-variable corresponding to $\mathbf{0} \in \mathbb{C}^n$ in s-variable,

$$z_{0,i} = -(n-i+1) + (n-i+\frac{1}{2})\frac{\pi\sqrt{-1}}{\log q}, \quad 1 \le i \le n, \tag{3.1}$$

$$w_m(t) = \prod_{i=1}^{m}(1-t^i),$$

$$Q_\lambda(z) = Q_\lambda(z; -q^{-1}, -q^{-2}), \quad Q_\lambda(z;t) = \sum_{\sigma \in W} \sigma\left(q^{-\langle \lambda, z\rangle} c(z,t)\right),$$

$$c(z;t) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z\rangle}}, \quad t_\alpha = \begin{cases} t_s = -q^{-1} & \text{if } \alpha \in \Sigma_s^+ \\ t_\ell = -q^{-2} & \text{if } \alpha \in \Sigma_\ell^+. \end{cases} \tag{3.2}$$

We prove Theorem 3.1 by using a general expression formula of spherical functions given in [H2, §2] based on the functional equations of spherical functions and data of the group G under certain condition. After checking the condition, which is rather troublesome, we use the results in §2, and formulate the formula.

Remark 3.2 We see the main part $Q_\lambda(z)$ of $\omega(x;z)$ is contained in $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$ by Theorem 2.4, and

$$P_\lambda(z; t_s, t_\ell) = \frac{1}{W_\lambda(t_s, t_\ell)} \cdot Q_\lambda(z; t_s, t_\ell) \tag{3.3}$$

is a specialization of Hall-Littlewood polynomial of type C_n, where $W_\lambda(t_s, t_\ell)$ is the Poincaré polynomial of the stabilizer subgroup W_λ of W at λ, and

$$W_\lambda(-q^{-1}, -q^{-2}) = \frac{\widetilde{w_\lambda}(-q^{-1})}{(1+q^{-1})^{n+1}}, \tag{3.4}$$

$$\widetilde{w_\lambda}(t) = \begin{cases} w_{m_0(\lambda)+1}(t) \cdot \prod_{\ell \ge 0} w_{m_\ell(\lambda)}(t) & \text{if } m_0 > 0 \\ \prod_{\ell \ge 1} w_{m_\ell(\lambda)}(t) & \text{if } m_0 = 0, \end{cases} \qquad m_\ell(\lambda) = \sharp\{i \mid \lambda_i = \ell\}.$$

Using $P_\lambda(z) = P_\lambda(z; -q^{-1}, -q^{-2})$ we have

$$\omega(x_\lambda; z) = \frac{(1-q^{-1})^n}{w_{2n+1}(-q^{-1})} \cdot \frac{1}{G(z)} \cdot \widetilde{w_\lambda}(-q^{-1}) \cdot q^{\langle \lambda, z_0\rangle} \cdot P_\lambda(z), \quad (\lambda \in \Lambda_n^+). \tag{3.5}$$

It is known (cf. [M2], [HK, Appendix B]) that the set $\{P_\lambda(z;t_s,t_\ell) \mid \lambda \in \Lambda_n^+\}$ forms an orthogonal $\mathbb{C}$-basis for $\mathcal{R}$, in particular $P_0(z) = 1$ (cf. (4.4) and (4.6)). In the present case, the root system of the group $G = U(j_{2n+1})$ is of type BC_n, but we can write the explicit formula for $\omega(x;z)$ as above by using $P_\lambda(z;t_s,t_\ell)$ of type C_n. We need a different specialization from the case of unitary hermitian forms of even size, which is a homogeneous space of the group $U(j_{2n})$ of type C_n and $(t_s,t_\ell) = (-q^{-1}, q^{-1})$ (cf. [HK]).

We have the following immediately from (3.5).

Corollary 3.3 *For* $x_0 = 1_{2n+1}$, *one has*

$$\omega(1_{2n+1};z) = \frac{(1-q^{-1})^n w_n(-q^{-1}) w_{n+1}(-q^{-1})}{w_{2n+1}(-q^{-1})} \times \frac{1}{G(z)}.$$

Remark 3.4 We give an interpretation of the constant z_0. For $v \in \mathbb{Z}^n$, let

$$\mathbf{t}^{\mathrm{ht}(v)} = \prod_{\beta \in \Sigma_s^+} t_s^{\langle v, \beta^\vee \rangle/2} \prod_{\beta \in \Sigma_\ell^+} t_\ell^{\langle v, \beta^\vee \rangle/2}, \tag{3.6}$$

where $\beta^\vee = 2\beta/\langle \beta, \beta \rangle$. Then for $v = \alpha \in \Sigma$, this coincides with the generalization of the heights of roots [M1]. On the other hand, (3.6) can be rewritten as

$$\mathbf{t}^{\mathrm{ht}(v)} = q^{\langle v, z_0 \rangle},$$

where z_0 is given by (3.1). Thus z_0 can be regarded as a generalization of the dual Weyl vector.

From this viewpoint, the constant z_0 in the even case is calculated as

$$z_{0,i} = -(n-i+\frac{1}{2}) + (n-i)\frac{\pi\sqrt{-1}}{\log q}, \quad 1 \le i \le n,$$

which corresponds to the change of variables

$$s_n = -z_n - \frac{1}{2} \tag{3.7}$$

in (0.3) with the same s_i $(1 \le i \le n-1)$ as before. This modification causes only the sign changes of $\omega(x_\lambda;z)$, that is, $\omega(x_\lambda;z)$ with (3.7) is the multiple by $(-1)^{|\lambda|}$ of the original $\omega(x_\lambda;z)$. In other words, on $G \cdot x_0$ the both coincide and on $G \cdot x_1$ the difference is the multiple by -1.

§4 Spherical Fourier transform and Plancherel formula on $\mathcal{S}(K\backslash X)$

We modify the spherical function by

$$\Psi(x;z) = \omega(x;z)/\omega(1_{2n};z) \in \mathcal{R} = \mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^W, \tag{4.1}$$

and define the spherical Fourier transform on the Schwartz space

$$\mathcal{S}(K\backslash X) \;=\; \{\varphi : X \longrightarrow \mathbb{C} \mid \text{left } K\text{-invariant, compactly supported}\},$$

by

$$\begin{array}{rcl} F:\ \mathcal{S}(K\backslash X) & \longrightarrow & \mathcal{R} \\ \varphi & \longmapsto & F(\varphi)(z) = \int_X \varphi(x)\Psi(x;z)dx, \end{array} \tag{4.2}$$

where dx is a G-invariant measure on X. The existence of a G-invariant measure is assured by the fact X is a union of two G-orbits and G is reductive, and we fix the normalization afterwards. Then, for each characteristic function ch_λ of $K\cdot x_\lambda$, $\lambda \in \Lambda_n^+$, we have

$$F(ch_\lambda)(z) = q^{\langle \lambda,\, z_0\rangle}\frac{\widetilde{w_\lambda}(-q^{-1})}{\widetilde{w_0}(-q^{-1})}\cdot v(K\cdot x_\lambda)P_\lambda(z), \tag{4.3}$$

where $\widetilde{w_\lambda}(-q^{-1})$ is defined in Remark 3.2, and $v(K\cdot x_\lambda)$ is the volume of $K\cdot x_\lambda$ with respect to dx. On the other hand, we regard $\mathcal{R}$ as an $\mathcal{H}(G,K)$-module through the Satake isomorphism λ_z (cf. (2.7)). Then we have the following.

Theorem 4.1 *The spherical Fourier transform F gives an $\mathcal{H}(G,K)$-module isomorphism*

$$\mathcal{S}(K\backslash X) \xrightarrow{\sim} \mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^W (= \mathcal{R}),$$

where $\mathcal{R}$ is regarded as $\mathcal{H}(G,K)$-module via λ_z. Especially, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 2^n.

Corollary 4.2 *All the spherical functions on X are parametrized by eigenvalues $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^n/W$ through $\mathcal{H}(G,K) \longrightarrow \mathbb{C}$, $f \longmapsto \lambda_z(f)$. The set $\{\Psi(x;z+u) \mid u \in \{0, \pi\sqrt{-1}/\log q\}^n\}$ forms a basis of the space of spherical functions on X corresponding to z.*

We introduce an inner product $\langle\ ,\ \rangle_{\mathcal{R}}$ on $\mathcal{R}$ by

$$\langle P, Q\rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z)\overline{Q(z)}d\mu(z), \quad P, Q \in \mathcal{R}. \tag{4.4}$$

Here

$$\mathfrak{a}^* = \left\{\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q}\mathbb{Z}\right)\right\}^n, \quad d\mu = \frac{1}{n!2^n}\cdot\frac{w_n(-q^{-1})w_{n+1}(-q^{-1})}{(1+q^{-1})^{n+1}}\cdot\frac{1}{|c(z)|^2}dz, \tag{4.5}$$

where dz is the Haar measure on $\mathfrak{a}^*$ with $\int_{\mathfrak{a}^*} dz = 1$ and $c(z) = c(z; -q^{-1}, -q^{-2})$ is defined in Theorem 3.1. Then, it is known

$$\langle P_\lambda, P_\mu \rangle_{\mathcal{R}} = \langle P_\mu, P_\lambda \rangle_{\mathcal{R}} = \delta_{\lambda,\mu} \cdot \frac{\widetilde{w_{\mathbf{0}}}(-q^{-1})}{\widetilde{w_\lambda}(-q^{-1})}, \quad (\lambda, \mu \in \Lambda_n^+). \tag{4.6}$$

On the other hand, one has

$$\frac{v(K \cdot x_\lambda)}{v(K \cdot x_\mu)} = \frac{q^{2\langle \mu, z_0 \rangle} \widetilde{w_\mu}(-q^{-1})}{q^{2\langle \lambda, z_0 \rangle} \widetilde{w_\lambda}(-q^{-1})}, \quad (\lambda, \mu \in \Lambda_n^+, \ |\lambda| \equiv |\mu| \,(\mathrm{mod} 2)). \tag{4.7}$$

Theorem 4.3 (Plancherel formula on $\mathcal{S}(K\backslash X)$) *Let $d\mu$ be the measure defined by (4.5). By the normalization of G-invariant measure dx such that*

$$v(K \cdot x_\lambda) = q^{-2\langle \lambda, z_0 \rangle} \frac{\widetilde{w_{\mathbf{0}}}(-q^{-1})}{\widetilde{w_\lambda}(-q^{-1})}, \qquad \lambda \in \Lambda_n^+, \tag{4.8}$$

one has for any $\varphi, \psi \in \mathcal{S}(K\backslash X)$

$$\int_X \varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*} F(\varphi)(z)\overline{F(\psi)(z)}d\mu(z). \tag{4.9}$$

Outline of a proof. We recall the G-orbit decomposition (Theorem 1.3) and normalize dx on each G-orbit as

$$v(K \cdot x_\lambda) = \begin{cases} 1 & \text{for } \lambda = \mathbf{0}, \\ q^{-2n} \frac{(1-(-q^{-1})^n)(1-(-q^{-1})^{n+1})}{1+q^{-1}} & \text{for } \lambda = (1, 0, \cdots, 0), \end{cases}$$

then the identity (4.8) follows from this and (4.7). Then, for any $\lambda, \mu \in \Lambda_n^+$, we see by (4.3), (4.6), and (4.8)

$$\int_X ch_\lambda(x)\overline{ch_\mu(x)}dx = \delta_{\lambda,\mu} q^{-2\langle \lambda, z_0 \rangle} \frac{\widetilde{w_{\mathbf{0}}}(-q^{-1})}{\widetilde{w_\lambda}(-q^{-1})} = \int_{\mathfrak{a}^*} F(ch_\lambda)(z)\overline{F(ch_\mu)(z)}d\mu(z).$$

Since $\{\mathrm{ch}_\lambda \mid \lambda \in \Lambda_n^+\}$ spans $\mathcal{S}(K\backslash X)$, we conclude the identity (4.9). ∎

The next corollary is an easy consequence of Theorem 4.3.

Corollary 4.4 (Inversion formula) *For any $\varphi \in \mathcal{S}(K\backslash X)$,*

$$\varphi(x) = \int_{\mathfrak{a}^*} F(\varphi)(z)\Psi(x; z)d\mu(z), \quad x \in X.$$

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