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Relative position of three subspaces in a Hilbert space (a summary)

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This is a summary of a preprint [1], which is a joint work with Masatoshi Enomoto.

1. Introduction.

We study the relative position of three subspaces in a separable infinite-dimensional Hilbert space. In the finite-dimensional case, Brenner described the general position of three subspaces completely. We extend it to a certain class of three subspaces in an infinite-dimensional Hilbert space. The relative position of one subspace of a Hilbert space is extremely simple and determined by the dimension and the co-dimension of the subspace. It is a well known fact that the relative position of two subspaces E and F in a Hilbert space H can be described completely up to unitary equivalence. The Hilbert space is the direct sum of five subspaces:

$$H = (E \cap F) \oplus (\text{the rest}) \oplus (E \cap F^{\perp}) \oplus (E^{\perp} \cap F) \oplus (E^{\perp} \cap F^{\perp}).$$

In the rest part, E and F are in generic position and the relative position is described only by "the angles" between them. We disregard "the angles" and study the still-remaining fundamental feature of the relative position of subspaces. This is the reason why we use bounded invertible operators instead of unitaries to define isomorphisms. Let H be a Hilbert space and $E_1, \dots E_n$ be n subspaces in H. Then we say that $\mathcal{S} = (H; E_1, \dots, E_n)$ is a system of n subspaces in H or an n-subspace system in H. Let $\mathcal{T} = (K; F_1, \dots, F_n)$ be another system of n-subspaces in a Hilbert space

K. We say that systems $\mathcal{S}$ and $\mathcal{T}$ are *isomorphic* if there is a bounded invertible operator $\varphi : H \to K$ satisfying that $\varphi(E_i) = F_i$ for $i = 1, \ldots, n$. We say that a system $\mathcal{S} = (H; E_1, E_2, E_3)$ of three subspaces in a Hilbert space H *forms a double triangle* if the family $\{H, E_1, E_2, E_3, 0\}$ is a double triangle lattice, (which is also called a diamond), that is,

$$E_i \vee E_j = H, \quad \text{and} \quad E_i \wedge E_j = 0, \quad (i \neq j, i, j = 1, 2, 3).$$

and each $E_i \neq H$, $E_i \neq 0$. We remark that the distributive law fails in any double triangle.

$$(E_1 \vee E_2) \wedge E_3 \neq (E_1 \wedge E_2) \vee (E_1 \wedge E_3).$$

S. Brenner gave a complete description of systems of three subspaces up to isomorphims when an ambient space H is finite-dimensional:

Theorem 1.(S. Brenner) Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be a system of three subspaces in a finite-dimensional Hilbert space H. Then $\mathcal{S}$ is isomorphic to the following $\mathcal{T} = (H; F_1, F_2, F_3)$ such that there exist subspaces $S, N_1, N_2, N_3, M_1, M_2, M_3, Q, L$ of H satisfying that Q has a form

$$(Q; Q_1, Q_2, Q_3) := (K \oplus K; K \oplus 0, 0 \oplus K, \{(x, x) \mid x \in K\})$$

of double triangle and

$$\begin{aligned}
H &= S \oplus N_1 \oplus N_2 \oplus N_3 \oplus M_1 \oplus M_2 \oplus M_3 \oplus Q \oplus L \\
F_1 &= S \oplus 0 \quad \oplus N_2 \oplus N_3 \oplus M_1 \oplus 0 \quad \oplus 0 \quad \oplus Q_1 \oplus 0 \\
F_2 &= S \oplus N_1 \oplus 0 \quad \oplus N_3 \oplus 0 \quad \oplus M_2 \oplus 0 \quad \oplus Q_2 \oplus 0 \\
F_3 &= S \oplus N_1 \oplus N_2 \oplus 0 \quad \oplus 0 \quad \oplus 0 \quad \oplus M_3 \oplus Q_3 \oplus 0
\end{aligned}$$

Remark. The above Brenner's theorem says that any system of three subspaces of a finite-dimensional Hilbert space is decomposed as a direct sum of a distributive part (or Boolean part)

$$S \oplus N_1 \oplus N_2 \oplus N_3 \oplus M_1 \oplus M_2 \oplus M_3 \oplus L$$

and a non-distributive part Q. The double triangle is the *only* obstruction of distributive law in finite-dimensional case.

2. Brenner type decomposition.

We study Brenner type of decomposition for a certain class of systems of three subspaces for an infinite-dimensional Hilbert space.

Definition. Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space H. Then $\mathcal{S}$ is said to have a *Brenner type decomposition* if $\mathcal{S}$ is isomorphic to a system $\mathcal{T} = (H; F_1, F_2, F_3)$ satisfying that there exist subspaces $S, N_1, N_2, N_3, M_1, M_2, M_3, Q, L$ of H such that $(Q; Q_1, Q_2, Q_3)$ forms a double triangle and

$$\begin{aligned}
H &= S \oplus N_1 \oplus N_2 \oplus N_3 \oplus M_1 \oplus M_2 \oplus M_3 \oplus Q \oplus L \\
F_1 &= S \oplus 0 \quad \oplus N_2 \oplus N_3 \oplus M_1 \oplus 0 \quad \oplus 0 \quad \oplus Q_1 \oplus 0 \\
F_2 &= S \oplus N_1 \oplus 0 \quad \oplus N_3 \oplus 0 \quad \oplus M_2 \oplus 0 \quad \oplus Q_2 \oplus 0 \\
F_3 &= S \oplus N_1 \oplus N_2 \oplus 0 \quad \oplus 0 \quad \oplus 0 \quad \oplus M_3 \oplus Q_3 \oplus 0
\end{aligned}$$

Theorem 2. Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space H. Then the followings are equivalent:

1. Linear sums $E_i + E_j$ and $(E_i \cap E_k) + (E_j \cap E_k)$ are closed for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$ and the quotient space $(E_3 \wedge (E_1 \vee E_2))/((E_3 \wedge E_1) \vee (E_3 \wedge E_2))$ is finite-dimensional.

2. $\mathcal{S}$ has a Brenner type decomposition with a finite-dimensional double triangle part Q.

Moreover if these equivalent conditions are satisfied, then the double triangle part Q is isomorphic to a typical form, i.e.

$$(Q; Q_1, Q_2, Q_3) \cong (K \oplus K; K \oplus 0, 0 \oplus K, \{(x, x) \mid x \in K\})$$

for some Hilbert space K.

Theorem 3. Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space H. Then the followings are equivalent:

1. Linear sums $(E_i \vee E_j) + E_k$ and $(E_i \cap E_j) + E_k$ are closed for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$.

2. $\mathcal{S}$ has a Brenner type decomposition.

Reference

[1]M. Enomoto and Y. Watatani, Relative position of three subspaces in a Hilbert space, preprint, arXiv;1407.6852v2 [Math.OA].

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