PULLBACKS OF HERMITIAN MAASS LIFTS

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1. INTRODUCTION

The purpose of this article is to report my talk at the conference "Automorphic Forms and Related Zeta Functions" on January 2014. We consider pullbacks of some lifts of elliptic cusp forms to get explicit formulas of critical values of certain automorphic L-functions in terms of these pullbacks. In this section, we give some examples of lifts and introduce previous works.

Let

$$\mathfrak{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(x) > 0\}$$

be the complex upper half space. We consider some lifts from modular forms on $\mathfrak{H}$.

Example 1.1. *Let g be a modular form on $\mathfrak{H}$ and $C \in \mathbb{Q}^{\times}$. Then we define a modular form $g \times g_C$ on $\mathfrak{H} \times \mathfrak{H}$ by*

$$(g \times g_C)(z_1, z_2) = g(z_1) \cdot g(z_2/C).$$

If $g \in S_k(\operatorname{SL}_2(\mathbb{Z}))$, then we have

$$g \times g_C \in S_k(\operatorname{SL}_2(\mathbb{Z})) \otimes S_k(d(C)^{-1}\operatorname{SL}_2(\mathbb{Z})d(C)),$$

where

$$d(C) = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} \in \operatorname{GL}_2(\mathbb{Q}).$$

Example 1.2 (Saito–Kurokawa lifts). *Let k be a positive odd integer and $f_1 \in S_{2k}(\operatorname{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then, f_1 gives a half integral weight modular form $h \in S^{+}_{k+1/2}(\Gamma_0(4))$ by the Shimura correspodence. Using the Fourier coefficients of h, we can construct a modular form F_{SK} on the Siegel upper half space*

$$\mathfrak{H}_2 = \{Z = X + \sqrt{-1}Y \in \mathrm{M}_2(\mathbb{C}) \mid {}^t Z = Z, \ Y > 0\}.$$

We have $F_{\mathrm{SK}} \in S_{k+1}(\operatorname{Sp}_2(\mathbb{Z}))$ and call F_{SK} the Saito–Kurokawa lift of f_1.

Example 1.3 (Hermitian Maass lifts). *Let $K/\mathbb{Q}$ be an imaginary quadratic field with discriminant $-D < 0$, and χ be the Dirichlet character corresponding to $K/\mathbb{Q}$. Let k be a positive integer and $f_2 \in S_{2k+1}(\Gamma_0(D), \chi)$ be a normalized Hecke eigenform. Using the Fourier coefficients of f_2, we can construct a modular form F_{M} on the hermitian upper half space*

$$\mathcal{H}_2 = \left\{ Z \in \mathrm{M}_2(\mathbb{C}) \ \middle| \ \frac{1}{2\sqrt{-1}}(Z - {}^t\overline{Z}) > 0 \right\}.$$

We have $F_{\mathrm{M}} \in S_{2k+2}(\mathrm{U}(\mathbb{Z}), \det^{-k-1})$ and call F_{M} the hermitian Maass lift of f_2. (See also the next section.)

Note that

$$\mathfrak{H} \times \mathfrak{H} \subset \mathfrak{H}_2 \subset \mathcal{H}_2.$$

We may consider the pullbacks and the period integrals

- $\langle F_\mathrm{M}|_{\mathfrak{H}_2}, F_\mathrm{SK} \rangle$;
- $\langle F_\mathrm{SK}|_{\mathfrak{H} \times \mathfrak{H}}, g \times g \rangle$;
- $\langle F_\mathrm{M}|_{\mathfrak{H} \times \mathfrak{H}}, g \times g \rangle$

for modular forms f_1, f_2 and g with suitable weights. Using these periods, we may get explicit formulas of critical values of certain L-functions.

Theorem 1.4 (Ichino–Ikeda (2008) [3]). *Let $k \in \mathbb{Z}_{>0}$ and $f_1 \in S_{4k+2}(\mathrm{SL}_2(\mathbb{Z}))$, $f_2 \in S_{2k+1}(\Gamma_0(D), \chi)$ be normalized Hecke eigenforms. We denote the lifts by $f_1 \rightsquigarrow h \rightsquigarrow F_\mathrm{SK}$ and $f_2 \rightsquigarrow F_\mathrm{M}$. Then, the identity*

$$\frac{\Lambda(4k+1, f_2 \times f_2 \times f_1)}{\langle f_1, f_1 \rangle^2} = -\frac{2^{8k+6}a_h(D)^2}{D^{4k+1}} \frac{\langle F_\mathrm{M}|_{\mathfrak{H}_2}, F_\mathrm{SK} \rangle^2}{\langle F_\mathrm{SK}, F_\mathrm{SK} \rangle^2}$$

holds. Here, $a_h(D)$ is the D-th Fourier coefficient of h.

The L-function $\Lambda(s, f_2 \times f_2 \times f_1)$ is completed and satisfies the functional equation

$$\Lambda(8k+2-s, f_2 \times f_2 \times f_1) = \Lambda(s, f_2 \times f_2 \times f_1).$$

Theorem 1.5 (Ichino (2005) [2]). *Let $k > 0$ be an odd integer and $f_1 \in S_{2k}(\mathrm{SL}(\mathbb{Z}))$, $g \in S_{k+1}(\mathrm{SL}(\mathbb{Z}))$ be normalized Hecke eigenforms. We denote the lifts by $f_1 \rightsquigarrow h \rightsquigarrow F_\mathrm{SK}$ and $g \rightsquigarrow g \times g$. Then, the identity*

$$\Lambda(2k, \mathrm{Sym}^2(g) \times f_1) = 2^{k+1} \frac{\langle f_1, f_1 \rangle}{\langle h, h \rangle} \frac{|\langle F_\mathrm{SK}|_{\mathfrak{H} \times \mathfrak{H}}, g \times g \rangle|^2}{\langle g, g \rangle^2}$$

holds.

The L-function $\Lambda(s, \mathrm{Sym}^2(g) \times f_1)$ is completed, and satisfies the functional equation

$$\Lambda(4k-s, \mathrm{Sym}^2(g) \times f_1) = \Lambda(s, \mathrm{Sym}^2(g) \times f_1).$$

In this article, we consider $F_\mathrm{M}|_{\mathfrak{H} \times \mathfrak{H}}$.

2. Hermitian Maass lifts

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D < 0$. We define the unitary group $\mathrm{U}(2,2)(\mathbb{Q})$ by

$$\mathrm{U}(2,2)(\mathbb{Q}) = \{ g \in \mathrm{GL}_4(K) \mid {}^t\bar{g}Jg = J \}$$

with

$$J = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \in \mathrm{GL}_4(K).$$

This group acts on $\mathcal{H}_2$ by

$$\gamma(Z) = (AZ + B)(CZ + D)^{-1}$$

for $Z \in \mathcal{H}_2$ and

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(2,2)(\mathbb{Q}).$$

Let χ be the Dirichlet character corresponding to $K/\mathbb{Q}$ and $\mathfrak{o}$ be the ring of integers of K. Using the Fourier coefficients of a normalized Hecke eigenform

$f \in S_{2k+1}(\Gamma_0(D), \chi)$ and an ideal $\mathfrak{c}$ of $\mathfrak{o}$ which is prime to D, we can construct a holomorphic function $F_{\mathfrak{c}}$ on $\mathcal{H}_2$ which satisfies

$$F_{\mathfrak{c}}(Z) = \det(\gamma)^{k+1} F_{\mathfrak{c}}(\gamma(Z)) \det(CZ + D)^{-(2K+2)}$$

for $Z \in \mathcal{H}_2$ and

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_K[\mathfrak{c}],$$

where

$$\Gamma_K[\mathfrak{c}] = \left\{ g \in \mathrm{U}(2,2)(\mathbb{Q}) \,\middle|\, g \begin{pmatrix} \mathfrak{o} \\ \mathfrak{c} \\ \mathfrak{o} \\ \overline{\mathfrak{c}}^{-1} \end{pmatrix} = \begin{pmatrix} \mathfrak{o} \\ \mathfrak{c} \\ \mathfrak{o} \\ \overline{\mathfrak{c}}^{-1} \end{pmatrix} \right\}.$$

We call $F_{\mathfrak{c}}$ the hermitian Maass lift of f which satisfies the Maass relation for $\mathfrak{c}$.

Remark 2.1. *Ichino and Ikeda* [3] *considered the pullbacks* $F_{\mathfrak{o}}|_{\mathfrak{H}_2}$. *Note that*

$$\Gamma_K[\mathfrak{o}] = \mathrm{U}(2,2)(\mathbb{Z}) = \mathrm{U}(2,2)(\mathbb{Q}) \cap \mathrm{GL}_4(\mathfrak{o}).$$

In this article, we consider $F_{\mathfrak{c}}|_{\mathfrak{H} \times \mathfrak{H}}$.
We have

$$F_{\mathfrak{c}}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right) \in S_{2k+2}(\mathrm{SL}(\mathbb{Z})) \otimes S_{2k+2}(d(C)^{-1}\mathrm{SL}(\mathbb{Z})d(C)),$$

where $C = N(\mathfrak{c}) \in \mathbb{Z}_{>0}$ is the ideal norm of $\mathfrak{c}$. For a normalized Hecke eigenform $g \in S_{2k+2}(\mathrm{SL}(\mathbb{Z}))$, we put

$$g_C(z) = g(z/C) \in S_{2k+2}(d(C)^{-1}\mathrm{SL}(\mathbb{Z})d(C)).$$

Then, we may consider the period integral

$$\langle F_{\mathfrak{c}}|_{\mathfrak{H} \times \mathfrak{H}}, g \times g_C \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the suitable Petersson inner product defined using the Lebesgue measure.

The following lemma is important.

Lemma 2.2. *Fix normalized Hecke eigenforms* $f \in S_{2k+1}(\Gamma_0(D), \chi)$ *and* $g \in S_{2k+2}(\mathrm{SL}(\mathbb{Z}))$. *Then, the map*

$$\mathfrak{c} \mapsto \frac{\langle F_{\mathfrak{c}}|_{\mathfrak{H} \times \mathfrak{H}}, g \times g_C \rangle}{\langle g, g \rangle \langle g_C, g_C \rangle}$$

depends only on the ideal class of $\mathfrak{c}$.

3. Main Theorem

Let $f = \sum_{n>0} a_f(n) q^n \in S_{2k+1}(\Gamma_0(D), \chi)$ and $g = \sum_{n>0} a_g(n) q^n \in S_{2k+2}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. First, we define L-functions $L(s, f \times g)$ and $L(s, f \times g \times \chi)$. We define the Satake parameters $\{\alpha_p, \chi(p)\alpha_p^{-1}\}$ of f and $\{\beta_p, \beta_p^{-1}\}$ normalized by

$$\alpha_p + \chi(p)\alpha_p^{-1} = p^{-k} a_f(p), \qquad \text{for } p \nmid D,$$

$$\beta_p + \beta_p^{-1} = p^{-(k+1/2)} a_g(p), \quad \text{for each prime } p.$$

For $p \mid D$, we put $\alpha_p = p^{-k} a_f(p)$. Then the Ramanujan conjecture proved by Deligne states that

$$|\alpha_p| = |\beta_p| = 1.$$

for each prime p. In paticular, we see that $a_f(D) \neq 0$. Put

$$A_p = \begin{cases} \begin{pmatrix} \alpha_p & \\ & \chi(p)\alpha_p^{-1} \end{pmatrix} & \text{if } p \nmid D \\ \alpha_p & \text{if } p \mid D \end{cases}, \quad B_p = \begin{pmatrix} \beta_p & \\ & \beta_p^{-1} \end{pmatrix}.$$

For $\mathrm{Re}(s) \gg 0$, we define the L-functions $L(s, f \times g)$ and $L(s, f \times g \times \chi)$ by

$$L(s, f \times g) = \prod_p \det(\mathbf{1}_r - A_p \otimes B_p \cdot p^{-s})^{-1},$$

$$L(s, f \times g \times \chi) = \prod_p \det(\mathbf{1}_r - A_p^{-1} \otimes B_p \cdot p^{-s})^{-1}.$$

Proposition 3.1.   (1) *The L-functions $L(s, f \times g)$ and $L(s, f \times g \times \chi)$ have holomorphic continuation for whole s-plane.*

(2) *Put $L_\infty(s) = \Gamma_{\mathbb{C}}(s + 2k + 1/2)\Gamma_{\mathbb{C}}(s + 1/2)$, where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then, they satisfy the functional equation*

$$L_\infty(s)L(s, f \times g) = -D^{1-2s+2k}a_f(D)^{-2}L_\infty(1-s)L(1-s, f \times g \times \chi).$$

(3) $L(s, f \times g) = \overline{L(\bar{s}, f \times g \times \chi)}.$

(4) $a_f(D)L(1/2, f \times g) \in \sqrt{-1}\mathbb{R}.$

Proof. (1) and (2) are well-known. (3) is a consequence of the Ramanujan conjecture. By (2), (3) and the Ramamujan conjecture, we have

$$(D^{-k}a_f(D))L(1/2, f \times g) = -(D^{-k}a_f(D))^{-1}L(1/2, f \times g \times \chi)$$
$$= -\overline{(D^{-k}a_f(D))L(1/2, f \times g)}.$$

This shows (4). $\square$

The main result is as follows.

Theorem 3.2. *Let $f \in S_{2k+1}(\Gamma_0(D), \chi)$ and $g \in S_{2k+2}(\mathrm{SL}(\mathbb{Z}))$ be normalized Hecke eigenforms. We denote the lifts by $f \rightsquigarrow F_c$ and $g \rightsquigarrow g \times g_C$. Then, the identity*

$$L\left(\frac{1}{2}, f \times g\right) = \frac{L(1, \chi)(4\pi)^{2k+1}}{a_f(D)(2k)!} \cdot \frac{1}{h_K} \sum_{[c] \in Cl_K} \frac{\langle F_c|_{\mathfrak{H} \times \mathfrak{H}}, g \times g_C \rangle}{\langle g_C, g_C \rangle}$$

holds. Here, Cl_K is the ideal class group of $K = \mathbb{Q}(\sqrt{-D})$ and $h_K = \#Cl_K$ is the class number of K.

4. Sckech of the proof

The proof consists of three steps as follows:

(1) Write down the hermitian Maass lifts in terms of the theta lift.
(2) Use the seesaw identity.
(3) Use the genus theory.

We explain more precisely.

Step(1). Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$. The modular forms f and g give automorphic forms $\mathbf{f}$ and $\mathbf{g}$ on $\mathrm{GL}_2(\mathbb{A})$:

$$f \rightsquigarrow \mathbf{f} \text{ on } \mathrm{GL}_2(\mathbb{A});$$

$$g \rightsquigarrow \mathbf{g} \text{ on } \mathrm{GL}_2(\mathbb{A}).$$

The family of hermitian Maass lifts $\{F_c\}$ gives an automorphic form $Lift^{(2)}(f)$ on $U(2,2)(\mathbb{A})$ (see [4]):

$$\{F_c\} \rightsquigarrow Lift^{(2)}(f) \text{ on } U(2,2)(\mathbb{A}).$$

Note that the strong approximation theorem does not hold for $U(2,2)$.

On the other hand, we let ψ be a standard additive character of $\mathbb{A}/\mathbb{Q}$ and $\omega = \omega_\psi$ be the Weil representation of $SL_2(\mathbb{A}) \times O(4,2)(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}^6)$ associated to ψ. For $\varphi \in \mathcal{S}(\mathbb{A}^6)$ with some condition, we put

$$\theta(\varphi)(\alpha, h) = \sum_{x \in \mathbb{Q}^6} [\omega(\alpha, h)\varphi](x)$$

for $\alpha \in SL_2(\mathbb{A})$ and $h \in O(4,2)(\mathbb{A})$. This is an automorphic form on $SL_2(\mathbb{A}) \times O(4,2)(\mathbb{A})$ which is called a theta function. Putting

$$\theta(\mathbf{f}, \varphi)(h) = \int_{SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})} \theta(\varphi)(\alpha, h)\mathbf{f}(\alpha)d\alpha$$

and extend this, we get an automorphic form $\theta(\mathbf{f}, \varphi)$ on $GSO(4,2)(\mathbb{A})$ (with trivial central character), which is called a theta lift:

$$(\mathbf{f}, \varphi) \rightsquigarrow \theta(\mathbf{f}, \varphi) \text{ on } PGSO(4,2)(\mathbb{A}).$$

It is known that there is an isomorphism on the algebraic groups
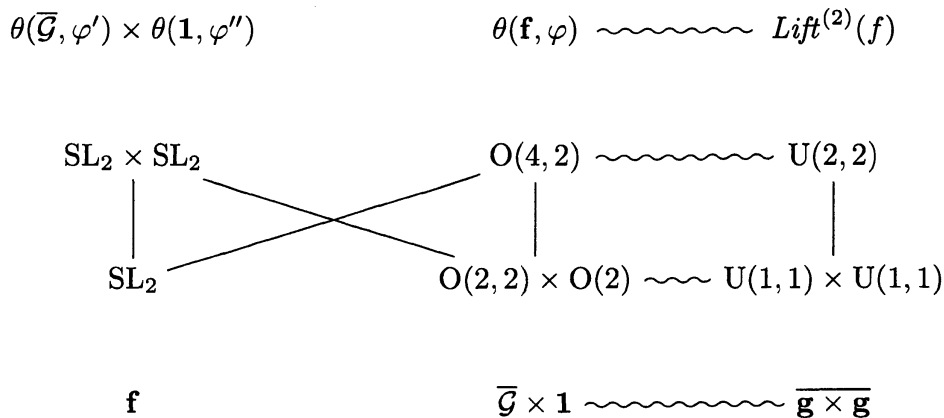
$$PGU(2,2) \xrightarrow{\sim} PGSO(4,2).$$

The key lemma is as follows:

Lemma 4.1. *We can find a function* $\varphi \in \mathcal{S}(\mathbb{A}^6)$ *and constants* $c_p \in \mathbb{Z}_{>0}$ *for* $p \mid D$ *explicitly, such that*

$$\theta(\mathbf{f}, \varphi) = \left(\prod_{p \mid D} c_p^{-1}\right) 2^{2k+2} a_f(D)^{-1} Lift^{(2)}(f)$$

via the map $U(2,2)(\mathbb{A}) \to PGU(2,2)(\mathbb{A}) \to PGSO(4,2)(\mathbb{A})$.

Step(2). Now, we consider the following seesaw:

$$\theta(\overline{\mathcal{G}}, \varphi') \times \theta(1, \varphi'') \qquad\qquad \theta(\mathbf{f}, \varphi) \rightsquigarrow Lift^{(2)}(f)$$

The right vertical line becomes a sum of period integrals. The sum comes from the integral on $O(2)$. On the other hand, the theta lifts $\theta(\overline{\mathcal{G}}, \varphi')$ and $\theta(1, \varphi'')$ are known:

Lemma 4.2 ([2] Lemma 5.1). *As an automorphic form of* $\mathrm{GL}_2(\mathbb{A})$, *we have*

$$\theta(\overline{\mathcal{G}}, \varphi') = 2^{2k+1} \xi_{\mathbb{Q}}(2)^{-2} \langle g, g \rangle \overline{\mathbf{g}},$$

where $\xi_{\mathbb{Q}}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ *is the completed Riemann zeta function.*

Proposition 4.3 (Siegel–Weil formula). *For* $\varphi'' \in \mathcal{S}(\mathbb{A}^2)$, *there is an Eisenstein series* $E(\alpha, s)$ *such that*

$$\theta(\alpha; \mathbf{1}, \varphi'') = \frac{1}{2} E(\alpha, 0)$$

for $\alpha \in \mathrm{SL}_2(\mathbb{A})$.

Due to this Eisenstein series, we can calculate the left vertical line of the seesaw. By the seesaw identity, we get the following equations:

Proposition 4.4. *Let* $\mathfrak{c}_0$ *be an ideal of* $\mathfrak{o}$. *Assume that the ideal norm* $C_0 = N(\mathfrak{c}_0)$ *is a square free integer which is prime to* D. *Then, the identity*

$$\sum_{Q \subset Q_D} \chi_Q(-C_0) a_{f_Q}(D) L(1/2, f_Q \times g)$$

$$= \frac{2L(1,\chi)(4\pi)^{2k+1}}{(2k)!} \frac{1}{\#(Cl_K^2)} \sum_{[\mathfrak{c}] \in Cl_K^2} \frac{\langle F_{\mathfrak{c}\mathfrak{c}_0} |_{\mathfrak{H} \times \mathfrak{H}}, g \times g_{CC_0} \rangle}{\langle g_{CC_0}, g_{CC_0} \rangle}$$

holds. Here,

- Q_D *is the set of prime divisors of* D;
- χ_Q *is a quadratic Dirichlet character defined using* χ *and* $Q \subset Q_D$;
- f_Q *is the quadratic twist of* f *by* χ_Q, *which is a normalized Hecke eigenform in* $S_{2k+1}(\Gamma_0(D), \chi)$;
- $Cl_K^2 = \{[\mathfrak{a}]^2 \mid [\mathfrak{a}] \in Cl_K\}$.

We remark that $\chi_\emptyset = 1$, $\chi_{Q_D} = \chi$, $f_\emptyset = f$ and

$$a_{f_{Q_D}}(n) = \overline{a_f(n)}$$

for all $n > 0$. We denote the equation in the above proposition by $I(\mathfrak{c}_0)$. Note that these equations are not the one of Main Theorem. To get Main Theorem, we use the genus theory.

Step(3). The genus theory implies the following lemma:

Lemma 4.5. *For* $Q \subset Q_D$, *the map*

$$\mathfrak{c}_0 \mapsto \chi_Q(N(\mathfrak{c}_0)) \in \{\pm 1\}$$

gives a character of Cl_K / Cl_K^2. *Moreover, this character is trivial on* Cl_K / Cl_K^2 *if and only if* $Q = \emptyset$ *or* $Q = Q_D$.

Consider the equation

$$(Cl_K : Cl_K^2)^{-1} \sum_{[\mathfrak{c}_0] \in Cl_K / Cl_K^2} I(\mathfrak{c}_0).$$

By the orthogonality relations, the left hand side is equal to

$$a_f(D) L(1/2, f \times g) - a_{f_{Q_D}}(D) L(1/2, f_{Q_D} \times g)$$

$$= a_f(D) L(1/2, f \times g) - \overline{a_f(D)} \, \overline{L(1/2, f \times g)}$$

$$= 2 a_f(D) L(1/2, f \times g),$$

since $\chi(-1) = -1$ and $a_f(D)L(1/2, f \times g) \in \sqrt{-1}\mathbb{R}$. On the other hand, in the right hand side, we have

$$\frac{1}{(Cl_K : Cl_K^2)} \frac{1}{\#(Cl_K^2)} \sum_{[\mathfrak{c}_0] \in Cl_K/Cl_K^2} \sum_{[\mathfrak{c}] \in Cl_K^2} = \frac{1}{h_K} \sum_{[\mathfrak{c}] \in Cl_K},$$

since the summands do not depend the choice of representatives of $[\mathfrak{c}]$. These give Main Theorem. $\square$

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