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EQUIVALENCE RELATIONS AMONG SOME INEQUALITIES ON OPERATOR MEANS

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ABSTRACT. We will consider about some inequalities for operator means for more than three operators, for instance, ALM and BMP geometric means will be considered. Moreover, log-Euclidean and logarithmic means for several operators will be treated.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space, and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator A is said to be positive semi-definite (resp. positive definite) if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ (resp. $\langle Ax, x\rangle > 0$ for all non-zero $x \in \mathcal{H}$). Let $\mathbb{P}$ and $\mathbb{S}$ be the sets of all positive definite and self-adjoint operators, respectively. From this, we can consider the order among $\mathbb{S}$, i.e., for $A, B \in \mathbb{S}$,

$$A \leq B \quad \text{if and only if} \quad 0 \leq B - A.$$

A real valued function f on an interval $J \subset \mathbb{R}$ is called the *operator monotone function* if and only if

$$A \leq B \quad \text{implies} \quad f(A) \leq f(B)$$

holds for all $A, B \in \mathbb{S}$ whose spectral are contained in J.

Kubo-Ando [7] have shown the following important result:

Theorem A ([7]). *For each operator connection σ, there exists a unique operator monotone function $f : (0, +\infty) \longrightarrow (0, +\infty)$ such that*

$$f(t)I = I\sigma(tI) \quad \text{for all } t \in (0, +\infty),$$

and for $A > 0$ and $B \geq 0$, the formula

$$A\sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

holds, where the right hand side is defined via the analytic functional calculus. Moreover if $f(1) = 1$, then an operator connection σ corresponding to f is an operator mean. An operator monotone function f is called a representing function of σ.

Typical examples of operator means are harmonic, geometric and arithmetic means denoted by !, # and ∇, respectively. Their representing functions are $[\frac{1}{2} + \frac{1}{2}t^{-1}]^{-1}$, $t^{\frac{1}{2}}$ and $\frac{1+t}{2}$, respectively.

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Extending Kubo-Ando theory to three or more operators was a long standing problem, in particular, we did not have any nice definition of geometric mean of three operators. Recently, Ando-Li-Mathias [2] have given a nice definition of geometric mean for n-tuple of positive operators. Then many authors study about operator means for n-tuple of positive operators, and now we have three definitions of geometric means which are called ALM, BMP and the Karcher means. Moreover, we have an extension of the Karcher mean which is called the power mean. It is defined by the unique positive solution of the following operator equation: For $t \in [-1,1]\setminus\{0\}$,

$$\sum_{i=1}^{n} w_i X \sharp_t A_i = X.$$

for $A_1, ..., A_n \in \mathbb{P}$ and probability vector $\omega = (w_1, ..., w_n) \in (0,1)^n$ i.e., $\sum_{i=1}^{n} w_i = 1$. If $t = 0$, we can consider the power mean as the Karcher mean.

M. Uchiyama and one of the authors have obtained equivalence relations between inequalities for the power and arithmetic means as applications of a converse of Loewner-Heinz inequality [16].

In this report, we shall investigate the previous research to other operator means for n-tuples of operators. In fact, we shall treat ALM and BMP means, moreover we shall discuss about some types of logarithmic means of several operators. This report is organized as follows: In Section 2, we will introduce some definitions and notations which will be used in the report. Then we shall consider the weighted operator means in the view point of their representing functions in Section 3. In Section 4, we shall consider about generalizations of the results by M. Uchiyama and one of the authors. Especially, we shall consider about log-Euclidean and logarithmic means. In the last section, we shall introduce some properties of the M-logarithmic mean which is generated from an arbitrary operator mean via integration.

2. Primarily

Let OM be the set of all operator monotone functions on $(0,\infty)$, and let $OM_1 = \{f \in OM : f(1) = 1\}$. For $f \in OM_1$, there exists an operator mean σ_f such that

$$A\sigma_f B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

for positive operators A and B. It is well known that if

$$A!B \leq A\sigma_f B \leq A\nabla B$$

holds for all positive operators A and B, where ! and ∇ mean harmonic and arithmetic means, then

$$\left(\frac{1+t^{-1}}{2}\right)^{-1} \leq f(t) \leq \frac{1+t}{2}.$$

holds for all $t > 0$.

For n-tuples of positive definite operators, the ALM and BMP (geometric) means are defined as follows:

Theorem B (ALM mean [2]). *For $\mathbb{A} = (A_1, A_2) \in \mathbb{P}^2$, the ALM (geometric) mean $\mathfrak{G}_{ALM}(\mathbb{A})$ of A_1 and A_2 is defined by $\mathfrak{G}_{ALM}(\mathbb{A}) = A_1 \sharp A_2$. Assume that the ALM* *(geometric) mean* $\mathfrak{G}_{ALM}(\cdot)$ *of* $(n-1)$*-tuples of positive definite operators is defined. Let* $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ *and* $\{A_i^{(r)}\}_{r=0}^{\infty}$ $(i = 1, ..., n)$ *be the sequences of positive definite operators defined by*

$$A_i^{(0)} = A_i \quad \textit{and} \quad A_i^{(r+1)} = \mathfrak{G}_{ALM}\left((A_j^{(r)})_{j \neq i}\right).$$

Then there exists $\lim_{r\to\infty} A_i^{(r)}$ $(i = 1, ..., n)$ *and it does not depend on* i*. The ALM (geometric) mean* $\mathfrak{G}_{ALM}(\mathbb{A})$ *is defined by* $\lim_{r\to\infty} A_i^{(r)}$.

Theorem C (BMP mean [4, 6, 10]). *For* $\mathbb{A} = (A_1, A_2) \in \mathbb{P}^2$ *and* $\omega = (1-w, w) \in (0,1)^2$*, the BMP (geometric) mean* $\mathfrak{G}_{BMP}(\omega; \mathbb{A})$ *of* A_1 *and* A_2 *is defined by* $\mathfrak{G}_{BMP}(\omega; \mathbb{A}) = A_1 \sharp_w A_2$*. Assume that the BMP (geometric) mean* $\mathfrak{G}_{BMP}(\cdot;\cdot)$ *of* $(n-1)$*-tuples of positive definite operators is defined. Let* $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ *and* $\omega = (w_1, ..., w_n)$ *be a probability vector. Define* $\{A_i^{(r)}\}_{r=0}^{\infty}$ $(i = 1, ..., n)$ *the sequences of positive definite operators defined by*

$$A_i^{(0)} = A_i \quad \textit{and} \quad A_i^{(r+1)} = \mathfrak{G}_{BMP}\left(\hat{\omega}_{\neq i}; (A_j^{(r)})_{j\neq i}\right) \sharp_{w_i} A_i^{(r)},$$

where $\hat{\omega}_{\neq i} = \sum_{j \neq i} w_j$*. Then there exists* $\lim_{r\to\infty} A_i^{(r)}$ $(i = 1, ..., n)$ *and it does not depend on* i*. The BMP (geometric) mean* $\mathfrak{G}_{BMP}(\omega; \mathbb{A})$ *is defined by* $\lim_{r\to\infty} A_i^{(r)}$.

We remark that it is not known any weighted ALM mean. Let $\mathbb{A} = (A_1, ..., A_n), \mathbb{B} = (B_1, ..., B_n) \in \mathbb{P}^n$ and probability vector $\omega = (w_1, ..., w_n)$. Here we denote the above geometric means of $\mathbb{A} = (A_1, ..., A_n)$ for the weight $\omega = (w_1, ..., w_n)$ by $\mathfrak{G}(\omega; \mathbb{A})$, and they have at least 10 basic properties as follows (in ALM mean case, we consider just only $\omega = (\frac{1}{n}, ..., \frac{1}{n})$ case):

(P1) If $A_1, ..., A_n$ commute with each other, then

$$\mathfrak{G}(\omega; \mathbb{A}) = \prod_{k=1}^{n} A_k^{w_k}.$$

(P2) For positive numbers $a_1, ..., a_n$,

$$\mathfrak{G}(\omega; a_1 A_1, ..., a_n A_n) = \mathfrak{G}(\omega; a_1, ..., a_n)\mathfrak{G}(\omega; \mathbb{A}) = \left(\prod_{k=1}^{n} a_k^{w_k}\right) \mathfrak{G}(\omega; \mathbb{A}).$$

(P3) For any permutation σ on $\{1, 2, ..., n\}$,

$$\mathfrak{G}(w_{\sigma(1)}, ..., w_{\sigma(n)}; A_{\sigma(1)}, ..., A_{\sigma(n)}) = \mathfrak{G}(\omega; \mathbb{A}).$$

(P4) If $A_i \leq B_i$ for $i = 1, ..., n$, then

$$\mathfrak{G}(\omega; \mathbb{A}) \leq \mathfrak{G}(\omega; \mathbb{B}).$$

(P5) $\mathfrak{G}(\omega; \cdot)$ is continuous on each operators. Especially,

$$d(\mathfrak{G}(\omega; \mathbb{A}), \mathfrak{G}(\omega; \mathbb{B})) \leq \sum_{i=1}^{n} w_i d(A_i, B_i),$$

where $d(\cdot, \cdot)$ means the Thompson metric.

(P6) For each $t \in [0,1]$,

$$(1-t)\mathfrak{G}(\omega;\mathbb{A}) + t\mathfrak{G}(\omega;\mathbb{B}) \leq \mathfrak{G}(\omega;(1-t)\mathbb{A}+t\mathbb{B}).$$

(P7) For any invertible $X \in B(\mathcal{H})$,

$$\mathfrak{G}(\omega; X^*A_1X, ..., X^*A_nX) = X^*\mathfrak{G}(\omega;\mathbb{A})X.$$

(P8) $\mathfrak{G}(\omega;\mathbb{A}^{-1})^{-1} = \mathfrak{G}(\omega;\mathbb{A})$, where $\mathbb{A}^{-1} = (A_1^{-1}, ..., A_m^{-1})$.

(P9) If every A_i is matrix, then $\det \mathfrak{G}(\omega;\mathbb{A}) = \prod_{i=1}^n \det A_i^{w_i}$.

(P10)

$$\left[\sum_{i=1}^n w_i A_i^{-1}\right]^{-1} \leq \mathfrak{G}(\omega;\mathbb{A}) \leq \sum_{i=1}^n w_i A_i.$$

3. Operator means of two variables

In this section, we shall consider the weighted operator means in the view point of their weight.

Theorem 1. *Let Φ and f be non-constant operator monotone functions on $(0,\infty)$ with $\Phi(1) = f(1) = 1$, and let σ be an operator mean whose representing function is Φ. Let $w \in (0,1)$. For self-adjoint operators A and B, they are mutually equivalent:*

1. *(1) $(1-w)A \leq wB$ iff $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ holds for all sufficiently small $\lambda > 0$,*
2. *(2) $\Phi'(1) = w$.*

Theorem 1 is an extension of the following Theorem D in [16]. It was shown as a converse of Loewner-Heinz inequality.

Theorem D ([16]). *Let $f(t)$ be an operator monotone function on $(0,\infty)$ with $f(1) = 1$, and let A and B be bounded self-adjoint operators. Let σ be an operator mean satisfying $! \leq \sigma \leq \nabla$. Then $A \leq B$ if and only if $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$.*

4. More than three operators case

Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}^n$. Define $\mathcal{A}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^n A_i$ and $\mathcal{H}(\mathbb{A}) = \left(\frac{1}{n}\sum_{i=1}^n A_i^{-1}\right)^{-1}$.
Let Δ_n be the set of all probability vectors, i.e.,

$$\Delta_n = \{\omega = (w_1, ..., w_n) \in (0,1)^n; \sum_{i=1}^n w_i = 1\}.$$

An an extension of the Karcher mean, the power mean is given by Lim-Páifia [11]. Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. For $t \in [-1,1] \setminus \{0\}$, the power mean $P_t(\omega;\mathbb{A})$ is defined by the unique positive definite solution of

$$X = \sum_{k=1}^n w_k X \sharp_t A_k.$$

We remark that $P_t(\omega;\mathbb{A})$ converges to the Karcher mean $\Lambda(\omega;\mathbb{A})$ as $t \to 0$, strongly. So we can consider $P_0(\omega;\mathbb{A})$ as $\Lambda(\omega;\mathbb{A})$. It is easy to see that $P_1(\omega;\mathbb{A}) = \mathcal{A}(\omega;\mathbb{A})$ and $P_{-1}(\omega;\mathbb{A}) = \mathcal{H}(\omega;\mathbb{A})$. Moreover $P_t(\omega;\mathbb{A})$ is increasing on $t \in [-1,1]$. Hence the power mean interpolates arithmetic-geometric-harmonic means. In [16], we have generalization of Theorem D as follows:

Theorem E ([16]). *Let $A_1, \dots, A_n$ be Hermitian matrices, and $\omega = (w_1, \dots, w_n) \in \Delta_n$. Let $f(t)$ be a non-constant operator monotone function on $(0,\infty)$ with $f(1) = 1$. Then the following are equivalent:*

(1) $\sum_{i=1}^{n} w_i A_i \leq 0$,

(2) $P_1(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) = \sum_{i=1}^{n} w_i f(\lambda A_i + I) \leq I$ *for all sufficiently small* $\lambda \geq 0$,

(3) *for each* $t \in [-1,1]$, $P_t(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \leq I$ *for all sufficiently small* $\lambda \geq 0$.

Here we shall generalize the above result into the following Theorem 2:

Theorem 2. *Let f be an strictly operator monotone function on $(0,\infty)$ with $f(1) = 1$, and let $\Phi(\omega;\mathbb{A},x) : \Delta_n \times \mathbb{S}^n \times \mathcal{H} \to \mathbb{R}^+$ satisfying*

$$\|\mathcal{H}(\omega;\mathbb{A})\| \leq \sup_{\|x\|=1} \Phi(\omega;\mathbb{A},x) \leq \|\mathcal{A}(\omega;\mathbb{A})\|. \tag{4.1}$$

for all $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{S}^n$ and $\omega \in \Delta_n$. Then they are mutually equivalent:

(1) $\sum_{i=1}^{n} w_i A_i \leq 0$,

(2) $\Phi(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I), x) \leq 1$ *for all sufficiently small $\lambda > 0$ and all unit vector $x \in \mathcal{H}$.*

From here we shall consider another geometric mean for n-tuples of operators which is called log-Euclidean mean $E(\omega;\mathbb{A})$. It is defined by

$$E(\omega;\mathbb{A}) = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right).$$

Unfortunately, log-Euclidean mean does not have the monotonicity property.

Corollary 3. *Let f be an strictly operator monotone function on $(0,\infty)$ with $f(1) = 1$. Let $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{S}^n$, $\omega = (w_1, \dots, w_n) \in \Delta_n$ and let $M(\omega;\mathbb{A})$ be ALM or weighted BMP or log-Euclidean mean (in the ALM mean case, ω should be $\omega = (\frac{1}{n}, \dots, \frac{1}{n})$). Then the following assertions are equivalent:*

(1) $\sum_{i=1}^{n} w_i A_i \leq 0$,

(2) $M(\omega; f(\lambda A_1 + I), \dots, f(\lambda A_n + I)) \leq I$ *for all sufficiently small* $\lambda > 0$.

5. Logarithmic means

We shall consider logarithmic means for more than 3-operators. Since the representing function of logarithmic mean is $\frac{t-1}{\log t}$, logarithmic mean $A\lambda B$ of two operators A and B can be considered as the following formula:

$$A\lambda B = \int_0^1 A\natural_t B dt.$$

So it is quite natural to consider the similar type of integrated means as follows:

Definition 1 (M-logarithmic mean). *Let $M: \Delta_n \times \mathbb{P}^n \to \mathbb{P}$. Then for $\mathbb{A} \in \mathbb{P}^n$, the M-logarithmic mean $L(M)(\mathbb{A})$ of $\mathbb{A} \in \mathbb{P}^n$ is defined by*

$$L(M)(\mathbb{A}) = \int_{\omega \in \Delta_n} M(\omega; \mathbb{A}) dp(\omega)$$

if there exists, where $dp(\omega)$ means an arbitrary probability measure.

In what follows, we consider the case of $dp(\omega) = (n-1)!d\omega$. Since the weighted Karcher mean $\Lambda(\omega; \mathbb{A})$ is continuous on the probability vector according to the Thompson metric [9], so $L(\Lambda)(\mathbb{A})$ exists.

Corollary 4. *Logarithmic mean $L(\Lambda)(\mathbb{A})$ satisfies the same assertion to Corollary 3.*

Proposition 5. *Let $M: \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ satisfy (P3), (P7), (P8) and (P10). Then M-logarithmic mean satisfies (P3) and (P7) if it exists. Especially, $L(M)$ satisfies (P10), i.e.,*

$$\mathcal{H}(\mathbb{A}) \leq L(M)(\mathbb{A}) \leq \mathcal{A}(\mathbb{A}).$$

We remark that $L(\mathcal{A})(\mathbb{A}) = \mathcal{A}(\mathbb{A})$, i.e., arithmetic mean is a fixed point for the map L. As for the preparation, we define some notations. Let S be the cyclic shift operator on $\mathbb{C}^n$ and let $\mathbb{S}$ be also the cyclic shift operator on $B(\mathcal{H})^n$; namely,

$$S(w_1, w_2, ..., w_n) = (w_2, w_3, ..., w_n, w_1).$$

Remark 6. Let $M: \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ be a map satisfying (P3), (P7), (P8) and (P10). We put

$$M_0(\omega; \mathbb{A}) := M\left((\frac{1}{n}, ..., \frac{1}{n}); M(\omega; \mathbb{A}), M(S\omega; \mathbb{A}), ..., M(S^{n-1}\omega; \mathbb{A}) \right).$$

Then M_0 satisfies the assumption of the above theorem. So $L(M_0)$ satisfies (P10). Moreover, the following inequalities hold

$$\mathcal{H}(\mathbb{A}) \leq L(M_0)(\mathbb{A}) \leq L(M)(\mathbb{A}) \leq \mathcal{A}(\mathbb{A}).$$

If M is a geometric mean, then it can be seen as an extension of harmonic-geometric-logarithmic-arithmetic means inequalities.

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