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Further Refinements of Zhan's Inequality for Unitarily Invariant Norms

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Abstract In this report, we show a further improvement of the integral Heinz mean inequality and prove

$$\frac{1}{2}\left|\!\left|\!\left|A^{2}X+2AXB+XB^{2}\right|\!\right|\!\right|\leq\frac{2}{t+2}\left|\!\left|\!\left|A^{2}X+tAXB+XB^{2}\right|\!\right|\!\right| \quad \text{for all} \quad t\in(-2,2].$$

Then we show some refinements of unitarily invariant norm inequalities, in particular we proved that: If $A, B, X \in M_n$ with A and B positive definite, and f, g are two continuous functions on $(0, \infty)$ such that $h(x)=\frac{f(x)}{g(x)}$ is Kwong, then

$$\left|\!\left|\!\left|A^{\frac{1}{2}}(f(A)Xg(B)+g(A)Xf(B))B^{\frac{1}{2}}\right|\!\right|\!\right|\leq\frac{k}{2}\left|\!\left|\!\left|A^{2}X+2AXB+XB^{2}\right|\!\right|\!\right|$$

holds for any unitarily invariant norm, where $k=\max\{\frac{f(\lambda)g(\lambda)}{\lambda}|\lambda\in\sigma(A)\cup\sigma(B)\}$.

1 Introduction

This report is based on [13] and the joint work of M.Fujii and Y.Seo.

A capital letter means an $n\times n$ matrix in the matrix algebra M_n, and $\sigma(A)$ is the set of all eigenvalues of $A\in M_n$. The Schur product of $A=(a_{ij})\in M_n$ and $B=(b_{ij})\in M_n$ is denoted by $A\circ B$, i.e., $A\circ B=(a_{ij}b_{ij})$, $|\!|\!|\cdot|\!|\!|$ denotes a unitarily invariant norm on M_n, that is, $|\!|\!|UAV|\!|\!|=|\!|\!|A|\!|\!|$ for all matrices A, U, V with U, V unitary.

Recent research on norm inequalities is very active. Such inequalities are not only of theoretical interest but also of practical importance. Here we have talk about some important inequalities on unitarily invariant norms.

The following double inequality due to Bhatia and Davis [2] asserts that

$$2\left|\!\left|\!\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right|\leq\left|\!\left|\!\left|A^{r}XB^{1-r}+A^{1-r}XB^{r}\right|\!\right|\!\right|\leq|\!|\!|AX+XB|\!|\!| \tag{1.1}$$

for $A, B, X\in M_n$ with A, B positive semidefinite and $0\leq r\leq 1$. Also (1.1) is equivalent to

$$2|\!|\!|AXB|\!|\!|\leq\left|\!\left|\!\left|A^{r}XB^{2-r}+A^{2-r}XB^{r}\right|\!\right|\!\right|\leq\left|\!\left|\!\left|A^{2}X+XB^{2}\right|\!\right|\!\right| \quad \text{for} \quad 0\leq r\leq 2. \tag{1.2}$$

Recently Kaur et al. [5] showed the following refinement of the Hermite-Hadamard inequality, which improved the left-hand side of (1.1).

Let $A, B, X \in M_n$ with A, B positive semidefinite. Then, for any real numbers α, β, we have

$$\left|\left|\left| A^{\frac{\alpha+\beta}{2}} X B^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}} X B^{\frac{\alpha+\beta}{2}} \right|\right|\right| \leq \frac{1}{|\alpha-\beta|} \left|\left|\left| \int_{\alpha}^{\beta} (A^v X B^{1-v} + A^{1-v} X B^v) dv \right|\right|\right|$$
$$\leq \frac{1}{2} \left|\left|\left| A^{\alpha} X B^{1-\alpha} + A^{1-\alpha} X B^{\alpha} + A^{\beta} X B^{1-\beta} + A^{1-\beta} X B^{\beta} \right|\right|\right| . (1.3)$$

The right-hand side of (1.1), called the Heinz inequality, was generalized by Zhan [12] in the following sense.

Theorem ZH Let $A, B, X \in M_n$ with A, B positive semidefinite. Then

$$|||A^r X B^{2-r} + A^{2-r} X B^r||| \leq \frac{2}{t+2} |||A^2 X + tAXB + XB^2||| \tag{1.4}$$

for any real numbers r, t satisfying $1 \leq 2r \leq 3$ and $-2 < t \leq 2$.
Note that, corresponding to the case r $= 1$, t $= 0$ of the above inequality, the inequality $|||AXB^*||| \leq |||A^*AX + XB^*B|||$ holds for any three matrices $A, B, X \in M_n$[2].
Singh and Vasudeva [10] showed that if $A, B, X \in M_n$ with A, B positive definite, and f is any matrix monotone function on $(0, \infty)$, then for $-2 < t \leq 2$

$$\left|\left|\left| A^{\frac{1}{2}} f(A) X f(B)^{-1} B^{\frac{3}{2}} + A^{\frac{3}{2}} f(A)^{-1} X f(B) B^{\frac{1}{2}} \right|\right|\right| \leq \frac{2}{t+2} |||A^2 X + tAXB + XB^2||| . \tag{1.5}$$

A continuous real-valued function f defined on an interval (a, b) with $a \geq 0$ is called a Kwong function [1] if the matrix

$$K_f = \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i + \lambda_j} \right)_{i,j=1,2,\cdots,n}$$

is positive semidefinite for any distinct real numbers $\lambda_1, \cdots, \lambda_n$ in (a, b). It is easy to see that if f is a non-zero Kwong function then f is positive and $\frac{1}{f}$ is Kwong. Kwong [7] showed that the set of all Kwong functions on $(0, \infty)$ is a closed cone and includes all non-negative operator monotone functions on $(0, \infty)$. Also Audenaert [1] gave a characterization of Kwong functions by showing that, for fixed $0 \leq a < b$, a function f on an interval (a, b) is Kwong if and only if the function $g(x) = \sqrt{x} f(\sqrt{x})$ is operator monotone on (a^2, b^2). Afterwards, Najafi [9] obtained a more generalized norm inequality of the Heinz inequality.

$$|||f(A)Xg(B) + g(A)Xf(B)||| \leq |||AX + XB|||$$

for any continous functions $f(x)$, $g(x)$ with $\frac{f(x)}{g(x)}$ Kwong and $f(x)g(x) \leq x$.

Recently, Fujii, Seo and Zuo [3] showed some improvements and generalizations of the unitarily invariant norm inequalities via Hadamard product, among others we show that if $A, B, X \in M_n$ such that A, B are positive definite, f and g are two continuous functions on $(0,\infty)$ such that $\frac{f(x)}{g(x)}$ is Kwong, and $k = \max\{\frac{f(\lambda)g(\lambda)}{\lambda} | \lambda \in \sigma(A) \cup \sigma(B)\}$, then for any $\beta > 0$

$$\left|\left|\left| A^{\frac{1}{2}}(f(A)Xg(B) + g(A)Xf(B))B^{\frac{1}{2}} \right|\right|\right| \\ \leq k \left|\left|\left| \beta_0 AXB + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXB + XB^2) \right|\right|\right|$$

holds for $-2 < t \leq 2\beta - 2$, where$\beta_0 = 2(1 - 2\beta + 2\beta r_0)$, $r_0 = \min\{\frac{1}{2} + |1-r|, 1 - |1-r|\}$. Moreover,

$$\left|\left|\left| A^{\frac{1}{2}}[f(A)Xg(B) + g(A)Xf(B)]B^{\frac{1}{2}} \right|\right|\right| \leq \frac{2k}{t+2} \left|\left|\left| A^2X + tAXB + XB^2 \right|\right|\right|$$

holds for $-2 < t \leq 2$. For a comprehensive inspection of the results concerning the above norm inequalities, the reader is referred to [4, 6, 8, 11].

In this report we show some further improvements of the above inequalities (1.1)–(1.5). Especially, if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\left|\left|\left| A^rXB^{2-r} + A^{2-r}XB^r \right|\right|\right| \leq \frac{1}{2} \left|\left|\left| A^2X + 2AXB + XB^2 \right|\right|\right| \leq \frac{2}{t+2} \left|\left|\left| A^2X + tAXB + XB^2 \right|\right|\right|$$

holds for $1 \leq 2r \leq 3$ and $-2 < t \leq 2$.

2 Refined Zhan's inequality

In this section, we show a refined version of Zhan's inequality (1.4). First of all, we study the function at the right-hand side of the inequality (1.4).

Theorem 2.1 *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite. Suppose that*

$$\Psi(t) = \frac{2}{t+2} \left|\left|\left| A^2X + tAXB + XB^2 \right|\right|\right|, \quad t \in (-2, 2].$$

Then $\Psi(t)$ is monotone decreasing on $(-2, 2]$. In particular,

$$\left|\left|\left| A^rXB^{2-r} + A^{2-r}XB^r \right|\right|\right| \leq \frac{1}{2} \left|\left|\left| A^2X + 2AXB + XB^2 \right|\right|\right| \\ \leq \frac{2}{t+2} \left|\left|\left| A^2X + tAXB + XB^2 \right|\right|\right| \tag{2.1}$$

holds for $1 \leq 2r \leq 3$ and $t \in (-2, 2]$.

Next we show a generalized Zhan's inequality, which contains the result (1.5) due to Singh-Vasudeva. For this, we need the following lemmas.

Lemma 2.2 *Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be any positive real numbers. If f and g are two continuous functions on $(0,\infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong, then the $n \times n$ matrix*

$$W = \left(\frac{f(\sigma_i)g^{-1}(\sigma_i) + f(\sigma_j)g^{-1}(\sigma_j)}{\sigma_i^2 + 2\sigma_i\sigma_j + \sigma_j^2} \right)_{i,j=1,2,\cdots,n}$$

is positive semidefinite.

Lemma 2.3 [6, p.343] *If $X = (x_{ij})$ is positive semidefinite, then for any matrix Y*

$$|||X \circ Y||| \leq \max_{1\leq i\leq n} x_{ii} \, |||Y||| \, .$$

Theorem 2.4 *Suppose that $A, B, X \in \mathbb{M}_n$ such that A, B are positive definite, and f, g are two continuous functions on $(0, \infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong. Then*

$$\left|\left|\left| A^{\frac{1}{2}}(f(A)Xg(B) + g(A)Xf(B))B^{\frac{1}{2}} \right|\right|\right| \leq \frac{k}{2} \, |||A^2X + 2AXB + XB^2|||$$

holds for $k = \max\{\frac{f(\lambda)g(\lambda)}{\lambda} | \lambda \in \sigma(A) \cup \sigma(B)\}$.

Remark 2.5 *Theorem 2.4 can be seen as a refinement of (1.5): Let f be an operator monotone function on $(0, \infty)$ and $g(x) = xf^{-1}(x)$. Since f is operator monotone, we have $\sqrt{x}(f/g)(\sqrt{x}) = f^2(\sqrt{x})$ is operator monotone. Hence it follows from Audenaert's result that $f(x)/g(x) = x^{-1}f^2(x)$ is Kwong [7] and $k = \max\{\frac{f(\lambda)g(\lambda)}{\lambda} | \lambda \in \sigma(A) \cup \sigma(B)\} = 1$, and this implies that for $-2 < t \leq 2$*

$$\begin{aligned} \left|\left|\left| A^{\frac{1}{2}}f(A)Xf(B)^{-1}B^{\frac{3}{2}} + A^{\frac{3}{2}}f(A)^{-1}Xf(B)B^{\frac{1}{2}} \right|\right|\right| &\leq \frac{1}{2} \, |||A^2X + 2AXB + XB^2||| \\ &\leq \frac{2}{t+2} \, |||A^2X + tAXB + XB^2||| \end{aligned}$$

holds for $A, B, X \in \mathbb{M}_n$ with A, B positive definite.

Using the method in the proof of Theorem 2.4 again we get the following theorem.

Theorem 2.6 *Suppose that $A, B, X \in \mathbb{M}_n$ such that A, B are positive definite. If f and g are two continuous functions on $(0, \infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong, then*

$$|||f(A)Xg(B) + g(A)Xf(B)||| \leq \frac{k'}{2} \, |||A^2X + 2AXB + XB^2|||$$

holds for $k' = \max\{\frac{f(\lambda)g(\lambda)}{\lambda^2} | \lambda \in \sigma(A) \cup \sigma(B)\}$.

Example 2.7 *Take $f(x) = \log(1 + x)$ and $g(x) = x$ defined on $(0, \infty)$. Then $f(x)g(x)^{-1}$ is not operator monotone but Kwong [9]. Theorem 2.4 leads to the following inequality:*

$$\left|\left|\left| A^{\frac{1}{2}}(\log(I + A)XB + AX\log(I + B))B^{\frac{1}{2}} \right|\right|\right| \leq \frac{\log(1 + \lambda_0)}{2} \, |||A^2X + 2AXB + XB^2|||$$

for $A, B, X \in M_n$ with A, B positive semidefinite, where $\lambda_0 = \max\{\lambda | \lambda \in \sigma(A) \cup \sigma(B)\}$.

Example 2.8 *Let $s, r \in \mathbb{R}$. Since $F(x) = x^{s-r}$ defined on $(0, \infty)$ is Kwong if and only if $-1 \leq s - r \leq 1$, it follows from Theorem 2.4 that if $A, B, X \in M_n$ with A, B positive semidefinite and $|s - r| \leq 1$, then*

$$\left|\left|\left| A^{\frac{1}{2}}(A^sXB^r + A^rXB^s)B^{\frac{1}{2}} \right|\right|\right| \leq \frac{k}{2} \, |||A^2X + 2AXB + XB^2|||$$

for $k = \max\{\lambda^{s+r-1} | \lambda \in \sigma(A) \cup \sigma(B)\}$. In particular, if we put $r \mapsto r - \frac{1}{2}$ and $s \mapsto \frac{3}{2} - r$ in the above inequality for $1 \leq 2r \leq 3$, then we have $|r - s| \leq 1$ and $k = 1$. Hence we have Zhan's inequality (1.4) by Theorem 2.1.

3 Refined integral Heinz mean inequality

Next we prove a stronger matrix version of the integral Heinz mean (1.3) by using the same integration technique as that in [5].

Theorem 3.1 *Let $A, B, X \in \mathbb{M}_n$ with A and B positive semidefinite. Then for any real positive numbers α and β,*

$$\frac{4}{|\alpha-\beta|}\left|\left|\left|\int_\alpha^\beta (A^vXB^{1-v}+A^{1-v}XB^v)dv\right|\right|\right| \leq$$
$$\left|\left|\left|2A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}}+2A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}+A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha+A^\beta XB^{1-\beta}+A^{1-\beta}XB^\beta\right|\right|\right|.$$

As a corollary of Theorem 3.1. we have a refinement of the left-hand sides of the inequality (1.2):

Corollary 3.2 *Let $A, B, X \in \mathbb{M}_n$ with A and B positive semidefinite and $r \in [\frac{1}{2}, \frac{3}{2}]$. Then*

$$\begin{aligned}
2\,|||AXB||| &\leq \frac{1}{|2-2r|}\left|\left|\left|\int_r^{2-r} (A^vXB^{2-v}+A^{2-v}XB^v)dv\right|\right|\right| \\
&\leq \frac{1}{2}\,|||2AXB+A^rXB^{2-r}+A^{2-r}XB^r||| \leq |||A^rXB^{2-r}+A^{2-r}XB^r|||\,.
\end{aligned}$$

Further,

$$\lim_{r\to 1}\frac{1}{|2-2r|}\left|\left|\left|\int_r^{2-r} (A^vXB^{2-v}+A^{2-v}XB^v)dv\right|\right|\right| = 2\,|||AXB|||\,.$$

By Theorem 3.1 we have the following integral inequalities as an improvement of inequality (1.1).

Corollary 3.3 *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite and $r \in [0,1]$. Then*

$$\begin{aligned}
2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leq \frac{1}{|1-2r|}\left|\left|\left|\int_r^{1-r} (A^vXB^{1-v}+A^{1-v}XB^v)dv\right|\right|\right| \\
&\leq \frac{1}{2}\left|\left|\left|2A^{\frac{1}{2}}XB^{\frac{1}{2}}+A^rXB^{1-r}+A^{1-r}XB^r\right|\right|\right| \\
&\leq \left|\left|\left|\alpha(A^rXB^{1-r}+A^{1-r}XB^r)+2(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \quad \textit{for} \quad \frac{1}{2}\leq\alpha\leq 1 \\
&\leq |||A^rXB^{1-r}+A^{1-r}XB^r|||\,.
\end{aligned}$$

By the symmetry of the integral function and Theorem 3.1 we have the following corollary:

Corollary 3.4 *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite and $r \in [0,1]$. Then*

$$\begin{aligned} & 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \\ \le\ & \frac{1}{|1-2r|}\left|\left|\left|\int_r^{1-r}(A^vXB^{1-v}+A^{1-v}XB^v)dv\right|\right|\right| \\ \le\ & \frac{1}{4}\left|\left|\left|4A^{\frac{1}{2}}XB^{\frac{1}{2}}+A^{1-r}XB^r+A^rXB^{1-r}+A^{\frac{1+2r}{4}}XB^{\frac{3-2r}{4}}+A^{\frac{3-2r}{4}}XB^{\frac{1+2r}{4}}\right|\right|\right| \\ \le\ & \left|\left|\left|A^{1-r}XB^r+A^rXB^{1-r}\right|\right|\right|. \end{aligned}$$

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