数理解析研究所講究録
第 1934 巻 2015 年 158-162

CONGRUENCE PROPERTIES OF SIEGEL MODULAR FORMS OF DEGREE 2 AND WEIGHT 47, 71, 89

SHO TAKEMORI

1. Introduction

Let X_{35} be a Siegel cusp form of degree 2 and weight 35. Kikuta, Kodama and Nagaoka [4] proved that $\det T\ a(T, X_{35}) \equiv 0 \bmod 23$ for every half integral positive symmetric matrix T.

In this paper, we give a finite number of examples of Hecke eigenforms of degree 2 and odd weights that have the same type of congruence relation above. We also introduce congruence relations for the Hecke eigenvalues of such eigenforms. We prove our main results by numerical computation. For the computation, we use Sage [5] and a Sage package for Siegel modular forms of degree two written by the author [6].

2. Definition

Let n be a positive integer. We define the Siegel modular group Γ_n of degree n by

$$\Gamma_n = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{Z}) \;\middle|\; {}^t g w_n g = w_n \right\},$$

where $w_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. Define the Siegel upper half space $\mathfrak{H}_n$ by

$$\mathfrak{H}_n = \left\{ Z \in \mathrm{Sym}_n(\mathbb{C}) \;\middle|\; \Im Z > 0 \right\}.$$

Let k be a non-negative integer. We denote by $M_k(\Gamma_n)$ the set of holomorphic functions F on $\mathfrak{H}_n$ satisfying the following condition:

$$F((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k F(z),$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. If $n = 1$, we add the cusp condition. We call an element of $M_k(\Gamma_n)$ a Siegel modular form of degree n and weight k.

For a Siegel modular form F of degree n, F has the Fourier expansion as follows:

$$F(Z) = \sum_{T \geq 0} a(T; F)\mathbf{e}(\mathrm{Tr}(TZ)).$$

Here $\mathbf{e}(z) = \exp(2\pi i z)$ and T runs over the set of half integral semi-positive definite symmetric matrices of degree n.

Department of the Mathematics, Kyoto University.

In particular if the degree n is equal to 2, then we have the following Fourier expansion:

$$F\left(\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\right) = \sum_{\substack{n,\, r,\, m \in \mathbb{Z} \\ n,\, m,\, 4nm - r^2 \geq 0}} a((n, r, m), F)\mathbf{e}(n\tau + rz + m\omega).$$

Here $\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathfrak{H}_2$.

We define $\Phi : M_k(\Gamma_2) \to M_k(\Gamma_1)$ by

$$\Phi(F)(z) = \sum_{n=0}^{\infty} a((n, 0, 0), F)e(nz).$$

Then we define the space of cusp forms $S_k(\Gamma_2)$ by $\ker(\Phi)$.

3. Theta operator and a theorem of Böcherer and Nagaoka

Let F be a Siegel modular form of degree n and

$$F(Z) = \sum_{T \geq 0} a(T; F)\mathbf{e}(\mathrm{Tr}(TZ))$$

be the Fourier expansion of F. Then we define the theta operator as follows:

$$\Theta(F) = \sum_{T \geq 0} (\det T)\, a(T; F)\mathbf{e}(\mathrm{Tr}(TZ)).$$

This operator is a generalization of the classical theta operator.

Theorem 1 (Böcherer and Nagaoka [2]). *Let p be a prime and assume that $p \geq n + 3$. Let $F \in M_k(\Gamma_n)$ and assume F has p-integral rational Fourier coefficients. Then there exists $G \in M_{k+p+1}(\Gamma_n)$ such that*

$$\Theta(F) \equiv G \mod p.$$

Here the congruence relation means the relation for all Fourier coefficients.

4. A congruence relation of the Igusa's cusp form of weight 35

We introduce a theorem of Kikuta, Kodama and Nagaoka [4]. Let $X_{35} \in S_{35}(\Gamma_2)$ be the Igusa's cusp form of weight 35. Here we normalize X_{35} so that $a((2, -1, 3), X_{35}) = 1$.

Theorem 2 (Kikuta, Kodama, Nagaoka '13).

$$(4nm - r^2)a((n, r, m), X_{35}) \equiv 0 \text{ mod } 23,$$

or equivalently,

$$\Theta(X_{35}) \equiv 0 \text{ mod } 23.$$

5. Statement of the main result

In this section, we state our main result. For a prime $p \equiv 3 \bmod 4$, we put

$$k'(p) = 2 + 3(p-1)/2.$$

Let $p = 23, 31, 47$ and 59. Then we have the following table.

p	23	31	47	59
$k'(p)$	35	47	71	89

We denote by $X_{k'(p)} \in S_{k'(p)}(\Gamma_2)$ a Hecke eigenform of degree 2 and weight $k'(p)$.

We normalize $X_{k'(p)}$ so that

$$a((2,-1,4), X_{47}) = 1, \qquad a((3,-1,4), X_{71}) = 1,$$
$$a((3,-1,5), X_{89}) = 1$$

Let $F \in M_k(\Gamma_2)$ be a Hecke eigenform. We denote by $\mathbb{Q}(F)$ the number field generated by the Hecke eigenvalues of F over $\mathbb{Q}$. For a positive integer m, we denote by $\lambda(m, F)$ the Hecke eigenvalue of $T(m)$. For a prime l, we denote by $Q_l(F, T)$ the Hecke polynomial of degree 4, that is $\prod_{l:\text{ prime}} Q_l(F, l^{-s})^{-1}$ is the spinor L function of F.

For a number field K, we denote by $Cl(K)$ the class group of K. Let $\chi : Cl(\mathbb{Q}(\sqrt{-p})) \to \mathbb{C}^\times$ be a character. For a prime $l \neq p$, we define a polynomial $F_l(\chi, T)$ by

$$F_l(\chi, T) = \begin{cases} (1-\chi(L_1)T)(1-\chi(L_2)T) & \text{if } (l) = L_1L_2 \text{ in } \mathbb{Q}(\sqrt{-p}), \\ 1 - T^2 & \text{if } \left(\frac{l}{p}\right) = -1. \end{cases}$$

Theorem 3. *Suppose $p = 23, 31, 47$ or 59. Then there exists a prime $\mathfrak{p}$ of $\mathbb{Q}(X_{k'(p)})$ above p such that*

$$\Theta(X_{k'(p)}) \equiv 0 \mod \mathfrak{p}.$$

Moreover, there exits a prime $\mathfrak{p}'$ of $\mathbb{Q}(X_{k'(p)})(\chi)$ above p and a non-trivial character $\chi : Cl(\mathbb{Q}(\sqrt{-p})) \to \mathbb{C}^\times$ such that

$$Q_l^{(2)}(X_{k'(p)}, T) \equiv F_l(\chi, T)F_l(\chi, lT) \mod \mathfrak{p}',$$

for any prime $l \neq p$.

Let $p = 23$. Then $\#Cl(\mathbb{Q}(\sqrt{-23})) = 3$. By the main result, for $l \neq 23$, we have the following congruence relations for Hecke eigenvalues of X_{35}.

(1) If $\left(\frac{l}{23}\right) = -1$,

$$\lambda(l, X_{35}) \equiv 0 \mod 23.$$

(2) If $\exists x, y \in \mathbb{Z}$ s.t. $l = x^2 + 23y^2$,

$$\lambda(l, X_{35}) \equiv 2(l+1) \mod 23.$$

(3) If $\left(\frac{l}{23}\right) = 1$ and $l \neq x^2 + 23y^2$ for all $x, y \in \mathbb{Z}$,

$$\lambda(l, X_{35}) \equiv -(l+1) \mod 23.$$

For a prime $l \leq 17$, the Hecke eigenvalue of $\lambda(l, X_{35})$ is as follows.

l	$\lambda(l, X_{35})$	$\lambda(l, X_{35}) \bmod 23$
2	−25073418240	20
3	−11824551571578840	19
5	9470081642319930937500	0
7	−10370198954152041951342796400	0
11	−8015071689632034858364818146947656	0
13	−20232136256107650938383898249808243380	9
17	118646313906984767985086867381297558266980	0

6. Sketch of the proof of the main result

In this section, we give a sketch of proof of the main result.

Fix a prime p with $p \equiv 3 \bmod 4$. Let $S = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ $(n, m \in \mathbb{Z}_{\geq 1}, r \in \mathbb{Z})$ be a half integral positive definite symmetric matrix with $4 \det S = p$. Put

$$\theta^{(2)}_{\det}(S) = \sum_{N \in M_2(\mathbb{Z})} \det N\, \mathbf{e}(\mathrm{Tr}\, {}^tNSNZ),$$

where $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$. Then $\theta^{(2)}(S)$ is a Siegel modular form of degree 2, level p, character $\left(\frac{-p}{*}\right)$ and weight 2.

Böcherer, Kodama and Nagaoka proved the following result.

Theorem 4 (Böcherer, Kodama, Nagaoka '13). *Let $p \equiv 3 \bmod 4$ be a prime and $S \in \mathrm{Sym}_2(\mathbb{Q})$ a half integral positive definite symmetric matrix with $4 \det S = p$. Then there exists $F \in S_{k'(p)+p-1}(\Gamma_2)$ such that*

$$F \equiv \theta^{(2)}_{\det}(S) \bmod p.$$

By the genus theory, we identify the set of the strict equivalent classes of half integral symmetric matrices S with $4 \det S = p$ with $Cl(\mathbb{Q}(\sqrt{-p}))$.

For a non-trivial character $\chi : Cl(\mathbb{Q}(\sqrt{-p})) \to \mathbb{C}^\times$, we put

$$\theta(\chi) = \sum_{a \in Cl(\mathbb{Q}(\sqrt{-p}))} \chi(a)\theta^{(2)}_{\det}(a).$$

By Theorem 4 and numerical computation, we can prove the following proposition.

Proposition 5. *Suppose $p = 23, 31, 47$ or 59. Then there exists a non-trivial character $\chi : Cl(\mathbb{Q}(\sqrt{-p})) \to \mathbb{C}^\times$, prime $\mathfrak{p}'$ of $\mathbb{Q}(X_{k'(p)})(\chi)$ above p and a constant $\alpha \in \mathbb{Z}[\chi]$ such that*

$$X_{k'(p)} \equiv \alpha\theta^{(2)}_{\det}(\chi) \bmod \mathfrak{p}'.$$

The main result follows from this proposition and [1, Theorem 15].

In the following, we give a sketch of the proof of Proposition 5. By [2], there exits $F \in S_{k'(p)+p-1}(\Gamma_2)$ such that $F \equiv X_{k'(p)} \bmod p$. On the other hand, by Theorem 4 there exits $G \in S_{k'(p)+p-1}(\Gamma_2)$ such that $G \equiv \theta(\chi) \bmod \mathfrak{p}''$ where $\mathfrak{p}''$ is a prime above p. Thus it is enough to prove $F \equiv G \bmod \mathfrak{p}'$ up to a constant. By the Sturm type theorem below, it is enough to check a finite number of the congruence relation among Fourier coefficients. Since it is easy to compute the Fourier coefficients of binary theta series $\theta^{(2)}_{\det}(\chi)$, we can check these congruence relations by [6].

Theorem 6 (Choi, Choie and Kikuta [3]). *Let $p \geq 5$ be a prime and $F \in M_k(\Gamma_2)$ with p-integral rational Fourier coefficients, then if $a((n,r,m),F) \equiv 0 \mod p$ for $n, m \leq [k/10]$, then $F \equiv 0 \bmod p$.*

Sho Takemori
Department of Mathematics,
Kyoto University
Kitashirakawa-Oiwake-Cho, Sakyo-Ku,
Kyoto, 606-8502, Japan
E-mail: takemori@math.kyoto-u.ac.jp

References

1. A Andrianov, *Action of Hecke operators on Maass theta series and zeta functions*, St. Petersburg Mathematical Journal **19** (2008), no. 5, 675–698.
2. S. Böcherer and S. Nagaoka, *On mod p properties of Siegel modular forms*, Mathematische Annalen **338** (2007), no. 2, 421–433.
3. D. Choi, Y. Choie, and T. Kikuta, *Sturm type theorem for Siegel modular forms of genus 2 modulo p*, ACTA ARITHMETICA **158** (2013), no. 2, 129–139.
4. T. Kikuta, H. Kodama, and S. Nagaoka, *Note on Igusa's cusp form of weight 35*, To appear in Rocky Mountain Journal of Mathematics (2013).
5. W. A. Stein et al., *Sage Mathematics Software (Version 5.9)*, The Sage Development Team, 2013, `http://www.sagemath.org`.
6. S. Takemori, *degree2*, `https://github.com/stakemori/degree2`, 2013.