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$1<|t|<2$ に対する Lawson-Lim-Pálfia による作用素冪平均について

Operator Power means due to Lawson-Lim-Pálfia for $1<|t|<2$

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1. Introduction

This paper is based on [6].

For a weight vector $\omega=(\omega_1,\ldots,\omega_n)$ such as $\omega_i\geq 0$ for $i=1,\ldots,n$ and $\sum_{i=1}^{n}\omega_i=1$ and positive invertible operators $\mathbb{A}=(A_1,\ldots,A_n)$, the power mean $P_t(\omega;\mathbb{A})$ for $t\in[-1,1]\backslash\{0\}$ due to Lawson-Lim-Pálfia [4, 5] is defined by the unique positive invertible solution of the following non-linear equation:

$$X=\sum_{i=1}^{n}\omega_i(X\sharp_tA_i)\qquad \text{for } t\in(0,1]$$

$$X=\left[\sum_{i=1}^{n}\omega_i(X^{-1}\sharp_{-t}A_i^{-1})\right]^{-1}\qquad \text{for } t\in[-1,0)$$

where $A\sharp_tB=A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}$ is the t-weighted geometric mean of A and B.

For a weight vector $\omega=(\omega_1,\ldots,\omega_n)$ and positive invertible operators $\mathbb{A}=(A_1,\ldots,A_n)$, the Karcher mean $G_{\mathrm{K}}(\omega;\mathbb{A})$ of $A_1,\ldots,A_n$ is defined by the unique positive invertible solution of the Karcher equation:

$$\sum_{i=1}^{n}\omega_i\log(X^{-\frac{1}{2}}A_iX^{-\frac{1}{2}})=0.$$

The power mean $P_t(\omega;\mathbb{A})$ is monotone increasing for t:

$$P_t(\omega;\mathbb{A})\leq P_s(\omega;\mathbb{A})\quad \text{for } -1\leq t\leq s\leq 1$$

and the Karcher mean is realized as the strong limit of the power means:

$$\mathrm{s-}\lim_{t\to 0}P_t(\omega;\mathbb{A})=G_{\mathrm{K}}(\omega;\mathbb{A})$$

under the strong-operator topology.

Problem: The range in which the power means $P_t(\omega;\mathbb{A})$ are defined, is $[-1,1]\backslash\{0\}$. However, the range in which the power arithmetic means $(\sum_{i=1}^{n}\omega_iA_i^t)^{1/t}$ are defined, is the set of all real numbers $\mathbb{R}$. It is then natural to ask the following question: Is it possible to extend the range in which the power means are defined?

The purpose of this paper is to extend the range of the definition of power means $P_t(\omega;\mathbb{A})$ defined by Lawson-Lim-Pálfia [4, 5].

2. PRELIMINARY

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm, $S(H)$ the set of all bounded self-adjoint operators, and $\mathbb{P} = \mathbb{P}(H)$ be the open convex cone of all positive invertible operators. For $X, Y \in S(H)$, we write $X \leq Y$ if $Y - X$ is positive, and $X < Y$ if $Y - X$ is positive invertible.

For $A, B \in \mathbb{P}$ and $t \in [0,1]$, the t-geometric operator mean is defined as

$$A \sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}.$$

For convenience, we use the notation $\natural_t$ for the binary operation

$$A \natural_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2} \qquad \text{for } t \notin [0,1],$$

whose formula is the same as $\sharp_t$. Though $A\sharp_t B$ for $t \in [0,1]$ has the monotonicity, $A\natural_s B$ for $s \notin [0,1]$ has not it.

Lemma 1. *Let $A, B, X, Y \in \mathbb{P}$ and $1 < t \leq 2$. Then*
(i) *If $X \leq Y$, then $Y \natural_t A \leq X \natural_t A$.*
(ii) *If $A \leq B$ with $m_1 \leq A \leq M_1$ and $m_2 \leq B \leq M_2$ and $m \leq X \leq M$, then*

$$X \natural_t A \leq K(m_i/M, M_i/m, t) X \natural_t B \quad \text{for } i = 1, 2,$$

where the generalized Kantorovich constant $K(m, M, t)$ is defined by

$$K(m,M,t) = \frac{mM^t - Mm^t}{(t-1)(M-m)}\left(\frac{t-1}{t}\frac{M^t - m^t}{mM^t - Mm^t}\right)^t \tag{1}$$

for any real number $t \in \mathbb{R}$, see [3, Theorem 2.53].
(iii) *If $m \leq A \leq M$, then*

$$\| X \|^{1-t} m^t \leq X \natural_t A \leq \| X^{-1} \|^{-(1-t)} M^t.$$

Proof. We only prove (i): For $1 < t \leq 2$

$$\begin{aligned} Y \natural_t A &= A \natural_{1-t} Y = A^{1/2}\left(A^{-1/2}YA^{-1/2}\right)^{1-t} A^{1/2} \\ &= A^{1/2}\left(A^{1/2}Y^{-1}A^{1/2}\right)^{t-1} A^{1/2} \\ &\leq A^{1/2}\left(A^{1/2}X^{-1}A^{1/2}\right)^{t-1} A^{1/2} \qquad \text{by } 0 < t-1 < 1 \text{ and } Y^{-1} \leq X^{-1} \\ &= X \natural_t A. \end{aligned}$$

□

The Thompson metric on $\mathbb{P}$ is defined by

$$d(A,B) = \log \max\{M(A/B), M(B/A)\}$$

where

$$M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \| B^{-1/2}AB^{-1/2} \| = r(B^{-1}A).$$

It is known that d is a complete metric on $\mathbb{P}$ and

$$d(A,B) = \| \log B^{-1/2}AB^{-1/2} \| = \| \log A^{-1/2}BA^{-1/2} \|,$$

see [7]. We list some basic properties of the Thompson metric:

Lemma 2 (see [1, 2]). *For $A, B, C, D \in \mathbb{P}$*
(i) *$d(A,B) = d(A^{-1}, B^{-1}) = d(T^*AT, T^*BT)$ for invertible $T \in B(H)$;*

(ii) $d(A+B, C+D) \leq \max\{d(A,C), d(B,D)\}$;
(iii) $d(A^t, B^t) \leq td(A,B)$ *for* $t \in [0,1]$;
(iv) $d(\alpha A, \alpha B) = d(A,B)$ *for positive real number* $\alpha > 0$;
(v) $d(A\sharp_t B, C\sharp_t D) \leq (1-t)d(A,C) + td(B,D)$ *for* $t \in [0,1]$.

For $A, B \in \mathbb{P}$, a map $\gamma_{A,B} : \mathbb{R} \mapsto \mathbb{P}$ defined by $\gamma_{A,B}(t) = A\natural_t B$ for $t \in \mathbb{R}$ is a path joining A and B. In particular, it is known that $\gamma_{A,B}(t)$ for $t \in [0,1]$ is a path joining A to B in $\mathbb{P}$. Then we have the following:

Theorem 3. *Let* $A, B \in \mathbb{P}$. *Then*

$$d(A\natural_s B, A\natural_t B) = |s-t| d(A,B) \qquad \text{for all } s,t \in \mathbb{R}.$$

Proof. By definition of the Thompson metric and Lemma 2

$$\begin{aligned} d(A\natural_s B, A\natural_t B) &= d((A^{-1/2}BA^{-1/2})^s, (A^{-1/2}BA^{-1/2})^t) \\ &= d((A^{-1/2}BA^{-1/2})^{s-t}, I) = \| \log(A^{-1/2}BA^{-1/2})^{s-t} \| \\ &= |s-t| \, \| \log A^{-1/2}BA^{-1/2} \| = |s-t| d(A,B). \end{aligned}$$

□

We have the following extimate in the case of $1 < t < 2$, which corresponds to (v) of Lemma 2:

Theorem 4. *Let* $A, B, C, D \in \mathbb{P}$ *such that* $m_1 A \leq C \leq M_1 A$ *and* $m_2 B \leq D \leq M_2 B$ *for some scalars* $0 < m_1 \leq M_1$ *and* $0 < m_2 \leq M_2$. *For each* $1 < t < 2$

$$d(A\natural_t B, C\natural_t D) \leq (t-1)d(A,C) + td(B,D) + \log K(t)$$

where $K(t) = \max\{K(m_1, M_1, t), K(m_2, M_2, t)\}$ *and the generalized Kantorovich constant* $K(m, M, t)$ *is defined by* (1).

3. MAIN RESULT

In this section, we extend to the range in which the power means due to Lawson-Lim-Pálfia are defined. For this, we need the following Lemma:

Lemma 5. *Let* $X, Y, A \in \mathbb{P}$ *and* $1 < t \leq 2$. *Then*

$$d(X\natural_t A, Y\natural_t A) \leq (t-1)d(X,Y).$$

Proof. For $1 < t \leq 2$,

$$\begin{aligned} d(X\natural_t A, Y\natural_t A) &= d(A\natural_{1-t} X, A\natural_{1-t} Y) \\ &= d((A^{1/2}X^{-1}A^{1/2})^{t-1}, (A^{1/2}Y^{-1}A^{1/2})^{t-1}) \quad \text{by (i) of Lemma 2} \\ &\leq (t-1)d(A^{1/2}XA^{1/2}, A^{1/2}YA^{1/2}) \quad \text{by (iii) of Lemma 2} \\ &= (t-1)d(X,Y) \quad \text{by (i) of Lemma 2.} \end{aligned}$$

□

Theorem 6. *Let* $A_1, A_2, \ldots, A_n \in \mathbb{P}$ *and a weight vector* $\omega = (\omega_1, \ldots, \omega_n)$. *Then for each* $1 < t < 2$, *the following equation has a unique positive invertible solution:*

$$X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i).$$

Proof. We will show that the map $f : \mathbb{P} \mapsto \mathbb{P}$ defined by $f(X) = \sum_{i=1}^n \omega_i(X\natural_t A_i)$ is a strict contraction with respect to the Thompson metric. Let $X, Y > 0$.

$$\begin{aligned} d(f(X), f(Y)) &\leq \max_{1\leq i\leq n}\{d(\omega_i(X\natural_t A_i), \omega_i(Y\natural_t A_i))\} \quad \text{by (ii) of Lemma 2} \\ &= \max_{1\leq i\leq n}\{d(X\natural_t A_i, Y\natural_t A_i)\} \quad \text{by (iv) of Lemma 2} \\ &\leq (t-1)d(X,Y) \quad \text{by Lemma 5.} \end{aligned}$$

Since $1 < t < 2$, it follows that f is a strict contraction and hence f has a unique fixed point. □

Definition 7. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ and a weight vector $\omega = (\omega_1, \ldots, \omega_n)$. For $t \in (1,2)$, we denote by $P_t(\omega; \mathbb{A})$ the unique positive invertible solution of*

$$X = \sum_{i=1}^n \omega_i(X \natural_t A_i).$$

For $t \in (-2,-1)$, we define $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$. In fact, $X = P_t(\omega; \mathbb{A})$ is the unique positive invertible solution of $X = \left(\sum_{i=1}^n \omega_i(X\natural_{-t}A_i)^{-1}\right)^{-1}$ and $X^{-1} = \sum_{i=1}^n \omega_i(X^{-1}\natural_{-t}A_i^{-1})$ if and only if $X^{-1} = P_{-t}(\omega; \mathbb{A}^{-1})$.

Remark 8. Let $t \in (1,2)$. Put $f : \mathbb{P} \mapsto \mathbb{P}$ defined by $f(X) = \sum_{i=1}^n \omega_i(X\natural_t A_i)$. By Theorem 6, f is a strict contraction for the Thompson metric and by the Banach fixed point theorem

$$\lim_{k\to\infty} f^k(X) = P_t(\omega; \mathbb{A}) \quad \text{for any } X \in \mathbb{P}.$$

Similarly, the map $g(X) = \left(\sum_{i=1}^n \omega_i(X\natural_{-t}A_i)^{-1}\right)^{-1}$ is a strict contraction for the Thompson metric and $\lim_{k\to\infty} g^k(X) = P_{-t}(\omega; \mathbb{A})$ for any $X \in \mathbb{P}$.

For $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n, M \in B(H), \omega = (\omega_1, \ldots, \omega_n)$ and for a permutation σ on n-letters, we set

$$M\mathbb{A}M^* = (MA_1M^*, \ldots, MA_nM^*), \quad A_\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$$
$$\hat{\omega} = \frac{1}{1-\omega_n}(\omega_1, \ldots, \omega_{n-1}).$$

We list some basic properties of $P_t(\omega; \mathbb{A})$ for $t \in (-2,2)\backslash[-1,1]$.

Proposition 9. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$, a weight vector $\omega = (\omega_1, \ldots, \omega_n)$ and let $t \in (-2,2)\backslash[-1,1]$.*

(i) $P_t(\omega; \mathbb{A}) = (\sum_{i=1}^n \omega_i A_i^t)^{1/t}$ *if the A_i's commute;*
(ii) $P_t(\omega_\sigma; \mathbb{A}_\sigma) = P_t(\omega; \mathbb{A})$ *for any permutation σ;*
(iii) $P_t(\omega; M\mathbb{A}M^*) = MP_t(\omega; \mathbb{A})M^*$ *for any invertible M;*
(iv) $P_t(\omega; \mathbb{A}^{-1})^{-1} = P_{-t}(\omega; \mathbb{A})$;
(v) $\sum_{i=1}^n \omega_i A_i \leq P_t(\omega; \mathbb{A})$ *for $t \in (1,2)$;*
(vi) $P_t(\omega; \mathbb{A}) \leq (\sum_{i=1}^n \omega_i A_i^{-1})^{-1}$ *for $t \in (-2,-1)$;*
(vii) *If $m \leq A_i \leq M$, then $m \leq P_t(\omega; \mathbb{A}) \leq m^{1-t}M^t$ for $t \in (1,2)$ and $m^{-t}M^{1+t} \leq P_t(\omega; \mathbb{A}) \leq M$ for $t \in (-2,-1)$;*
(viii) *For $t \in (1,2)$, $P_t(\omega; A_1, \ldots, A_{n-1}, X) = X$ if and only if $X = P_t(\hat{\omega}; A_1, \ldots, A_{n-1})$.*

Proof. Proofs from (i) to (iv) and (vii) are similar to those of [5].
(v): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (1, 2)$. Since $(1-t)A + tB \leq A \natural_t B$ for $1 < t < 2$, we have

$$\begin{aligned} X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i) &\geq \sum_{i=1}^{n} \omega_i ((1-t)X + tA_i) \\ &= (1-t)X + t \sum_{i=1}^{n} \omega_i A_i \end{aligned}$$

and hence $X \geq \sum_{i-1}^{n} \omega_i A_i$.

(vi): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (-2, -1)$. Since $X = \left(\sum_{i=1}^{n} \omega_i (X^{-1} \natural_{-t} A_i^{-1})\right)^{-1}$, it follows that

$$\begin{aligned} X^{-1} = \sum_{i=1}^{n} \omega_i (X^{-1} \natural_{-t} A_i^{-1}) &\geq \sum_{i=1}^{n} \omega_i ((1+t)X^{-1} + (-t)A_i^{-1}) \\ &= (1+t)X^{-1} - t \sum_{i=1}^{n} \omega_i A_i^{-1} \end{aligned}$$

and hence $X \leq (\sum_{i=1}^{n} \omega_i A_i^{-1})^{-1}$ for $t \in (-2, -1)$. □

Theorem 10. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ such that $0 < m \leq A_i \leq M$ for some scalars $0 < m \leq M$ and a weight vector $\omega = (\omega_1, \ldots, \omega_n)$. Let $1 < t \leq s < 2$. Then*

$$\begin{aligned} &d(P_t(\omega; \mathbb{A}), P_s(\omega; \mathbb{A})) \\ &\leq \frac{s-t}{(2-s)(2-t)} \left[t\Delta(\mathbb{A}) + \log K\left(m/M, (M/m)^t, t\right) \right], \end{aligned}$$

where the generalized Kantorovich constant $K(m, M, t)$ is defined by (1) and $\Delta(\mathbb{A}) = \max_{1 \leq i,j \leq n}\{d(A_i, A_j)\}$ denotes the d-diameter of $\mathbb{A} = (A_1, \ldots, A_n)$.

Proof. Put $X = P_t(\omega; \mathbb{A})$ and $Y = P_s(\omega; \mathbb{A})$, then by definition it follows that $X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i)$ and $Y = \sum_{i=1}^{n} \omega_i (Y \natural_s A_i)$. Therefore

$$\begin{aligned} d(X, Y) = d(Y, X) &= d\Big(\sum_{i=1}^{n} \omega_i (Y \natural_s A_i), \sum_{i=1}^{n} \omega_i (X \natural_t A_i)\Big) \\ &\leq \max_{1 \leq i \leq n} \{ d(Y \natural_s A_i, X \natural_t A_i) \} \\ &\leq \max_{1 \leq i \leq n} \{ d(Y \natural_s A_i, X \natural_s A_i) + d(X \natural_s A_i, X \natural_t A_i) \} \\ &\leq \max_{1 \leq i \leq n} \{ (s-1) d(Y, X) + (s-t) d(X, A_i) \} \\ &\leq (s-1) d(X, Y) + (s-t) \left[\frac{t}{2-t} \Delta(\mathbb{A}) + \frac{1}{2-t} \log K\left(m/M, (M/m)^t, t\right) \right] \end{aligned}$$

and hence we have

$$d(X, Y) \leq \frac{s-t}{2-s} \left[\frac{t}{2-t} \Delta(\mathbb{A}) + \frac{1}{2-t} \log K\left(m/M, (M/m)^t, t\right) \right].$$

□

Theorem 11. *Let* $\mathbb{A} = (A_1, \ldots, A_n)$ *and* $\mathbb{B} = (B_1, \ldots, B_n)$ *such that* $0 < m_1 \le A_i \le M_1$ *and* $0 < m_2 \le B_i \le M_2$ *for* $i = 1, \ldots, n$ *for some scalars* $0 < m_1 \le M_1$ *and* $0 < m_2 \le M_2$. *Then for each* $1 < t < 2$

$$d(P_t(\omega;\mathbb{A}), P_t(\omega;\mathbb{B})) \le \frac{t}{2-t} \max_{1\le i\le n}\{d(A_i, B_i)\} + \frac{1}{2-t} \log K_1(t),$$

where

$$K_1(t) = \max\{K(m_2/m_1^{1-t}M_1^t, m_2^{1-t}M_2^t/m_1, t), K(m_2/M_1, M_2/m_1, t)\}.$$

Proof. Put $X = P_t(\omega;\mathbb{A})$ and $Y = P_t(\omega;\mathbb{B})$. Then it follows that

$$\begin{aligned} d(X,Y) &= d(\sum_{i=1}^n \omega_i(X\natural_t A_i), \sum_{i=1}^n \omega_i(Y\natural_t B_i)) \\ &\le \max_{1\le i\le n}\{d(X\natural_t A_i, Y\natural_t B_i)\} \\ &\le \max_{1\le i\le n}\{(t-1)d(X,Y) + td(A_i,B_i) + \log K_1(t)\} \\ &= (t-1)d(X,Y) + t\max_{1\le i\le n}\{d(A_i,B_i)\} + \log K_1(t) \end{aligned}$$

and hence we have

$$d(X,Y) \le \frac{t}{2-t} \max_{1\le i\le n}\{d(A_i, B_i)\} + \frac{1}{2-t} \log K_1(t).$$

□

Conclusion and problems: We were able to extend the range of the power means $P_t(\omega;\mathbb{A})$ to the $1 < |t| < 2$. Unfortunately, we do not know whether the power means are defined for $t \ge 2$ or not. For example, we put $t = 2$ and $\mathbb{A} = (A, B, C)$. Then the power mean $P_2(\omega;\mathbb{A})$ is the unique positive invertible solution of

$$X = \omega_1 AX^{-1}A + \omega_2 BX^{-1}B + \omega_3 CX^{-1}C.$$

What is X?

Moreover, we do not know whether the power means are monotone increasing or not for $1 < t < 2$:

$$P_t(\omega;\mathbb{A}) \le P_s(\omega;\mathbb{A}) \quad \text{for } 1 \le t \le s < 2$$

holds or not.

References

[1] R. Bhatia, *On the exponential metric increasing property*, Linear Algebra Appl., **375** (2003), 211–220.

[2] G. Corach, H. Porta and L. Recht, *Convexity of the geodesic distance on spaces of positive operators*, Illinois J. Math. **38**(1994), 87–94.

[3] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[4] J. Lawson and Y. Lim, *Karcher means and Karcher equations of positive definite operators*, Trans. Amer. Math. Soc., Series B, **1** (2014), 1–22.

[5] Y. Lim and M. Pálfia, *Matrix power means and the Karcher mean*, J. Funct. Anal., **262** (2012), 1498–1514.

[6] Y. Seo, *Operator power means due to Lawson-Lim-Pálfia for* $1 < t < 2$, Linear Algebra Appl., **459** (2014), 342–356.

[7] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., **14**(1963), 438–443.

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